Mon. Not. R. astr. Soc. (1970) **148**, 249–260.

THE DYNAMICS OF CLUSTERS OF GALAXIES IN UNIVERSES WITH NON-ZERO COSMOLOGICAL CONSTANT, AND THE VIRIAL THEOREM MASS DISCREPANCY

J. C. Jackson

(Communicated by R. J. Tayler)

(Received 1970 January 28)

SUMMARY

The effects on groups of galaxies of a non-zero cosmological constant are considered. A version of the virial theorem which includes a cosmological term is derived. It is shown that a negative value of λ, in the region $-10^{-34}\,\mathrm{s}^{-2}$ to $-10^{-33}\,\mathrm{s}^{-2}$, can account quite well for the well-known virial theorem mass discrepancy in actual clusters. The theorem predicts that a graph of log M_{VT}/M against log (cluster mean density) should be a straight line with slope $-0{\cdot}5$; the observed value is $-0{\cdot}43 \pm 0{\cdot}08$. It is proved that this effect could be responsible for the discrepancy only in universes with curvature index $k = -1$. The Universe ages corresponding to the above range of λ values are calculated; they are shorter than currently accepted values, but, it is argued, not impossibly so.

1. INTRODUCTION

An outstanding problem of extra-galactic research is that of the so-called 'missing mass' from clusters of galaxies (Zwicky 1933; Smith 1936; *Astronomical Journal* 1961). The total mass of a cluster can be estimated in two ways. Knowing the motions of its member galaxies, the virial theorem gives one estimate, M_{VT} say. The second is obtained by separately estimating the mass of each individual member, and summing these masses, to give a total M say. Almost without exception, it is found that M_{VT} is considerably greater than M, a typical value of M_{VT}/M being about 20. The work reported here attempts to explain this phenomenon, although the results obtained should be useful even if the discrepancy is removed in some entirely different manner.

This observation has been interpreted in various ways. The hypothesis which has received most attention is that the mass M evaluated above seriously underestimates the real mass of the cluster. Recent observations of M 87 (de Vaucouleurs 1969; Arp & Bertola 1969) suggest that individual galaxies are more massive than previously supposed. This object appears to have a large, tenuous halo of stars. Another suggestion is that large quantities of 'invisible' matter, presumably hydrogen, reside in the space between the cluster galaxies. Observations by no means preclude this possibility, but they are largely of a negative nature. A summary of some of these can be found in Burbidge & Burbidge (1967); X-ray observations have been interpreted by Felten *et al.* (1966); Woolf (1967); Friedman & Byram (1967).

Mass determination using the virial theorem assumes that the system is in equilibrium, and Ambartsumian (1961) has questioned this assumption. He postulates that the clusters are unstable, and violently expanding. This hypothesis is difficult to test (Rood, Rothman & Turnrose 1969).

The validity of Newton's inverse square law over distances ~ 1 Mpc (the symbol $\sim$ will be used throughout this paper to mean 'approximately equal to') has been questioned by Finzi (1963), who suggests arbitrary modifications which could stabilize the clusters. In effect, the mechanism put forward here falls into the same category. However, the 'modification' of gravitational law is one which has a natural place within the framework of general relativity, and one with which cosmologists have long been familiar, namely the inclusion of a cosmological term in Einstein's equations. We show that if the cosmological constant λ is not zero, then values of M_{VT}/M not equal to unity are inescapable. The effect we require, namely $M_{VT}/M > 1$, is produced by negative values of λ, while $\lambda > 0$ gives $M_{VT}/M < 1$. This is easy to understand heuristically. Positive λ is often said to add a 'cosmic repulsion' to Newton's law, and similarly negative λ adds a 'cosmic attraction'. In the latter case, the effective attractive force at large distances is greater than that given by the inverse square law, and the velocity dispersions necessary for equilibrium correspondingly greater. It is also proved that only in certain well-defined world models can the effect be as large as the one observed.

The next section is devoted to developing a version of the virial theorem which includes the effects of a cosmological constant, after which the theoretical predictions are compared with observations. Finally, the general cosmological implications of the hypothesis are considered.

