The dynamics of neutron star crusts: Lagrangian perturbation theory for a relativistic superfluid-elastic system

N. Andersson[1], B. Haskell[2], G.L. Comer[3] and L. Samuelsson[4]

[1] *Mathematical Sciences and STAG Research Centre,*
University of Southampton, Southampton SO17 1BJ, United Kingdom

[2] *Nicolaus Copernicus Astronomical Center,*
Polish Academy of Sciences, ul. Bartycka 18, 00-716 Warsaw, Poland

[3] *Department of Physics, Saint Louis University, St. Louis, MO, 63156-0907, USA*

[4] *Nordita, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden*

Abstract

The inner crust of a mature neutron star is composed of an elastic lattice of neutron-rich nuclei penetrated by free neutrons. These neutrons can flow relative to the crust once the star cools below the superfluid transition temperature. In order to model the dynamics of this system, which is relevant for a range of problems from pulsar glitches to magnetar seismology and continuous gravitational-wave emission from rotating deformed neutron stars, we need to understand general relativistic Lagrangian perturbation theory for elastic matter coupled to a superfluid component. This paper develops the relevant formalism to the level required for astrophysical applications.

I. INTRODUCTION

Astrophysical observations of neutron stars provide an important probe of the state of matter under extreme conditions. Shortly after the star is born, the outer layers freeze to form an elastic crust and the temperature of the high-density core drops below the level where superfluid and superconducting components are expected to be present. The different phases of matter impact on the observed phenomenology in a variety of ways. The crust region is important as it anchors the star's magnetic field (and provides specific channels for the gradual field evolution [1]), leading to an immediate connection between observed quasi- periodic oscillations in the tails of magnetar flares [2] and the dynamics of the elastic nuclear lattice. A detailed understanding of the properties of the crust is essential for efforts to match the theory to observed seismology features [3, 4]. In a different context, the ability of the crust to sustain elastic strain is key to the formation of asymmetries which may lead to detectable gravitational waves from a mature spinning neutron star. Continuous gravitational-wave searches with the LIGO-Virgo network of interferometers is beginning to set interesting upper limits for such signals for a number of known pulsars [5], in some instances reaching significantly below the expected maximum “mountain” size estimated from state of the art molecular dynamics simulations of the crustal breaking strain [6, 7]. Finally, the ability of the neutron-rich nuclei in the crust to pin superfluid vortices is also a key part of the standard explanation for observed glitches in young pulsars [8, 9]. All things considered, the crust region is crucial for an understanding of neutron star phenomenology and we need to make sure that our theoretical models incorporate as much of the relevant physics as possible.

As an important step towards the development of an appropriate theoretical framework, we will extend Lagrangian perturbation theory to the coupled superfluid-elastic crust system in the context of general relativity (extending the framework used in [10] to discuss the gravitational-wave driven instability of rotating relativistic stars, see [11] for a review). This is a formal development (similar in spirit to the efforts in [12] and [13]), but it should be immediately relevant to efforts aimed at modelling specific astrophysical scenarios. It is natural to use a Lagrangian framework since the perturbations of the elastic component becomes much “simpler” when considered in a frame that is co-moving with the crust. Moreover, it is essential that the problem is considered in the framework of general relativity as this is a pre-requisite for any quantitative analysis based on a realistic matter description. However, as the combined superfluid-elastic problem is still rather complex (and it is helpful to make the development as clear and intuitive as possible), we will construct the theory step by step, starting with a review of a single perfect fluid, considering next a pure elastic crust, and then adding the anticipated superfluid neutron component (as well as the associated entrainment effect [14]).

We take as our starting point the series of papers by Karlovini and Samuelsson [15–18], which build on earlier work by Carter and Quintana [19] (see also [20–23]), and the convective variation approach to relativistic fluid dynamics [24, 25]. The main focus of our discussion is the intimate connection between relativistic elasticity and Lagrangian perturbation theory. This link has not previously been explored in detail, yet we will demonstrate that Lagrangian variations capture neutron star crust physics in a natural fashion.

Throughout the discussion we assume a spacetime represented by a metric g_{ab} with signature $(-,+,+,+)$ and use early Latin letters, $a, b, c, d\dots$ to denote abstract spacetime indices. The Einstein summation convention applies, unless otherwise stated.

II. THE VARIATIONAL APPROACH

Our main focus may be on elastic matter, but it is nevertheless natural to begin by reviewing the variational approach to relativistic fluid dynamics [24, 25]. This is useful for two reasons: First of all, it is important to understand this formulation in order to extend it beyond simple perfect fluid models, e.g. add elasticity and additional fluid components that should be present when the system becomes superfluid. Secondly, the variational derivation already involves Lagrangian variations. Hence, the derivation of the fluid equations of motion provides some of the results we need if we want to study Lagrangian perturbations of a more general system.

In order to avoid undue confusion, let us consider the simplest model: a single barotropic fluid. In this case the matter equation of state can be expressed in terms of an energy functional that is a function of a single parameter. We take this parameter to be the particle number density, n, and assume that the dynamics is governed by a Lagrangian $\Lambda(n)$. The relation between this Lagrangian and the energy of the system will become clear shortly. The matter flux is represented by a (conserved) flux n^a, such that $n^2 = -g_{ab}n^a n^b$. In effect, this means that the Lagrangian depends on the flux and the spacetime metric.

An arbitrary variation of $\Lambda = \Lambda(n^2) = \Lambda(n^a, g_{ab})$ then gives (ignoring terms that can be written as total derivatives, that is, "surface terms", in the action [25])

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left[\ \mu_a \delta n^a + \frac{1}{2}\left(\Lambda g^{ab} + n^a \mu^b\right)\delta g_{ab}\right] \ , \tag{1}$$

where g is the determinant of the spacetime metric and μ_a is the canonical momentum defined by

$$\mu_a = \frac{\partial \Lambda}{\partial n^a} = -2\frac{\partial \Lambda}{\partial n^2} g_{ab} n^b = \mathcal{B} n_a \ . \tag{2}$$

We have also used

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{ab} \delta g_{ab} \ . \tag{3}$$

The result in (1) shows why a variational derivation of fluid dynamics is nontrivial. As it stands, the variation of Λ suggests that the equations of motion should be $\mu_a = 0$. In essence, the fluids would not carry energy or momentum. This is obviously not what we are looking for. To resolve this issue, we need a constrained variation. We need to insist that the matter flux is conserved. That is, we want to ensure that

$$\nabla_a n^a = 0 \ . \tag{4}$$

A natural way to do this is to make use of a three-dimensional "matter space" [24]. The coordinates of this matter space, X^A where $A = \{1, 2, 3\}$, serve as labels that distinguish individual fluid element worldlines [25]. These labels are assigned at the initial time of the evolution, say $t = 0$. The matter space coordinates can be considered as scalar fields on spacetime, with a unique map (obtained by a pull-back construction) relating them to the spacetime coordinates.

The variational construction then involves three steps. First we note that the conservation of the individual fluxes is ensured provided the dual three-form

$$n_{abd} = \epsilon_{abde} n^e \quad , \quad n^a = \frac{1}{3!}\epsilon^{abde} n_{bde} \ , \tag{5}$$

(where ϵ_{abde} is the usual volume form associated with the spacetime) is closed, i.e.

$$\nabla_{[a} n_{bde]} = 0 \quad \longrightarrow \quad \nabla_a n^a = 0 \ . \tag{6}$$

In the second step we make use of the matter space to construct three-forms that are automatically closed on spacetime, i.e.

$$n_{abd} = \psi^A_{[a} \psi^B_b \psi^D_{d]} n_{ABD} \ , \tag{7}$$

(the square brackets indicate anti-symmetrization, as usual) where the map is given by

$$\psi^A_a = \frac{\partial X^A}{\partial x^a} \ , \tag{8}$$

and the Einstein summation convention applies to repeated matter-space indices $\{A, B, ...\}$. The volume form n_{ABD}, which is anti-symmetric, provides matter space with a geometric structure (we elaborate on this in the Appendix). If integrated over a volume in matter space it provides a measure of the number of particles in that volume. With this definition, the three form (7) is closed if n_{ABD} is a function only of the X^A. In other words, the scalar fields (in spacetime) X^A are taken to be fundamental variables. A consequence of this construction is that n_{ABD} is "fixed" on matter space[1].