2. MODIFICATION OF THE VIRIAL THEOREM

We shall consider an isolated, spherically symmetric, stable cluster, situated in de Sitter space. (Until further notice, relativistic units will be used, in which $G = c = 1$.) Space-time outside this is described by the metric

$$ds^2 = (1-2M/r-\tfrac{1}{3}\lambda r^2)\,dt^2-(1-2M/r-\tfrac{1}{3}\lambda r^2)^{-1}\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2).$$

It will be assumed that the cluster is not particularly relativistic, so that if R is its radius the terms $2M/R$, $\frac{1}{3}\lambda R^2$ are much smaller than unity. We expect the cosmological term to be important if $\frac{1}{3}\lambda R^2 \sim 2M/R$, i.e. $\lambda/\bar{\mu} \sim 1$, where $\bar{\mu}$ is the cluster's mean density. The relevance of this parameter will be verified by the detailed calculations.

The usual spherically symmetric static metric will be used inside the cluster

$$ds^2 = e^{\nu}\,dt^2 - e^{\sigma}\,dr^2 - r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

and no approximations will be made initially. The program is to describe the galaxies, which are treated as identical (it is easy to remove this restriction, which is retained for the sake of simplicity), collisionless point-particles, by a distribution function, and to write down the general-relativistic Boltzmann equation governing it. This can be integrated over velocity space, to yield an equation which is used in conjunction with Einstein's field equations to give the required form of the virial theorem. Introduce the following orthonormal frame of covariant vectors

$$\overset{0}{\lambda}_{\mu} = e^{\nu/2}\delta_{\mu}{}^{0} \qquad \overset{1}{\lambda}_{\mu} = e^{\sigma/2}\delta_{\mu}{}^{1}$$
$$\overset{2}{\lambda}_{\mu} = r\delta_{\mu}{}^{2} \qquad \overset{3}{\lambda}_{\mu} = r\sin\theta\,\delta_{\mu}{}^{3}.$$

Let u^μ be the four-velocity of a typical galaxy, satisfying $u^\mu u_\mu = 1$, with tetrad components $\overset{\alpha}{u} = u^\mu \overset{\alpha}{\lambda}_\mu$ (Greek letters have the range 0–3, and Roman ones 1–3; early Greek letters denote tetrad components). The relativistic Boltzmann equation in tetrad form is (Lindquist 1966)

$$\overset{\alpha}{u}\, \underset{\alpha}{\lambda}^\mu \frac{\partial f}{\partial x^\mu} + \gamma^{\mathrm{a}}_{\beta\gamma} \overset{\beta}{u} \overset{\gamma}{u} \frac{\partial f}{\partial \overset{\mathrm{a}}{u}} = 0 \tag{1}$$

where $f(x^\mu, \overset{\mathrm{a}}{u})$ is the distribution function and $\gamma^\alpha_{\beta\gamma} = \overset{\alpha}{\lambda}_{\mu;\nu} \underset{\beta}{\lambda}^\mu \underset{\gamma}{\lambda}^\nu$ are the rotation coefficients. Writing

$$\overset{0}{u} = u_t \qquad \overset{1}{u} = u_r$$
$$\overset{2}{u} = u_\theta \qquad \overset{3}{u} = u_\phi$$

equation (1) becomes, in view of the symmetry (i.e. the only coordinate dependence of f is upon r)

$$u_r \frac{\partial f}{\partial r} - \left(\frac{1}{2} u_t^2 \frac{\partial \nu}{\partial r} - \frac{u_\theta^2 + u_\phi^2}{r}\right) \frac{\partial f}{\partial u_r}$$
$$- \frac{1}{r} u_r \left(u_\theta \frac{\partial f}{\partial u_\theta} + u_\phi \frac{\partial f}{\partial u_\phi}\right)$$
$$- \frac{1}{r} e^{\sigma/2} u_\phi \cot\theta \left(u_\theta \frac{\partial f}{\partial u_\phi} - u_\phi \frac{\partial f}{\partial u_\theta}\right) = 0. \tag{2}$$

The symmetry demands that the coefficient of $\cot\theta$ be zero, which implies that f is a function of r, u_r, and $u_\theta^2 + u_\phi^2$. Equation (2) is now multiplied by $m u_r\, d\omega$, where m is the mass of each galaxy, and $d\omega = du_r\, du_\theta\, du_\phi / u_t$ is the invariant volume element of velocity space. If f vanishes sufficiently rapidly as the velocities $\to \pm\infty$, we have, on integrating over velocity space

$$r \frac{\partial}{\partial r} (\mu \overline{u_r^2}) + \tfrac{1}{2} \mu (\overline{u_t^2} + \overline{u_r^2}) r \frac{\partial \nu}{\partial r} - \mu (\overline{u_\theta^2} + \overline{u_\phi^2} - 2\overline{u_r^2}) = 0 \tag{3}$$

where, at each point, $\overline{u_r^2}$ = average value of u_r^2 etc., and μ is the mass density. The Newtonian form of equation (3) can be found in Ogorodnikov (1965). Equation (3) is now multiplied by $4\pi r^2$ and integrated over the cluster, to give

$$- \int_0^R 4\pi\mu (\overline{u_r^2} + \overline{u_\theta^2} + \overline{u_\phi^2}) r^2\, dr + \frac{1}{2} \int_0^R 4\pi r^3 \mu (\overline{u_t^2} + \overline{u_r^2}) \frac{\partial \nu}{\partial r}\, dr = 0. \tag{4}$$

At this stage, we introduce Einstein's equations

$$-8\pi T_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu}. \tag{5}$$

The conventions defining the various signs are

$$u_{\mu;\nu\sigma} - u_{\mu;\sigma\nu} = R^\epsilon{}_{\mu\nu\sigma} u_\epsilon$$
$$R_{\mu\nu} = R^\epsilon{}_{\mu\nu\epsilon}.$$

The energy–momentum tensor $T_{\mu\nu}$ can be written in terms of the distribution function as

$$T_{\mu\nu} = \int f m u_\mu u_\nu\, d\omega. \tag{6}$$

It is easy to show that in this situation equations (5) and (6) imply

$$4\pi\mu(\overline{u_t^2}+\overline{u_r^2}+\overline{u_\theta^2}+\overline{u_\phi^2}) = e^{-\sigma}\left\{\frac{\nu''}{2}+\frac{\nu'}{r}-\frac{1}{4}\,\nu'(\sigma'-\nu')\right\}+\lambda \tag{7}$$

where $\nu' \equiv \dfrac{\partial\nu}{\partial r}$ etc.

It is convenient here to introduce our approximations, namely that σ and ν are small, so that the quadratic terms in equation (7) may be neglected, and that the galaxies have velocities much smaller than that of light, so that $\overline{u_r^2}$, $\overline{u_\theta^2}$, $\overline{u_\phi^2} \ll \overline{u_t^2}$ and $\overline{u_t^2} \sim 1$. These are usually the conditions under which Newtonian theory is valid, and certainly apply to actual clusters. From now on, the calculations are essentially Newtonian. We have chosen to use relativity at the start so that the cosmological constant can be introduced in an unambiguous way. Equations (4) and (7) become

$$-2K+\frac{1}{2}\int_0^R 4\pi r^3\mu\,\frac{\partial\nu}{\partial r}\,dr = 0 \tag{8}$$

$$4\pi\mu = \frac{1}{2}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\nu}{\partial r}\right)+\lambda \tag{9}$$

where K is the total kinetic energy of the galaxies. The gravitational potential energy Ω of the system is defined by

$$\Omega = -\int_0^R \frac{M_r}{r}\,dM_r$$

where M_r is the mass out to radius r, so that $dM_r = 4\pi r^2\mu\,dr$. Equation (9) has a first integral

$$M_r = \frac{1}{2}\,r^2\,\frac{\partial\nu}{\partial r}+\frac{1}{3}\,\lambda r^3$$

and we find

$$\frac{1}{2}\int_0^R 4\pi r^3\mu\,\frac{\partial\nu}{\partial r}\,dr = -\Omega-\frac{1}{3}\,\lambda I$$

where

$$I = \int_0^R r^2 dM_r$$

is the moment of inertia of the system. Using this in equation (8) gives the final result, i.e. the new form of the virial theorem, which is

$$2K+\Omega+\tfrac{1}{3}\lambda I = 0.$$

The above equation must be translated into an expression for M_{VT}/M. Introduce the radii R_Ω and R_I, defined by

$$R_\Omega = \frac{M^2}{\displaystyle\int_0^R \frac{M_r}{r}\,dM_r}$$

$$R_I = \left\{\frac{\displaystyle\int_0^R r^2 dM_r}{M}\right\}^{1/2}$$

so that

$$\Omega = -M^2/R_\Omega$$
$$I = MR_I^2. \tag{10}$$

The definition of M_{VT} is

$$2K = M_{VT}^2/R_\Omega$$

and if these expressions are substituted into the new virial theorem, we find

$$M_{VT}/M = \left(1-\frac{\lambda}{4\pi\bar{\rho}}\right)^{1/2}$$

where $\bar{\rho} = M/(\frac{4}{3}\pi R_\Omega R_I^2)$ is a quantity related to the mean density $\bar{\mu}$ of the cluster. We can write $\bar{\rho} = \alpha\bar{\mu}$, where α is a factor of order unity, its precise value depending upon the density distribution. For example, if the galaxies are concentrated in a thin shell so that $\mu = \delta(r-R)$, or if the cluster is a sphere of uniform density so that μ is constant out to $r = R$, α is exactly unity. Transforming to more usual units, this equation becomes