The final step involves introducing the Lagrangian displacement, ξ^a, and linking back to the spacetime perturbations. The displacement tracks the movement of a given fluid element. From the standard definition of Lagrangian variations in the relativistic context, we have

$$\Delta X^A = \delta X^A + \mathcal{L}_\xi X^A = 0 \ , \tag{9}$$

where δX^A is the Eulerian variation and $\mathcal{L}_\xi$ is the Lie derivative along ξ^a. We see that, convective variations are such that (since X^A is a scalar field on spacetime)

$$\delta X^A = -\mathcal{L}_\xi X^A = -\xi^a \frac{\partial X^A}{\partial x^a} = -\xi^a \psi^A_a \ . \tag{10}$$

For later benefit, it is worth noting that this leads to

$$\Delta \psi^A_a = 0 \ . \tag{11}$$

After some algebra, one finds

$$\Delta n_{abd} = 0 \ , \tag{12}$$

which in turn implies

$$\delta n^a = n^b \nabla_b \xi^a - \xi^b \nabla_b n^a - n^a \left(\nabla_b \xi^b + \frac{1}{2} g^{bd} \delta g_{bd} \right) = -\mathcal{L}_\xi n^a - n^a \left(\nabla_b \xi^b + \frac{1}{2} g^{bd} \delta g_{bd} \right) \ . \tag{13}$$

[1] Each point in matter space is associated with a particular worldline in spacetime. The matter space coordinates are a set of three scalar fields on spacetime, such that their values do not change along their particular worldline. As the fluid evolves in spacetime, any object in matter space which depends on only its own matter space coordinates (i.e. a tensor) will therefore not change its value at each coordinate. In this sense, it is "fixed", even though the object itself can vary across matter space points. As a side note, Andersson and Comer [26] have shown that when n_{ABD} is no longer fixed, an action principle incorporating dissipation can be built.

This is the main result of the exercise.

Now we can return to the variation of the matter Lagrangian. By expressing the variation of Λ in terms of the displacement ξ^a we ensure that the flux conservation is accounted for in the equations of motion. We get

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left\{\frac{1}{2}\left[\left(\Lambda - n^d\mu_d\right)g^{ab} + n^a\mu^b\right]\delta g_{ab} + f_a\xi^a\right\} , \tag{14}$$

and it follows that the equations of motion are given by

$$f_b \equiv 2n^a\nabla_{[a}\mu_{b]} = 0 . \tag{15}$$

Meanwhile, the stress-energy tensor follows as

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\Lambda\right)}{\delta g_{ab}} = \left(\Lambda - n^d\mu_d\right)g^{ab} + n^a\mu^b . \tag{16}$$

The final results may seem somewhat unfamiliar, but it is easy to recast them in a more commonly used form. All we need is a bit of thermodynamics. First we introduce the matter four-velocity such that $n^a = nu^a$. Then it follows that the chemical potential is, c.f. Eq. (2),

$$-u^a\mu_a = \mu = n\mathcal{B} , \tag{17}$$

Moreover, an observer moving with the matter flow would measure the mass-energy

$$\varepsilon = u_a u_b T^{ab} = -\Lambda , \tag{18}$$

which means that

$$\mu = \frac{d\varepsilon}{dn} , \tag{19}$$

as expected.

The fundamental relation [27]

$$p = -\varepsilon + n\mu = \Lambda - n^a\mu_a , \tag{20}$$

which defines the pressure, means that we have

$$T^{ab} = pg^{ab} + n^a\mu^b = \varepsilon u^a u^b + ph^{ab} , \tag{21}$$

where we have introduced the standard spacetime projection

$$h^{ab} = g^{ab} + u^a u^b . \tag{22}$$

Not surprisingly, Eq. (21) is the usual perfect fluid stress-energy tensor.

Next, let us consider the equations of motion (15). Making use of our various definitions, the momentum equation can be written

$$\mu \dot{u}_b + h^a_b \nabla_a \mu = 0 \ , \tag{23}$$

where $\dot{u}_b = u^a \nabla_a u_b$ is the four acceleration. Again making use Eq. (20), we arrive at the standard relativistic Euler equation. That is,

$$\dot{u}_b + \frac{1}{p+\varepsilon} h^a_b \nabla_a p = 0 \ . \tag{24}$$

An easy way to see that this result was inevitable is to note that

$$\nabla_a T^{ab} = f^b + \nabla^b \Lambda - \mu_a \nabla^b n^a = f^b = 0 \ . \tag{25}$$

The second equality follows from i) the fact that Λ is a function only of n^a and g_{ab}, and ii) the definition of the momentum μ_a.

III. LAGRANGIAN PERTURBATIONS

By introducing the displacement ξ^a, effectively tracking the fluid elements, we have prepared the ground for a study of Lagrangian perturbations. In fact, we see immediately from (13) that

$$\Delta n^a = -n^a \left(\nabla_b \xi^b + \frac{1}{2} g^{bd} \delta g_{bd} \right) = -\frac{1}{2} n^a \left(g^{bd} \Delta g_{bd} \right) \ , \tag{26}$$

where

$$\Delta g_{ab} = \delta g_{ab} + 2 \nabla_{(a} \xi_{b)} \ , \tag{27}$$

(the parentheses indicate symmetrization). Eq. (26) has a natural interpretation: The variation of a fluid worldline with respect to its own Lagrangian displacement has to be along the worldline and can only measure the changes of the volume of its own fluid element. This is one of the advantages of the Lagrangian variation approach, alluded to earlier. It also follows that [10]

$$\Delta n = -\frac{n}{2} h^{ab} \Delta g_{ab} \ , \tag{28}$$

and

$$\Delta u^a = \frac{1}{2} u^a u^b u^d \Delta g_{bd} \ . \tag{29}$$

For any given equation of state $\Lambda(n)$, we can now express the perturbed equations of motion in terms of the displacement vector ξ^a and the Eulerian variation of the metric δg_{ab}. In doing this it is worth noting that the usual approach to relativistic stellar perturbations is to work with this combination of variables (see for example [28]). Essentially, we need the Eulerian perturbation of the Einstein field equations and the Lagrangian variation of the momentum equation (15). The description of the perturbed Einstein equations is standard, so we focus on the fluid aspects here.

The perturbations of (15) are easy to work out once we note that the Lagrangian variation commutes with the exterior derivative. We immediately get

$$(\Delta n^a)\nabla_{[a}\mu_{b]} + n^a\nabla_{[a}\Delta\mu_{b]} = 0 \ . \tag{30}$$

This simplifies further if we use (26) and assume that the background is such that (15) is satisfied. The first term then vanishes, and we are left with

$$n^a\nabla_{[a}\Delta\mu_{b]} = 0 \ . \tag{31}$$

To complete this expression, we need to work out $\Delta\mu_a$. This is a straightforward task given the above results, and we find

$$\Delta\mu_a = \left(\mathcal{B} + n\frac{d\mathcal{B}}{dn}\right) g_{ab}\Delta n^b + \left(\mu^b\delta^d_a - \frac{d\mathcal{B}}{dn^2}n_a n^b n^d\right)\Delta g_{bd} \ . \tag{32}$$

For later convenience, we note that this expression can be written [25]

$$\Delta\mu_a = \mathcal{B}_{ab}\Delta n^b + \frac{1}{2}g^{db}\left(\delta^e_a\mu_b + \mathcal{B}_{ab}n^e\right)\Delta g_{de} \ , \tag{33}$$

where

$$\mathcal{B}_{ab} = \mathcal{B}g_{ab} - 2\frac{d\mathcal{B}}{dn^2}n_a n_b \ . \tag{34}$$

If we insert Eq. (26) into (33) we find

$$\begin{aligned}\Delta\mu_a &= -\frac{1}{2}\mathcal{B}_{ab}n^b\left(g^{de}\Delta g_{de}\right) + \frac{1}{2}g^{db}\left(\delta^e_a\mu_b + \mathcal{B}_{ab}n^e\right)\Delta g_{de} \\ &= \frac{1}{2}\left[\delta^e_a\mu^d + \mathcal{B}_{ab}\left(g^{db}n^e - g^{de}n^b\right)\right]\Delta g_{de} \ . \end{aligned}\tag{35}$$

That is, Δn^a has been completely replaced by Δg_{ab} in the fluid equations, thus completing the point about the advantage of Lagrangian displacements. Finally, in order to interpret the perturbed momentum we note that, in the single-fluid case the speed of sound follows from

$$c_s^2 = \frac{dp}{d\varepsilon} = \frac{n}{\mu}\left(\mathcal{B} + n\frac{d\mathcal{B}}{dn}\right) \ . \tag{36}$$

In principle, we now have all the results we need in order to express the perturbed equations of motion (31) in terms of ξ^a and δg_{ab}. In doing this, it is worth noting that (31) is orthogonal to u^b, which means that the problem only has three fluid degrees of freedom. This is natural since the conservation of particle number was guaranteed by the construction of the framework.