$$M_{VT}/M = \left(1-\frac{\lambda}{4\pi G\bar{\rho}}\right)^{1/2} \tag{11}$$

λ is measured in units of (seconds)$^{-2}$. We see that negative values of λ are required to give $M_{VT}/M > 1$. If $M_{VT}/M > 3$, which is true for most clusters, the term unity can be neglected with little loss of accuracy, which is equivalent to saying $\Omega \ll \frac{1}{3}\lambda I$. Then

$$\log\frac{M_{VT}}{M} = -\frac{1}{2}\log\bar{\rho}+\frac{1}{2}\log\left(\frac{-\lambda}{4\pi G}\right)$$

so that a graph of $\log M_{VT}/M$ against $\log\bar{\rho}$ should be a straight line with slope -0.5. In the next section we shall, in fact, plot $\log M_{VT}/M$ against $\log(3M/[4\pi R_\Omega^3])$, and not $\log\bar{\rho}$, for actual clusters, because the information necessary to evaluate each R_I is not available at present. However, we can write $R_I = \beta R_\Omega$, where β is a factor of order unity whose exact value depends upon the density distribution, giving

$$\log\frac{M_{VT}}{M} = -\frac{1}{2}\log\left(\frac{3M}{4\pi R_\Omega^3}\right)+\frac{1}{2}\log\left(\frac{-\lambda\beta^2}{4\pi G}\right) \tag{12}$$

β should not vary greatly from cluster to cluster, and this curve should still be a straight line with slope -0.5.

3. COMPARISON WITH OBSERVATIONS

In order to test the theory well, information about large numbers of well-populated, approximately spherically symmetric clusters would be required, including details of their density distributions. The Coma cluster fits this description very well, and has been extensively studied. However, most of the data available relates to clusters with few members. We shall make use of the analysis of the results of Holmberg (1964) and de Vaucouleurs (1968) given by Rood *et al.* (1969). These authors assign to each cluster a dimension R_Ω which is defined by equation (10), although we cannot, of course, claim that this is precisely the parameter we require, as the clusters they treat are not in general spherically symmetric. They give values of M based on estimates of the mass of each galaxy obtained by measuring

TABLE I

The numbers in the first column are taken from the paper of Rood et al. *In the last column,* $E-32$ *means* 10^{-32} etc.

Group	R_Ω (Mpc)	M ($10^{10}M_\odot$)	$\frac{3M}{4\pi R_\Omega^3}$ (g cm^{-3})	$\frac{M_{VT}}{M}$	$-\lambda$ (sec^{-2})
	Logarithms				
NGC 3031	−0·60	1·50	−27·49	0·66	5·34E−33
Leo	−0·27	1·94	−28·04	0·59	1·07E−33
Virgo	0·38	2·92	−29·01	1·41	5·35E−33
1	0·19	0·97	−30·39	1·94	2·57E−33
2	−0·44	0·89	−28·58	1·44	1·65E−32
3	0·29	1·07	−30·59	1·98	1·95E−33
4	0·52	1·85	−30·50	0·54	2·89E−36
5	−0·42	1·27	−28·26	1·01	4·73E−33
6	0·35	0·75	−31·09	2·34	3·23E−33
7	0·25	1·31	−30·23	1·97	4·26E−33
9	−0·17	1·46	−28·82	0·83	5·62E−34
10	0·17	1·25	−30·05	1·21	1·94E−34
11	−0·75	1·87	−26·67	−0·12	−7·70E−34
13	−0·12	1·95	−28·48	1·53	3·15E−32
18	0·21	1·63	−29·79	2·83	6·16E−31
19	0·44	2·56	−29·55	1·30	9·29E−34
20	0·57	2·00	−30·50	2·45	2·09E−32
26	0·04	1·97	−28·94	2·01	9·98E−32
31	0·76	1·84	−31·23	1·95	3·88E−34
33	0·17	1·97	−29·33	1·30	1·54E−33
34	0·32	1·53	−30·22	0·95	3·92E−35
41	0·49	1·83	−30·43	1·82	1·35E−33
42	0·84	1·70	−31·61	2·04	2·45E−34
47	−0·63	1·71	−27·19	0·40	2·84E−33
49	−0·55	2·05	−27·09	0·94	5·05E−32
50	−0·12	2·01	−28·42	1·42	2·18E−32
51	0·68	1·72	−31·11	2·59	9·76E−33

its luminosity, and multiplying by a suitable mass to luminosity ratio. We have selected groups containing five or more members, with five or more measured velocities. The choice of five is quite arbitrary, being the largest number compatible with having more than a small sample. The relevant information is given in Table I (I have corrected the original figures for projection effects, using the correction factors quoted by Rood *et al.*), with the value of λ needed to explain each value of M_{VT}/M, estimated using

$$\lambda = -3GM/R_\Omega^3(M_{VT}^2/M^2-1) \qquad (13)$$

which is exact if $\beta = 1$.