Before we proceed it may also be useful to note that (33) can be written

$$\Delta\mu_a = -\frac{1}{2n}\beta u_a h^{bd}\Delta g_{bd} + \mu\left(\delta_a^b u^d + \frac{1}{2}u_a u^b u^d\right)\Delta g_{bd} \ , \tag{37}$$

where we have defined the bulk modulus β as

$$\beta = n\frac{dp}{dn} = (p+\varepsilon)\frac{dp}{d\varepsilon} = (p+\varepsilon)\check{c}_s^2 \ , \tag{38}$$

where we have used the fundamental relation $p + \varepsilon = n\mu$. It also follows that

$$\Delta p = -\frac{\beta}{2}h^{ab}\Delta g_{ab} \ . \tag{39}$$

Later, we will extend the analysis to problems with several distinct fluid flows, as required to describe, for example, heat flux and/or systems with superfluid components [25]. In addition, we want to account for the possibility that one of these components is elastic rather than fluid. At the end of the day, we want to arrive at a formulation that allows us to model the dynamics of a realistic neutron star crust. To reach this point, we need to extend the formalism in two directions: We need to i) account for the crust elasticity and ii) allow for the presence of a superfluid neutron component. For practical reasons, it makes sense to first consider the elasticity.

IV. RELATIVISTIC ELASTICITY

With some of the formalities out of the way, let us return to the variational derivation of the fluid equations, with the intention of extending the analysis to account for elasticity. The motivation for this exercise is that "force-balance equations" like (15) are readily (as we will see later) adapted to multi-fluid settings, where it is necessary to have individual momentum equations for the different constituents [25]. Some of these equations can be extracted from the stress energy tensor, but this route is not as elegant and additional information would still be required.

The modern view of elasticity builds on the comparison of an actual matter configuration to an unstrained reference shape. In order to keep track of the reference state relative to which the strain is measured, we introduce a positive definite and symmetric tensor field k_{ab} [15]. Intuitively, this tensor encodes the (3-)geometry of the solid (as seen by the solid itself). The tensor k_{ab} is similar to n_{abd} in the sense that it is flow-line orthogonal, $u^a k_{ab} = 0$, and fixed in matter space. Moreover, as discussed in the Appendix, key properties of k_{ab} are established by introducing the corresponding matter space object, $k_{AB}(= k_{BA})$, through

$$k_{ab} = \psi^A_a \psi^B_b k_{AB} \ . \tag{40}$$

First of all, the Lagrangian variation of k_{ab} vanishes [c.f. Eq. (157)]. This means that k_{ab}, in addition to being a natural quantity for describing the elastic configuration, is useful in the development of Lagrangian perturbation theory. In particular,

$$\mathcal{L}_u k_{ab} = 0 \ . \tag{41}$$

Next, by assuming that n_{ABD} is the volume form associated with k_{AB} [c.f. Eq. (149)], one can show that k (the determinant of k_{ab}) is such that $k = n^2$ [15], even though k_{ab} does not depend on the number density n.

Letting the Lagrangian Λ depend also on this new tensor (in essence, incorporating the energy associated with elastic strain) we have

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left[\mu_a \delta n^a + \left(\frac{1}{2}\Lambda g^{ab} + \frac{\partial \Lambda}{\partial g_{ab}}\right)\delta g_{ab} + \frac{\partial \Lambda}{\partial k_{ab}}\delta k_{ab}\right] \ . \tag{42}$$

We proceed as before and replace δn^a with the Lagrangian displacement ξ^a. In addition, we have from Eq. (157) in the Appendix

$$\delta k_{ab} = -\xi^d \nabla_d k_{ab} - k_{db} \nabla_a \xi^d - k_{ad} \nabla_b \xi^d \ . \tag{43}$$

Again ignoring surface terms, we have (as k_{ab} is symmetric)

$$\frac{\partial \Lambda}{\partial k_{ab}}\delta k_{ab} = \xi^a \left[2\nabla_b\left(\frac{\partial \Lambda}{\partial k_{bd}} k_{ad}\right) - \frac{\partial \Lambda}{\partial k_{bd}}\nabla_a k_{bd}\right] \ . \tag{44}$$

Making use of this result, we arrive at

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left\{\left[\frac{1}{2}\left(\Lambda - n^d\mu_d\right)g^{ab} + \frac{\partial\Lambda}{\partial g_{ab}}\right]\delta g_{ab} + \tilde{f}_a\xi^a\right\} \ , \tag{45}$$

where

$$\tilde{f}_a = 2n^b\nabla_{[a}\mu_{b]} + 2\nabla_b\left(\frac{\partial\Lambda}{\partial k_{bd}}k_{ad}\right) - \frac{\partial\Lambda}{\partial k_{bd}}\nabla_a k_{bd} = 0 \ . \tag{46}$$

As in the fluid case, this result provides the equations of motion for the system. However, we need to do a bit of work in order to get the result into a user-friendly form. To start with, we read off the stress-energy tensor from (45):

$$T^{ab} = \left(\Lambda - n^d\mu_d\right)g^{ab} + 2\frac{\partial\Lambda}{\partial g_{ab}} \ . \tag{47}$$

The next step involves giving physical meaning to k_{ab}. As we want to model elasticity, we need to quantify the deviation of a given state from a relaxed configuration. In order to do this, it is convenient to follow Karlovini and Samuelsson [15] and introduce one further matter space tensor, η_{AB}. This object depends on n, and relates directly to the relaxed state. Its defining characteristic is that, in the relaxed configuration, it is the inverse to

$$g^{AB} = \psi^A_a\psi^B_b g^{ab} = \psi^A_a\psi^B_b h^{ab} \tag{48}$$

That is, for this specific state, we have

$$g^{AC}\eta_{CB} = \delta^A_B \ . \tag{49}$$

The spacetime counterpart is

$$\eta_{ab} = \psi^A_a\psi^B_b\eta_{AB} \ . \tag{50}$$

and, as outlined in the Appendix, one can show that [15]

$$\eta_{ab} = n^{-2/3}k_{ab} \ . \tag{51}$$

This relation is important, as we have already established that k_{ab} is a fixed matter space tensor and this will be crucial when we consider Lagrangian perturbations.

Let us now imagine that the system evolves away from the relaxed state. This means that (49) no longer holds: η_{AB} retains the value set by the initial state, but g^{AB} evolves along with the spacetime. This leads to the build up of elastic strain, simply quantified in terms of the strain tensor

$$s_{ab} = \frac{1}{2}(h_{ab} - \eta_{ab}) = \frac{1}{2}\left(h_{ab} - n^{-2/3}k_{ab}\right) \ . \tag{52}$$

In the relaxed configuration, we have $\eta_{ab} = h_{ab}$ by construction so it is obvious that s_{ab} vanishes.

This description is quite intuitive, but in practice it is more natural to work with scalars formed from η_{ab} (which can be viewed as "invariant"). This makes the model less abstract. Hence we introduce the strain scalar s^2 as a suitable combination (see below) of the invariants of η_{ab}:

$$\begin{aligned} I_1 &= \eta^a{}_a = g^{AB}\eta_{AB} \ , \\ I_2 &= \eta^a{}_b\eta^b{}_a = g^{AD}g^{BE}\eta_{EA}\eta_{DB} \ , \\ I_3 &= \eta^a{}_b\eta^b{}_d\eta^d{}_a = g^{AE}g^{BF}g^{DG}\eta_{EB}\eta_{FD}\eta_{GA} \ . \end{aligned} \tag{53}$$

However, because of the Cayley-Hamilton theorem [29], the number density n also can be seen to be a combination of invariants, i.e.

$$k = n^2 = \frac{1}{3!}\left(I_1^3 - 3I_1I_2 + 2I_3\right) \ . \tag{54}$$

Thus, it makes sense to replace one of the I_N ($N = 1-3$) with n which now becomes one of the required invariants. Then we define s^2 to be a function of two of the other invariants. We can choose different combinations, but we must ensure that s^2 vanishes for the relaxed state. For example, Karlovini and Samuelsson [15] work with

$$s^2 = \frac{1}{36}\left(I_1^3 - I_3 - 24\right) \ . \tag{55}$$

In the limit $\eta_{ab} \to h_{ab}$ we have $I_1, I_3 \to 3$ and therefore the combination for s^2 in Eq. (55) vanishes.