Log M_{VT}/M is plotted against log $3M/4\pi R_\Omega^3$ in Fig. 1. There is strong evidence of linear correlation between these quantities. The correlation coefficient is $-0{\cdot}72$; the probability of two uncorrelated variables giving |correlation coefficient| $\geqslant 0{\cdot}72$ is less than 10^{-5} for a sample of this size. The slope of the least-squares straight line (i.e. the line which minimizes the sum of squares of its perpendicular distances from the points, which, as there is observational error in both ordinate and abscissa, is the appropriate 'line of best fit', rather than a regression line) is $-0{\cdot}43$, with

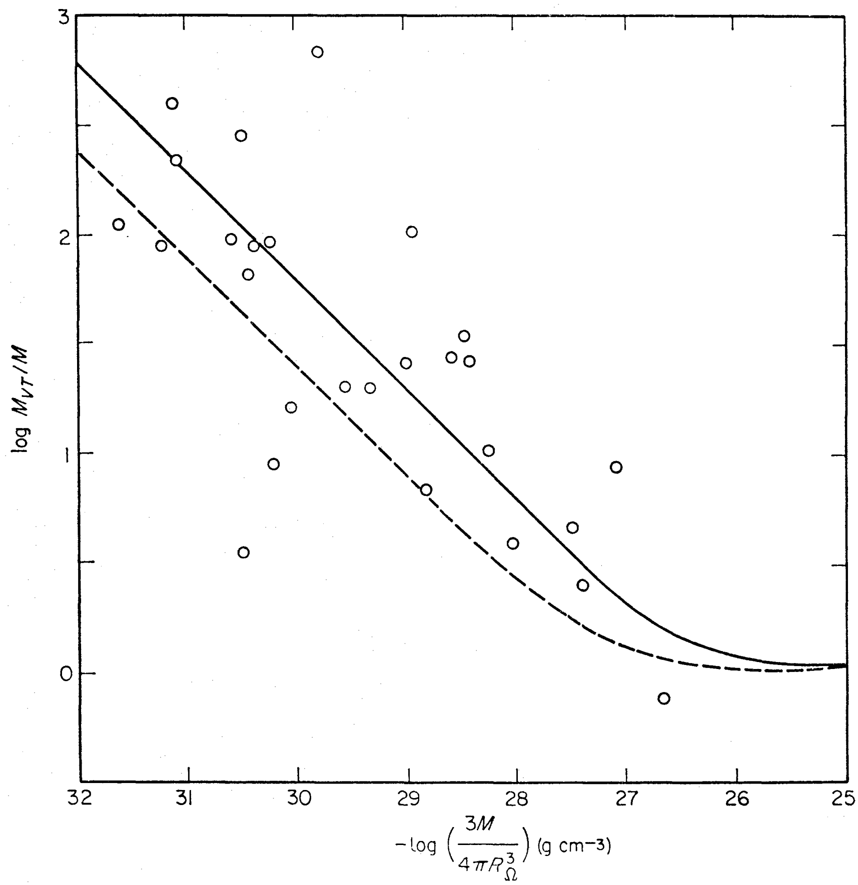

FIG. 1. *Solid and dashed curves are equation* (12), *with* $\lambda = -2{\cdot}9\times10^{-33}$ and -5×10^{-34} sec^{-2} respectively, taking $\beta = 1$.

a standard deviation of 0·08, which compares well with the theoretical prediction of − 0·5. The centre of mass of the points is at

$$\log M_{VT}/M = 1{\cdot}46 \qquad \log\frac{3M}{4\pi R_\Omega{}^3} = -29{\cdot}39$$

and the upper curve in Fig. 1 is the theoretical line passing through this point. It corresponds to $\lambda = -2{\cdot}9\times10^{-33}$ s^{-2}. This should almost certainly be regarded as a lower limit, for the following reasons:

(1) The masses and dimensions given in Table I are functions of Hubble's constant H_0, although the ratio M_{VT}/M is independent of this parameter. Both M and R_Ω are proportional to $1/H_0$, so that from equation (13) λ is proportional to $H_0{}^2$. The figures used here are for $H_0 = 100$ km s^{-1} Mpc^{-1}, which is probably too high. Sandage (1958) gives $H_0 = 75\pm25$ km s^{-1} Mpc^{-1}, and taking his lower limit would reduce the estimate of $-\lambda$ by a factor of 4. This uncertainty has an entirely systematic effect; the least-squares line slope is not a function of H_0.