Next, we assume that the Lagrangian of the system depends on s^2, rather than the tensor k_{ab}. In doing this, we need to keep in mind that Eqs. (51) and (53) show that the invariants I_N depend on n (and hence both n^a and g_{ab}) as well as k_{ab}.

So far, the description is nonlinear, but in most situations of astrophysical interest it should be sufficient to consider a slightly deformed configuration[2]. Then we may focus on a Hookean model, such that

$$\Lambda = -\check{\varepsilon}(n) - \check{\mu}(n)s^2 = -\varepsilon \ , \tag{56}$$

where $\check{\varepsilon}$ is the energy from (18) and $\check{\mu}$ is the shear modulus (not to be confused with the chemical potential). As mentioned earlier, the checks indicate that quantities are calculated

[2] Note that this assumption is distinct from that of linear perturbations describing the dynamics.

for the unstrained state, with the specific understanding that $s^2 = 0$, and it should be apparent from (56) that we have an expansion in a supposedly small s^2. Since the strain scalar is given in terms of invariants, as in (55), it might be tempting to suggest a change of variables such that $s^2 = s^2(I_1, I_3)$. Our final equations of motion will, indeed, reflect this, but it would be premature to make the change at this point.

Instead we note that we now have for the momentum

$$\mu_a = \frac{\partial \Lambda}{\partial n^a} = \frac{\partial n^2}{\partial n^a}\frac{\partial \Lambda}{\partial n^2} = -\frac{1}{n}\frac{\partial \Lambda}{\partial n} g_{ab} n^b = \frac{1}{n}\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn} s^2 + \check{\mu}\frac{\partial s^2}{\partial n} \right) g_{ab} n^b \ , \tag{57}$$

while

$$\frac{\partial \Lambda}{\partial g_{ab}} = -\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn} s^2 + \check{\mu}\frac{\partial s^2}{\partial n} \right)\frac{\partial n}{\partial g_{ab}} - \check{\mu}\frac{\partial s^2}{\partial g_{ab}} \ . \tag{58}$$

Here we need (note that n^a is held fixed in the partial derivative)

$$\frac{\partial n}{\partial g_{ab}} = -\frac{1}{2n} n^a n^b \ , \tag{59}$$

and it is useful to note that

$$\frac{\partial s^2}{\partial g_{ab}} = -g^{ad} g^{be} \frac{\partial s^2}{\partial g^{de}} \ . \tag{60}$$

Also, when working out this derivative, we need to hold n fixed [as is clear from (58)]. At the end of the day, we have for the stress-energy tensor

$$\begin{aligned} T^{ab} &= \left[\Lambda + n\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn} s^2 + \check{\mu}\frac{\partial s^2}{\partial n} \right) \right] g^{ab} + \frac{1}{n}\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn} s^2 + \check{\mu}\frac{\partial s^2}{\partial n} \right) n^a n^b + 2\check{\mu} g^{ad} g^{be}\frac{\partial s^2}{\partial g^{de}} \\ &= \Lambda g^{ab} + n\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn} s^2 + \check{\mu}\frac{\partial s^2}{\partial n} \right) h^{ab} + 2\check{\mu} g^{ad} g^{be}\frac{\partial s^2}{\partial g^{de}} \ . \end{aligned} \tag{61}$$

Let us now effect the change of variables we hinted at previously. To be specific, let us consider a situation where s^2 depends only on I_1. Then we need

$$I_1 = \eta^a{}_a = n^{-2/3} g^{ab} k_{ab} \ , \tag{62}$$

$$\left(\frac{\partial s^2}{\partial n} \right)_1 = -\frac{2 I_1}{3n}\frac{\partial s^2}{\partial I_1} \ , \tag{63}$$

$$\left(\frac{\partial \Lambda}{\partial k_{ab}} \right)_1 = -\check{\mu}\frac{\partial s^2}{\partial k_{ab}} = -\check{\mu} n^{-2/3} g^{ab}\frac{\partial s^2}{\partial I_1} \ , \tag{64}$$

(recall comment on the partial derivative from before) and

$$\left(\frac{\partial s^2}{\partial g^{de}} \right)_1 = \frac{\partial s^2}{\partial I_1}\eta_{de} \ . \tag{65}$$

Making use of these results, we readily find

$$T^{ab} = -\varepsilon g^{ab} + n\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2\right)h^{ab} + 2\check{\mu}\frac{\partial s^2}{\partial I_1}\left(\eta^{ab} - \frac{1}{3}I_1 h^{ab}\right)$$
$$= -\varepsilon g^{ab} + n\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2\right)h^{ab} + 2\check{\mu}\frac{\partial s^2}{\partial I_1}\eta^{\langle ab\rangle} \ , \quad (66)$$

where the $\langle\ldots\rangle$ brackets indicate the symmetric, trace-free part of a tensor with two free indices. In our case, we have

$$\eta_{\langle ab\rangle} = \eta_{(ab)} - \frac{1}{3}\eta^d{}_d h_{ab} \ . \quad (67)$$

Comparing this result to the standard decomposition of the stress-energy tensor,

$$T^{ab} = \varepsilon u^a u^b + \bar{p}h^{ab} + \pi^{ab} \ , \qquad \text{where} \qquad \pi^a{}_a = 0 \ , \quad (68)$$

and $\bar{p}$ is the isotropic pressure (which differs from the fluid pressure, $\check{p}$, as it accounts for the elastic contribution, see below). We see that elasticity introduces an anisotropic contribution

$$\pi^1_{ab} = 2\check{\mu}\frac{\partial s^2}{\partial I_1}\eta_{\langle ab\rangle} \ . \quad (69)$$

A similar analysis for the other two invariants, I_2 and I_3, leads to

$$I_2 = n^{-4/3}g^{ad}g^{be}k_{ea}k_{db} \ , \quad (70)$$

$$\left(\frac{\partial s^2}{\partial n}\right)_2 = -\frac{4I_2}{3n}\frac{\partial s^2}{\partial I_2} \ , \quad (71)$$

$$\left(\frac{\partial s^2}{\partial g^{de}}\right)_2 = 2\frac{\partial s^2}{\partial I_2}\eta^f{}_d\eta_{ef} \quad (72)$$

$$\left(\frac{\partial \Lambda}{\partial k_{ab}}\right)_2 = -2\check{\mu}n^{-4/3}k^{ab}\frac{\partial s^2}{\partial I_2} \ , \quad (73)$$

and

$$I_3 = n^{-6/3}g^{ae}g^{bf}g^{dg}k_{ga}k_{eb}k_{fd} \ , \quad (74)$$

$$\left(\frac{\partial s^2}{\partial n}\right)_3 = -\frac{6I_3}{3n}\frac{\partial s^2}{\partial I_3} \ , \quad (75)$$

$$\left(\frac{\partial s^2}{\partial g^{de}}\right)_3 = 3\frac{\partial s^2}{\partial I_3}\eta^f{}_d\eta_{eg}\eta^g{}_f \ , \quad (76)$$

$$\left(\frac{\partial \Lambda}{\partial k_{ab}}\right)_3 = -3\check{\mu}n^{-6/3}k^{da}k^b{}_d\frac{\partial s^2}{\partial I_3} \ . \quad (77)$$

Recalling the definition in Eq. (67), these lead to

$$\pi^2_{ab} = 4\check{\mu}\frac{\partial s^2}{\partial I_2}\eta_{d\langle a}\eta_{b\rangle}{}^d \ , \quad (78)$$

and

$$\pi^3_{ab} = 6\check{\mu}\frac{\partial s^2}{\partial I_3}\eta^{de}\eta_{d\langle a}\eta_{b\rangle e} \ , \tag{79}$$

respectively. Combining these results with (55), we have

$$\pi_{ab} = \sum_N \pi^N_{ab} = \frac{\check{\mu}}{6}\left[\left(\eta^d{}_d\right)^2\eta_{\langle ab\rangle} - \eta^{de}\eta_{d\langle a}\eta_{b\rangle e}\right] \ , \tag{80}$$

which agrees with equation (128) from [15].

Now consider the final stress-energy tensor. Note first of all that, if we consider n and s^2 as the independent variables of the energy functional, then the isotropic pressure should follow from

$$\bar{p} = n\left(\frac{\partial\varepsilon}{\partial n}\right)_{s^2} - \varepsilon = \check{p} + \left(\frac{n}{\check{\mu}}\frac{d\check{\mu}}{dn} - 1\right)\check{\mu}s^2 \ , \tag{81}$$

where

$$\check{p} = n\frac{d\check{\varepsilon}}{dn} - \check{\varepsilon} \ , \tag{82}$$

is identical to the fluid pressure from before. However, we may also introduce a corresponding momentum, such that

$$\bar{\mu}_a = -\left(\frac{\partial\Lambda}{\partial n^a}\right)_{s^2} = \left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2\right)n_a \ , \tag{83}$$

which leads to

$$\bar{p} = \Lambda - n^a\bar{\mu}_a = \check{p} + \left(\frac{n}{\check{\mu}}\frac{d\check{\mu}}{dn} - 1\right)\check{\mu}s^2 \ . \tag{84}$$

Finally, in order to obtain the equations of motion for the system we can either take the divergence of (68) or return to (46) and make use of our various definitions. The results are the same (as they have to be). After a little bit of work we find that (46) leads to

$$2n^b\nabla_{[b}\bar{\mu}_{a]} + h^d_a\left(\nabla^b\pi_{bd} - \check{\mu}\nabla_d s^2\right) = 0 \ . \tag{85}$$

where it is worth noting that the combination in the parentheses is automatically flow line orthogonal.

V. LAGRANGIAN PERTURBATIONS OF AN UNSTRAINED MEDIUM

The results in the previous section prepare the ground for a discussion of Lagrangian perturbations of elastic matter. In fact, we have already done most of the required work. In particular, we already know that

$$\Delta k_{ab} = 0 \ . \tag{86}$$

We now want to make maximal use of this fact.

If we assume that the background configuration is relaxed, i.e. that $s^2 = 0$ vanishes for the configuration we are perturbing with respect to, then the fluid results from Section III together with (86) make the elastic perturbation problem straightforward (although it still involves a fair bit of algebra).

Consider, first of all, the strain scalar. A few simple steps leads to

$$\Delta s^2 = 0 \ . \tag{87}$$

To see this, recall that s^2 is a function of the invariants, I_N. Express these in terms of the number density n, the spacetime metric and k_{ab}. Once this is done, make use of (86) and the fact that the background is unstrained, i.e. $\eta_{ab} = h_{ab}$, to see that $\Delta I_N = 0$. Intuitively, this result makes sense. Since the strain scalar is quadratic, linear perturbations away from a relaxed configuration should vanish. An important implication of this result is that the last term in (85) does not contribute to the perturbed equations of motion.

This strategy leads to

$$\Delta\eta_{ab} = \frac{1}{3}\eta_{ab}h^{de}\Delta g_{de} \tag{88}$$

and

$$\Delta\eta^{ab} = \left[-2g^{a(e}\eta^{d)b} + \frac{1}{3}\eta^{ab}h^{de}\right]\Delta g_{de} \tag{89}$$

It then follows from (52) and (80), that

$$\Delta\pi_{ab} = -2\check{\mu}\Delta s_{ab} \ , \tag{90}$$

where

$$2\Delta s_{ab} = \left(h^e{}_a h^d{}_b - \frac{1}{3}h_{ab}h^{de}\right)\Delta g_{de} \ . \tag{91}$$

It is worth noting that the final result for an isotropic material agrees with, for example, [30] where the relevant strain term is simply added to the stress-energy tensor (without particular justification).

Finally, let us turn to the perturbed equations of motion. In the case of an unstrained background, it is easy to see that the argument that led to (31) still holds. This gives us the perturbation of the first term in (85) (after replacing $\mu_a \to \bar{\mu}_a$). Similarly, since π_{ab} vanishes in the background, the Lagrangian variation commutes with the covariant derivative in the second term. Thus, we end up with a perturbation equation of form

$$2n^a\nabla_{[a}\Delta\bar{\mu}_{b]} + \nabla^a\Delta\pi_{ab} = 0 \ . \tag{92}$$

This is the final result, but in order to arrive at an explicit expression for the perturbed momentum, we need to make use of (83) and the (unstrained) fluid result (37).

When we consider perturbations of an elastic medium we need to pay careful attention to the magnitude of the deviation away from the relaxed state. If the perturbation is too large, the material will yield [6]. It may fracture or behave in some other fashion that is not appropriately described by the equations of perfect elasticity. We need to quantify the associated breaking strain. In applications involving neutron stars, this is important if we want to consider star quakes in a spinning down pulsar, establish to what extent crust quakes in a magnetar lead to the observed flares [2] and whether the crust breaks due to the tidal interaction in an inspiralling binary [31, 32].

A commonly used criterion to discuss elastic yield strains in engineering involves the von Mises stress, defined as

$$\Theta_{\rm vM} = \sqrt{\frac{3}{2} s_{ab} s^{ab}} \tag{93}$$

When this scalar exceeds some critical value $\Theta_{\rm vM} > \Theta_{\rm vM}^{\rm crit}$, say, the material no longer behaves elastically and the framework we have developed needs to be amended. In order to work out the dominant contribution to the von Mises stress in general we need to (at least formally) consider second order perturbation theory. This is due to the positive definite nature of (93) which implies that the first order perturbation is zero for unstrained backgrounds. We could perturb (93) directly to second order, but it turns out to be simpler (and more elegant) to expand the trace of the squared strain tensor separately and then calculate the von Mises stress. This works because the von Mises stress is not a primary variable needed to solve the perturbation equations, but rather a quantity that can be estimated in post-processing.

Hgher-order perturbations, when the strain s_{ab} is considered "small" in the sense that $\Theta_{\rm vM} << \Theta_{\rm vM}^{\rm crit}$, then formally involve the substitution

$$s_{ab} \to s_{ab} + \Delta s_{ab} + \Delta^{(2)} s_{ab} + \dots \tag{94}$$

$$s^{ab} \to s^{ab} + \Delta s^{ab} + \Delta^{(2)} s^{ab} + \dots \ . \tag{95}$$

This leads to the trace of the squared strain tensor

$$s_{ab}s^{ab} \to s_{ab}s^{ab} + (s_{ab}\Delta s^{ab} + s^{ab}\Delta s_{ab}) + (\Delta s_{ab}\Delta s^{ab} + s_{ab}\Delta^{(2)}s^{ab} + s^{ab}\Delta^{(2)}s_{ab}) + \dots \tag{96}$$

where we have grouped the terms according to the perturbative order. Although we cannot say anything about the relative size of s_{ab} and Δs_{ab} (this involves choice), we do know that

$\Delta^{(2)} s_{ab} \ll \Delta s_{ab}$ (and similar for the contravariant stress tensor) so the last two terms can be neglected compared to the linear order terms. Making use of this, we have

$$\Theta_{\rm vM} \approx \sqrt{\frac{3}{2} s_{ab} s^{ab} + \frac{3}{2}(s^{\langle ab\rangle} - 2 s^a{}_c s^{cb}) \Delta g_{ab} + \frac{3}{8} h^{a\langle c} h^{d\rangle b} \Delta g_{ab} \Delta g_{cd}}$$
$$\approx \sqrt{\frac{3}{2} s_{ab} s^{ab} + \frac{3}{2} s^{\langle ab\rangle} \Delta g_{ab} + \frac{3}{8} h^{a\langle c} h^{d\rangle b} \Delta g_{ab} \Delta g_{cd}} \quad (97)$$

where the smallness of the strain tensor was used in the last step.