(2) It is very difficult when observing a group of galaxies to sort out real members from background and foreground objects (Holmberg 1961; Rood *et al.* 1969), and it seems inevitable that some of the clusters listed have included fictitious members. This increases the measured velocity dispersion, and considerably increases the calculated value of M_{VT}/M, which is proportional to the square of the velocity dispersion. The effect on the required value of λ is even more marked, as it is proportional to the fourth power of the velocity dispersion. This largely systematic

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error can perhaps be eliminated by taking the lower envelope of the points in Fig. 1, which would correspond to those clusters whose observed members are, by chance, all real members. The dashed curve is the theoretical line for $\lambda = -5\times10^{-34}\,\mathrm{s}^{-2}$, and coincides quite well with the lower envelope of the main body of points.

Taking into account the uncertainty in H_0, our estimates of λ are

$$-2{\cdot}9\times10^{-33}\leqslant\lambda\leqslant-7{\cdot}3\times10^{-34}\quad(\mathrm{s}^{-2})$$
(from centre of mass of points)

$$-5\times10^{-34}\leqslant\lambda\leqslant-1{\cdot}3\times10^{-34}\quad(\mathrm{s}^{-2})$$
(from lower envelope)

4. COSMOLOGICAL IMPLICATIONS

In this section we shall show that the theory only makes sense in those cosmological models with curvature index $k = -1$, and the above values of λ will then be translated into figures for the age of the Universe and its deceleration parameter.

The field equations for Friedman models are

$$8\pi G\rho = 3kc^2/R^2+3(\dot{R}/R)^2-\lambda \tag{14}$$
$$8\pi Gp/c^2 = -kc^2/R^2-2\ddot{R}/R-(\dot{R}/R)^2+\lambda \tag{15}$$

where $R = R(t)$ has its usual significance (see for example Bondi 1961), and should not be confused with the cluster radius R used previously; ρ is the mean universal matter density and p its pressure. If $k\geqslant0$, the inequality

$$8\pi G\rho>-\lambda$$

follows from equation (14). This inequality and equation (11) imply

$$M_{VT}/M<(1+2\rho/\bar{\rho})^{1/2}.$$

As a cluster is by definition a region of the Universe where the density is considerably greater than ρ, we see that under these conditions M_{VT}/M could only be marginally greater than unity. The hypothesis, that large mass discrepancies can be explained by introducing a negative cosmological constant, is tenable only if $8\pi G\rho\ll-\lambda$, which is possible only if $k = -1$. The Universe must be ‘ empty’, and the dynamics of the matter in it dominated not by its mutual attraction but by the cosmological term. A well-known feature of present estimates of ρ is, of course, that the Universe does appear to be ‘empty’ (Sandage 1961). We shall proceed assuming $k = -1$, and approximate by setting $\rho = 0$, $p = 0$. If the dimensionless variables

$$q_0 = -\ddot{R}_0/H_0^2R_0 \quad \text{(the deceleration parameter)}$$
$$\lambda_0 = +\tfrac{1}{3}\lambda/H_0^2$$

are introduced, the above approximations imply

$$q_0 = -\lambda_0$$

(see, for example, Tomita & Hayashi 1963). Equations (14) and (15) can be integrated explicitly, yielding

$$R = (-\tfrac{1}{3}\lambda)^{1/2}\sin\{(-\tfrac{1}{3}\lambda)^{1/2}t\}$$
(assuming $\lambda<0$)

and

$$H_0t_0 = q_0^{-1/2}\cot^{-1}(q_0^{-1/2})$$

where t_0 is the present age of the Universe. The values of λ determined from the centre of mass of points give

$$q_0 = 92{\cdot}3 \qquad H_0t_0 = 0{\cdot}153$$
$$t_0 = 1{\cdot}5\times 10^9 \text{ years if } H_0 = 100 \text{ km s}^{-1}\text{ Mpc}^{-1}$$
$$t_0 = 3{\cdot}0\times 10^9 \text{ years if } H_0 = 50 \text{ km s}^{-1}\text{ Mpc}^{-1}$$

while the lower envelope gives

$$q_0 = 15{\cdot}9 \qquad H_0t_0 = 0{\cdot}332$$
$$t_0 = 3{\cdot}3\times 10^9 \text{ years if } H_0 = 100 \text{ km s}^{-1}\text{ Mpc}^{-1}$$
$$t_0 = 6{\cdot}5\times 10^9 \text{ years if } H_0 = 50 \text{ km s}^{-1}\text{ Mpc}^{-1}.$$

These ages are clearly too short. Studies of globular clusters (Sandage 1962; Demarque & Larson 1964) show that there are stars in the galaxy at least 16×10^9 years old. However, there is great difficulty in fitting any cosmological model to this timescale; according to Sandage (1961) the age of the Universe is $5{\cdot}6-11{\cdot}2\times 10^9$ years if λ is zero, and Tomita & Hayashi (1963) were not able to improve the situation by assuming positive values of λ. This conflict must be resolved, perhaps by another revision of the distance scale; viewed in this light, our figures of $3{\cdot}0-6{\cdot}5\times 10^9$ years are not wholly unreasonable. The above q_0 values are also in conflict with observations, although Sandage's estimate ($q_0 = 2{\cdot}5$) (1961) is not relevant here, since he assumes that λ is zero. It is in principle possible to determine both q_0 and λ_0 by examining the precise shape of the redshift—magnitude diagram. In this way, Solheim (1966) finds that the most probable values of these parameters are

$$\lambda_0 = +4{\cdot}59 \qquad q_0 = -0{\cdot}06.$$

The question answered by Sandage is: if $\lambda_0 = 0$, what is the most probable value of q_0? The relevant question here is: if the density is zero, so that $\lambda_0 = -q_0$, what value of q_0 best fits the observational data? According to Solheim, the answer is $q_0 \sim 5$. Although this figure is not as high as the ones demanded by this paper, it is interesting to note that observations give a higher value of q_0 if $\lambda_0 = -q_0$ than they do if $\lambda_0 = 0$.

This section will be concluded with a discussion of the state of affairs which will exist if the viral theorem mass discrepancy is eventually removed in some 'conventional' manner, say by finding large amounts of intergalactic matter. The purpose of this paper would then be to place limits on the cosmological constant. For example, if for clusters with mean density $\sim 10^{-28}$ g cm^{-3} agreement is found to the extent of $|M_{VT}/M-1|<0{\cdot}1$, equation (11) would enable us to say $|\lambda|<1{\cdot}7\times 10^{-35}$ s^{-2}. (The one non-cosmological observation used so far to produce limits for λ, namely planetary motions, gives $|\lambda|<$ about 10^{-24} s^{-2}.)

It is perhaps worth pointing out that equation (11) raises considerable difficulties for Lemaître cosmologies, which require positive λ. Interest in these models has been stimulated by the discovery of large numbers of quasar absorption line redshifts concentrated at $z\sim 1{\cdot}95$ (Petrosian, Salpeter & Szekeres 1967; Shklovsky 1967; Kardashev 1967). Enough cluster intergalactic matter to 'over-eliminate' the discrepancy, i.e. to produce values of $M_{VT}/M<1$, would have to be found if such models are to be considered. Suppose that the smoothed out matter density during the quasi-Einstein stage of such a universe is ρ_L, which requires a value of $\lambda\sim 4\pi G\rho_L$; equation (11) shows that if λ has this value, stable clusters with $\bar{\rho}$ less than ρ_L cannot exist at any epoch. This cannot be subjected to observational test, since the real value of $\bar{\rho}$ for actual clusters is not known. However, for a cluster with known M_{VT} and dimensions, it is possible to calculate the value of M which is compatible with the above value of λ. This value is given by

$$M_{VT}/M = -\delta + (1+\delta^2)^{1/2} \tag{16}$$

where

$$\delta = \tfrac{1}{2}\rho_L/\bar{\rho}_{VT}$$

and

$$\bar{\rho}_{VT} = \frac{M_{VT}}{\frac{4}{3}\pi R_\Omega R_I^2}$$

is the viral theorem mean density. Fig. 2 compares the predictions of this equation with figures for the clusters listed in Table I, using Kardashev's estimate of ρ_L (1967), namely $\rho_L \sim 4{\cdot}8 \times 10^{-29}$ g cm^{-3}. The real mass of the low density clusters would have to be about $1000 \times$ luminous mass, if the Lemaître model is to be taken seriously.