The relevant comparison is between the size of the background strain tensor and the spatially projected trace-free part of the perturbed metric. If either is much larger than the other, then (97) simplifies. For instance, if the background is unstrained (or weakly strained) we have

$$\Theta_{\rm vM} = \sqrt{\frac{3}{2} \Delta s_{ab} \Delta s^{ab}} = \sqrt{\frac{3}{8} h^{a\langle c} h^{d\rangle b} \Delta g_{ab} \Delta g_{cd}} \quad (98)$$

A very neat result, indeed. If, on the other hand, we consider perturbations of a strained background the relevant expression is

$$\Theta_{\rm vM} \approx \sqrt{\frac{3}{2} \left(s^{ab} s_{ab} + s^{\langle ab\rangle} \Delta g_{ab} \right)} \quad (99)$$

As an aside, it is worth noting that this expression neatly demonstrates the interpretation of perturbed spacetime as a strain.

VI. ADDING AN ENTRAINED SUPERFLUID COMPONENT

In order to develop a model for the coupled superfluid neutrons-elastic nucleon crust we need to account for the presence of two (coupled) fluxes[3]. We take these to be $n_{\rm c}^a$ and $n_{\rm f}^a$, where the constituent indices x = c and f distinguish the "confined" baryons in the lattice from the "free" (superfluid) neutrons[4]. Now we have two distinct four velocities, such that $n_{\rm c}^a = n_{\rm c} u_{\rm c}^a$ and $n_{\rm f}^a = n_{\rm f} u_{\rm f}^a$. The simplest relevant model for the matter Lagrangian of this system assumes that the elastic contribution is unaffected by the presence of the interpenetrating fluid component. Assuming a Hookean model, we then have [c.f. (56)]

$$\Lambda = \Lambda_{\rm liq}(n_{\rm c}^a, n_{\rm f}^a, g_{ab}) + \Lambda_{\rm sol}\left(n_{\rm c}, s^2\right) \quad (100)$$

[3] We are assuming that the second fluid contribution represents superfluid neutrons, but from a formal point of view it could equally well correspond to a dynamical thermal component [25].

[4] The careful reader will note that, in order to avoid confusion we have left out the letters c and f as spacetime indices from this point.

where we have made a "minimal coupling" assumption for the elastic contribution, $\Lambda_{\rm sol}$. That is, the corresponding matter component is associated with the constituent index c, such that $\check{\mu} = \check{\mu}(n_{\rm c})$. The liquid contribution is, of course, different from before. In general, $\Lambda_{\rm liq}$ is a function of three scalar densities [25]:

$$n_{\rm f}^2 = -n^{\rm f}_a n^a_{\rm f} \ , \quad n_{\rm c}^2 = -n^{\rm c}_a n^a_{\rm c} \ , \quad n_{\rm fc}^2 = -n^{\rm f}_a n^a_{\rm c} \ . \tag{101}$$

The last of these represents effects due to the relative flow between the two components. While this flow is generally expected to be small in magnitude, its contribution is nevertheless significant since it encodes the entrainment effect [23, 33–35].

At this point it makes sense to point out that, in the neutron star crust, the distinction between the two dynamical components is somewhat ambiguous. Throughout most of the crust (beyond neutron drip), we have a fraction of neutrons bound in nuclei but there is also a "gas" of free neutrons. In static situations, neutrons can be assigned to either component depending on the nature of the ions in the lattice [36]. However, when we turn to dynamical settings, it is no longer clear to what extent the "confined" neutrons are able to move [20, 37, 40]. The answer depends on the extent to which they can tunnel through the relevant interaction potentials, an effect that can be expressed in terms of the entrainment. As a result, while it is clear that we must deal with a two-component model, it is conceptually less clear how one determines the parameters of the system. Formally, one may consider the problem in terms of different chemical "gauges" [20]. This is tricky. Fortunately, while the choice of chemical gauge affects the interpretation of the involved quantities (number densities, etc), the two-fluid model remains conceptually unaffected [41]. The upshot is that one has to exercise a level of care in practical applications where the model is combined with a detailed microphysical equation of state.

A. The background dynamics

Given the form of Λ, the variational procedure determines the momenta that are canonically conjugate to the two fluxes. We have

$$\mu^{\rm f}_a = \frac{\partial \Lambda}{\partial n^a_{\rm f}} = \frac{\partial \Lambda_{\rm liq}}{\partial n^a_{\rm f}} \ , \qquad \mu^{\rm c}_a = \frac{\partial \Lambda}{\partial n^a_{\rm c}} \ . \tag{102}$$

and the stress-energy tensor takes the form

$$T^a{}_b = (\Lambda - n^d_{\rm f}\mu^{\rm f}_d - n^d_{\rm c}\mu^{\rm c}_d)\delta^a{}_b + n^a_{\rm f}\mu^{\rm f}_b + n^a_{\rm c}\mu^{\rm c}_b + \pi^a{}_b \ , \tag{103}$$

where the anisotropic pressure, π_{ab}, is still given by (80). We also need the generalized pressure

$$\Psi = \Lambda - n_{\rm c}^a \mu_a^{\rm c} - n_{\rm f}^a \mu_a^{\rm f} \ . \tag{104}$$

The stress-energy tensor serves as source for Einstein's equations. Moreover, in the case where the two fluxes are individually conserved, i.e., when we are not accounting for reactions (for example, when the dynamical timescale is much faster than that associated with reactions), we have

$$\nabla_a n_{\rm c}^a = \nabla_a n_{\rm f}^a = 0 \ . \tag{105}$$

In this case, we obtain two equations of motion

$$2 n_{\rm f}^a \nabla_{[a} \mu_{b]}^{\rm f} = 0 \ , \tag{106}$$

$$2 n_{\rm c}^a \nabla_{[a} \mu_{b]}^{\rm c} + \nabla^a \pi_{ab} = 0 \ . \tag{107}$$

Given the previous discussion, the form of these equations should come as no surprise.

It should be noted that Equations (106) and (107), which represent the Euler equations, combine to ensure the conservation of energy momentum, $\nabla^a T_a^b = 0$. This information is, of course, also encoded in the Einstein equations. Thus, it is sufficient to consider a combination of the Einstein equations and one of the Euler equations. An often used strategy, especially in work on neutron star oscillations is to focus on the Einstein equations which, for a single component fluid, contain all required information. In the two-fluid case this strategy will not completely specify the problem [42, 43]. We also need information from (106) and/or (107). From the formal point of view, the tidiest approach may be to use both Euler equations and a smaller subset of the Einstein field equations. A key reason for this is that one can then develop the model in such a way that many of the equations are "symmetric" in the constituent indices. This makes the description economical, and has the advantage that the inclusion of additional fluid components is straightforward. Of course, it comes at the price of having to work with the constituent indices at a more abstract level.

To complete the model, and obtain explicit equations, we need to determine the fluid momenta. Adapting the notation from [25] we have

$$\mu_a^{\rm f} = \mathcal{B}^{\rm f} n_a^{\rm f} + \mathcal{A}^{\rm cf} n_a^{\rm c} \tag{108}$$

$$\mu_a^{\rm c} = \mathcal{B}^{\rm c} n_a^{\rm c} + \mathcal{A}^{\rm cf} n_a^{\rm f} \tag{109}$$

where

$$\mathcal{B}^{\rm f} = -2\frac{\partial \Lambda}{\partial n_{\rm f}^2} \ , \quad \mathcal{B}^{\rm c} = -2\frac{\partial \Lambda}{\partial n_{\rm c}^2} \ , \quad \mathcal{A}^{\rm cf} = -\frac{\partial \Lambda}{\partial n_{\rm cf}^2} \ . \tag{110}$$

B. The perturbation equations

Let us now turn to the Lagrangian perturbations of this system. In principle, this problem is straightforward given the previous developments. It is natural to work with two matter spaces [25] and, hence, two distinct displacements $\xi^a_{\rm f}$ and $\xi^a_{\rm c}$ (see [44] for a discussion of the corresponding Newtonian problem).

From the single-fluid results in Section III, it is easy to see that we will have

$$\Delta_{\rm x} n^a_{\rm x} = -n^a_{\rm x}\left(\nabla_b \xi^b_{\rm x} + \frac{1}{2} g^{bd}\delta g_{bd}\right) = -\frac{1}{2} n^a_{\rm x}\left(g^{bd}\Delta_{\rm x} g_{bd}\right) \ , \tag{111}$$

where the constituent index x represents either f or c, and

$$\Delta_{\rm x} g_{ab} = \delta g_{ab} + 2\nabla_{(a}\xi^{\rm x}_{b)} \ . \tag{112}$$

This leads to

$$\Delta_{\rm x} n_{\rm x} = -\frac{n_{\rm x}}{2} h^{ab}_{\rm x} \Delta_{\rm x} g_{ab} \ , \tag{113}$$

where

$$h^{ab}_{\rm x} = g^{ab} + u^a_{\rm x} u^b_{\rm x} \ , \tag{114}$$

is the projection orthogonal to $u^a_{\rm x}$. We also get

$$\Delta_{\rm x} u^a_{\rm x} = \frac{1}{2} u^a_{\rm x} u^b_{\rm x} u^d_{\rm x} \Delta_{\rm x} g_{bd} \ . \tag{115}$$

Moreover, the argument that led to the perturbed equations of motion remains valid (as long as we assume that the elastic background is isotropic and unstrained) and we have

$$2n^a_{\rm f}\nabla_{[a}\Delta_{\rm f}\mu^{\rm f}_{b]} = 0 \ , \tag{116}$$

$$2n^a_{\rm c}\nabla_{[a}\Delta_{\rm c}\mu^{\rm c}_{b]} + \nabla^a \Delta_{\rm c}\pi_{ab} = 0 \ . \tag{117}$$

These are the main results. Of course, given the two-fluid context, the perturbed momenta are more complicated than before.

Starting from (108)–(109), one can show that [25]

$$\Delta_{\rm x}\mu^{\rm x}_a = \left(\mathcal{B}^{\rm x}_{ab} + \mathcal{A}^{\rm xx}_{ab}\right)\Delta_{\rm x} n^b_{\rm x} + \left(\chi^{\rm xy}_{ab} + \mathcal{A}^{\rm xy}_{ab}\right)\Delta_{\rm x} n^b_{\rm y} + \frac{1}{2} g^{db}\left[\delta^e_a \mu^{\rm x}_b + \left(\mathcal{B}^{\rm x}_{ab} + \mathcal{A}^{\rm xx}_{ab}\right) n^e_{\rm x} + \left(\chi^{\rm xy}_{ab} + \mathcal{A}^{\rm xy}_{ab}\right) n^e_{\rm y}\right]\Delta_{\rm x} g_{ed} \ , \tag{118}$$

where we have introduced another constituent index, $\mathrm{y} \neq \mathrm{x}$. We have also defined

$$\mathcal{B}^{\mathrm{x}}_{ab} = \mathcal{B}^{\mathrm{x}} g_{ab} - 2\frac{\partial \mathcal{B}^{\mathrm{x}}}{\partial n^2_{\mathrm{x}}} n^{\mathrm{x}}_a n^{\mathrm{x}}_b \ , \tag{119}$$

in obvious analogy with (34),

$$\chi^{\mathrm{xy}}_{ab} = -2\frac{\partial \mathcal{B}^{\mathrm{x}}}{\partial n^2_{\mathrm{y}}} n^{\mathrm{x}}_a n^{\mathrm{y}}_b \ , \tag{120}$$

$$\mathcal{A}^{\mathrm{xx}}_{ab} = -\frac{\partial \mathcal{B}^{\mathrm{x}}}{\partial n^2_{\mathrm{xy}}} \left(n^{\mathrm{x}}_a n^{\mathrm{y}}_b + n^{\mathrm{x}}_b n^{\mathrm{y}}_a \right) - \frac{\partial \mathcal{A}^{\mathrm{xy}}}{\partial n^2_{\mathrm{xy}}} n^{\mathrm{y}}_a n^{\mathrm{y}}_b \ , \tag{121}$$

and

$$\mathcal{A}^{\mathrm{xy}}_{ab} = \mathcal{A}^{\mathrm{xy}} g_{ab} - \frac{\partial \mathcal{B}^{\mathrm{x}}}{\partial n^2_{\mathrm{xy}}} n^{\mathrm{x}}_a n^{\mathrm{x}}_b - \frac{\partial \mathcal{B}^{\mathrm{y}}}{\partial n^2_{\mathrm{xy}}} n^{\mathrm{y}}_a n^{\mathrm{y}}_b - \frac{\partial \mathcal{A}^{\mathrm{xy}}}{\partial n^2_{\mathrm{xy}}} n^{\mathrm{y}}_a n^{\mathrm{x}}_b \ . \tag{122}$$

From these expressions, it is apparent that we also need

$$\Delta_{\mathrm{x}} n^a_{\mathrm{y}} = \Delta_{\mathrm{y}} n^a_{\mathrm{y}} + (\xi^b_{\mathrm{x}} - \xi^b_{\mathrm{y}}) \nabla_b n^a_{\mathrm{y}} - n^b_{\mathrm{y}} \nabla_b (\xi^a_{\mathrm{x}} - \xi^a_{\mathrm{y}}) \ , \tag{123}$$

and it is useful to note that

$$\Delta_{\mathrm{y}} g_{ab} = \Delta_{\mathrm{x}} g_{ab} - 2 \left[\nabla_{(a} \xi^{\mathrm{x}}_{b)} - \nabla_{(a} \xi^{\mathrm{y}}_{b)} \right] \ . \tag{124}$$

These relations provide all the information we need in order to express the perturbations in terms of the two displacement vectors ξ^a_{x} and the perturbed metric δg_{ab}. Once an equation of state is provided (so that we can work out the action), all required coefficients can be calculated and we have a description of a generic situation.

C. A couple of steps towards applications

Even though our main aim is to establish formal aspects of the Lagrangian perturbation problem, it makes sense to make a few comments on applications. In particular, it is worth noting that the magnitude of the relative velocity between the two components is likely to be small in most situations of practical relevance. As an example, consider the typical velocity lag (between superfluid and crust) required to explain the observed pulsar glitches. In terms of angular velocity we have $\Delta\Omega/\Omega \approx 10^{-6}$ for the glitches seen in the Vela pulsar. This leads to

$$v \sim R\Delta\Omega \sim 10^{-6}\Omega R \lesssim 10^{-6} \left(\frac{M}{R^3} \right)^{1/2} R \approx 10^{-7} \tag{125}$$

where R is the radius of the star.

In order to quantify the relative flow, let us focus on the frame associated with one of the fluids. Taking the four velocity $u^a = u^a_{\rm c}$ as our reference, the relative velocity v^a follows from

$$n^a_{\rm f} = \gamma n_{\rm f}(u^a + v^a) \ , \qquad u^a v_a = 0 \ , \qquad \gamma = (1-v^2)^{-1/2} \ , \qquad v^2 = v^a v_a \ . \tag{126}$$

Assuming that v^2 is small, it makes sense to work with an expansion using this as a small parameter.

With this in mind, and considering the variables that we used in the derivation of the equations of motion, we may follow [4, 45] and expand the Lagrangian as

$$\Lambda(n^2_{\rm f}, n^2_{\rm c}, n^2_{\rm fc}) \approx \sum_{i=0}^{N} \lambda_i(n^2_{\rm f}, n^2_{\rm c}) \left[n^2_{\rm fc} - n_{\rm f} n_{\rm c}\right]^i \ . \tag{127}$$

Since

$$n^2_{\rm fc} - n_{\rm f} n_{\rm c} \approx \frac{1}{2} n_{\rm f} n_{\rm c} v^2 \ , \tag{128}$$

it should be sufficient to retain the first couple of terms in the expansion. For example, if we accept errors of order v^2 in the equations of motion then we need to keep the first three terms, up to $N = 2$. At this level of precision, we get

$$\mathcal{B}^{\rm f} \approx -\frac{1}{n_{\rm f}} \frac{\partial \lambda_0}{\partial n_{\rm f}} + \frac{n_{\rm c}}{n_{\rm f}} \lambda_1 \ , \tag{129}$$

$$\mathcal{B}^{\rm c} \approx -\frac{1}{n_{\rm c}} \frac{\partial \lambda_0}{\partial n_{\rm c}} + \frac{n_{\rm f}}{n_{\rm c}} \lambda_1 \ , \tag{130}$$

and

$$\mathcal{A}^{\rm fc} \approx -\lambda_1 \ . \tag{131}$$

To completely specify the model, we need to provide the λ_i coefficients. In order to illustrate how different features enter at different levels of complexity, we can start by considering models where the two fluids co-move in the background. This will be the case if one insists that the background configuration is in both dynamical and chemical equilibrium.