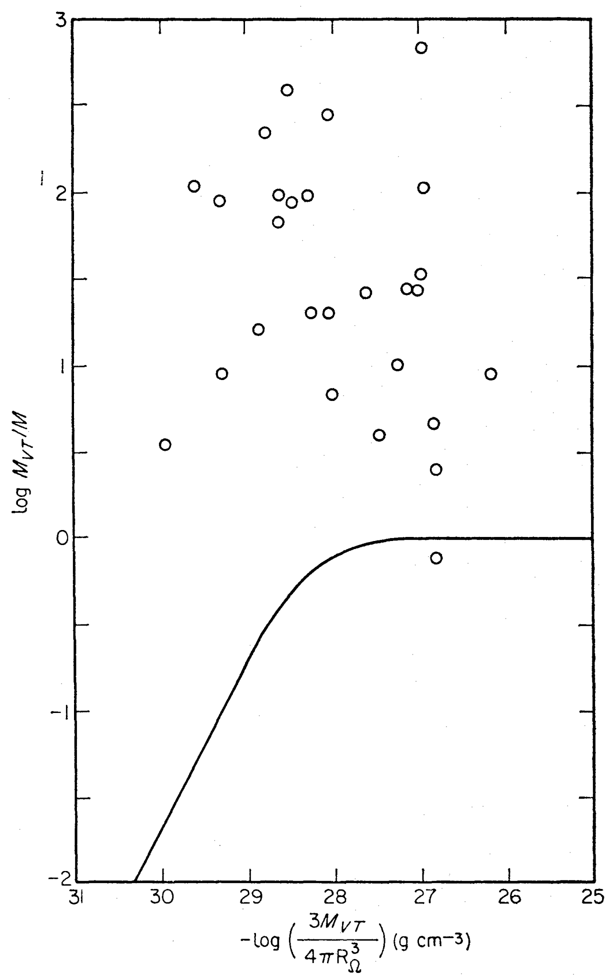

FIG. 2. *The continuous curve is equation* (16), *and represents the behaviour expected in the Lemaître model. The open circles represent the clusters listed in Table I.*

5. CONCLUSIONS

By choosing a suitable λ, the theory discussed here can be made to account quite well for the observed mass discrepancies. However, the real significance of the discussion at the end of Section 3 is that we cannot explain the whole of the mass discrepancy by introducing a negative cosmological constant, and at the same time avoid ages of the Universe which are unreasonably short. Only by assuming that some of the discrepancy is due to the 'fictitious member' observational effect, can ages which are at all reasonable be obtained. This observational effect undoubtedly exists, but its magnitude is unknown. Circumstances which would alter these conclusions are conceivable. Another radical reduction in the determined value of Hubble's constant could resolve the timescale difficulty, although not the q_0 conflict. The discovery of some cluster intergalactic matter, say enough to multiply estimates of M by a factor of 2 or 3, would still leave large discrepancies to be explained, which could then be achieved using smaller values of $-\lambda$. In this case, more credible values of both t_0 and q_0 would be obtained.

The universal mass defect, in the sense of Sandage (1961), and the cluster one are intimately related, in that finding the latter would more or less account for the former. In Section 4 we are in effect saying that the hypothesis put forward here can also solve both problems. Sandage (1961) has already pointed out that negative values of λ could account for the cosmological discrepancy.

Perhaps the most important aspect of this work is that it points out that cosmological constants, with magnitudes which would not be excluded by cosmological observations, have observable effects upon groups of galaxies, and that it shows how to convert the relevant cluster observations into estimates of λ.

ACKNOWLEDGMENTS

I am extremely grateful to Professor T. L. Page, for helping me find the information upon which Section 3 of this paper is based, and to Dr H. J. Rood, for sending me a preprint of his paper.

This work has benefitted from many conversations with members of the Astronomy Centre of the University of Sussex, in particular Drs A. C. Edwards and J. Hazlehurst, and Professor W. H. McCrea who also read the manuscript and suggested several valuable improvements in presentation. My thanks also go to Dr R. C. Smith, for advice about the statistical treatment of the observational data.

Astronomy Centre, University of Sussex, Brighton BN1 9QH.

Received in original form 1969 *November* 10.

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NOTE ADDED IN PROOF

Since this work was completed, a note has appeared (Forman, W. R., 1970. *Astrophys. J.*, **159,** 719) in which ideas similar to the ones here are discussed. However, the author's approach is somewhat different from the one given in this paper, and he obtains an expression for M_{VT}/M which is not the same as equation (11) above. The reason for this conflict is still under discussion.