If we take the two fluids to move together in the background we have $u^a_{\rm f} = u^a_{\rm c} = u^a$ and the problem simplifies. The Lagrangian is simply given by $\Lambda = \lambda_0$ and it makes sense (as in the single-fluid problem) to work with the energy density $\check{\rho} = -\lambda_0$. We also find that the pressure is given by

$$\check{p} = -\check{\rho} + n_{\rm c}\mu_{\rm c} + n_{\rm f}\mu_{\rm f} = -\check{\rho} + n_{\rm c} \frac{\partial \check{\rho}}{\partial n_{\rm c}} + n_{\rm f} \frac{\partial \check{\rho}}{\partial n_{\rm f}} \ . \tag{132}$$

Combining this with (118), we find that the perturbed momenta can be written

$$\Delta_{\rm x}\mu^{\rm x}_a = -\frac{1}{2}\left(n_{\rm x}\frac{\partial^2\check\rho}{\partial n_{\rm x}^2} + n_{\rm y}\frac{\partial^2\check\rho}{\partial n_{\rm x}\partial n_{\rm y}}\right)u_a h^{de}\Delta_{\rm x}g_{de} + \frac{\partial\check\rho}{\partial n_{\rm x}}\left(\delta^e_a u^d + \frac{1}{2}u_a u^e u^d\right)\Delta_{\rm x}g_{ed}$$
$$- \frac{\partial^2\check\rho}{\partial n_{\rm x}\partial n_{\rm y}}\left[\psi^d_{\rm xy}\nabla_d n_{\rm y} - n_{\rm y}h^{cd}\nabla_c\psi^{\rm xy}_d\right]u_a \ , \qquad {\rm y}\neq{\rm x}\ . \quad (133)$$

Alternatively, introducing

$$\check\beta_{\rm x} = n_{\rm x}\mu_{\rm x} = n_{\rm x}\frac{\partial\check p}{\partial n_{\rm x}} \ , \quad (134)$$

we have

$$\Delta_{\rm x}\mu^{\rm x}_a = -\frac{\check\beta_{\rm x}}{2n_{\rm x}}u_a h^{de}\Delta_{\rm x}g_{de} + \mu_{\rm x}\left(\delta^e_a u^d + \frac{1}{2}u_a u^e u^d\right)\Delta_{\rm x}g_{ed}$$
$$- \frac{\partial\mu_{\rm x}}{\partial n_{\rm y}}\left(\psi^d_{\rm xy}\nabla_d n_{\rm y} - n_{\rm y}h^{cd}\nabla_c\psi^{\rm xy}_d\right)u_a \ , \qquad {\rm y}\neq{\rm x}\ . \quad (135)$$

The single-fluid result (123) provides a useful sanity check on this result. Comparing, we see that when the equation of state depends on the composition (and nuclear physics parameters like the symmetry energy), i.e. when

$$\frac{\partial\mu_{\rm x}}{\partial n_{\rm y}} \neq 0 \ , \quad (136)$$

there are two key differences. First of all, variations in the composition affect $\check\beta_{\rm x}$. In the case when the two components are coupled also at the perturbative level, this change leads to the presence of g-modes etcetera [46]. Secondly, when the two fluids are free to move relative one another, the associated displacements are "chemically" coupled through the last two terms in (135).

The problem gets significantly more involved if the two fluids are not flowing together in the unperturbed configuration. The general expression for the perturbed momenta, (118), remains valid but the involved terms are more complex. Having said that, one should be able to neglect all quadratic terms in the relative velocity in most situations of practical interest. To make the dependence on the relative velocity, v^a, explicit it may be useful express the perturbation equations in a prefered frame. However, this strategy breaks the "symmetry" with respect to the constituent indices that we have relied upon so far.

Let us opt to work in the frame associated with the crust component, taking $u^a = u^a_{\rm c}$. To linear order in the relative velocity, we then have $u^a_{\rm f} \approx u^a + v^a$. It follows that the two unperturbed momenta are given by

$$\mu^{\rm c}_a \approx -\frac{\partial\lambda_0}{\partial n_{\rm c}}u_a - n_{\rm f}\lambda_1 v_a \ , \quad (137)$$

and

$$\mu_a^{\rm f} \approx -\frac{\partial \lambda_0}{\partial n_{\rm f}}(u_a + v_a) - n_{\rm c}\lambda_1 v_a \ , \tag{138}$$

Using these results, it is straightforward to show that (up to terms of order v^2) the energy density $\check{\rho}$ and the pressure $\check{p}$ remain as in the previous model problem. This tells us that the problem does not deviate too far from the co-moving situation as long as we neglect higher order terms in the relative velocity.

In this model, the entrainment enters through the λ_1 coefficient. However, it is often useful to represent the effect in terms of an effective nucleon mass. This makes sense intuitively, and it also relates to a quantity that can be determined from detailed microphysics [47]. In recent years, there has been an effort to determine the effective neutron mass for the crust superfluid. Perhaps surprisingly, this work [23, 37] suggests that the effective mass may be very different from the bare nucleon mass. The effect is still under debate (see, for example, [38, 39] for counter arguments) but it needs to be taken seriously. In particular, it would mean that the inclusion of entrainment in the treatment of the crust superfluid is essential [40, 48].

Let us see how the effective mass arises in a relativistic model [4]. We can do this by considering the momentum in a local inertial frame associated with one of the fluids, say the crust component. Then we have

$$u_{\rm c}^a = [1, 0, 0, 0] \ , \quad u_{\rm f}^a = [\gamma, \gamma v^a] \ , \tag{139}$$

with $\gamma = (1 - v^2)^{-1/2}$ as before. This leads to

$$\mu_{\rm f}^0 = \mathcal{B}^{\rm f} n_{\rm f} \gamma + \mathcal{A}^{\rm fc} n_{\rm c} = \gamma m_0 \ , \tag{140}$$

where m_0 is the baryon rest mass (we assume that $m_{\rm f} = m_{\rm c} = m_0$ here) and

$$\mu_{\rm f}^i = \mathcal{B}^{\rm f} n_{\rm f} \gamma v^i \equiv m_{\rm f}^* \gamma v^i \ , \quad i = 1 - 3 \ , \tag{141}$$

where $m_{\rm f}^*$ is the effective neutron mass. It follows that

$$\mathcal{A}^{\rm fc} = \frac{\gamma}{n_{\rm c}}(m_0 - m_{\rm f}^*) \ , \tag{142}$$

which reduces to the usual Newtonian result [47]

$$\mathcal{A}^{\rm fc} = \frac{1}{n_{\rm c}}(m_0 - m_{\rm f}^*) \ , \tag{143}$$

in the limit $v^2 \ll c^2$.

In order to make contact with the low-velocity expansion of the Lagrangian (127), we note that

$$\mathcal{A}^{\rm fc} = -\frac{\partial \Lambda}{\partial n_{\rm fc}^2} = -\frac{1}{n_{\rm f} n_{\rm c}} \frac{\partial \Lambda}{\partial \gamma} \ , \tag{144}$$

where

$$\left. \frac{\partial \Lambda}{\partial \gamma} \right|_{n_{\rm f}, n_{\rm c}} = n_{\rm f} \gamma (m_{\rm f}^* - m_0) \ . \tag{145}$$

In general, one would expect the effective mass to depend on γ, preventing us from integrating to get an expression for Λ. Assuming that this dependence is weak enough that it can be ignored, we have

$$\Lambda = \Lambda_0(n_{\rm f}, n_{\rm c}) + \frac{1}{2} n_{\rm f} (m_{\rm f}^* - m_0) \gamma^2 \ , \tag{146}$$

which leads to

$$\lambda_1 = \frac{n_{\rm f}(m_{\rm f}^* - m_0)}{2} \ . \tag{147}$$

Finally, it is worth noting that we could (obviously) have introduced an analogous effective mass, $m_{\rm c}^*$, for the crust nucleons. Because of the symmetry of the entrainment terms, the two effective quantities must be related by

$$\mathcal{A}^{\rm fc} = \frac{1}{n_{\rm c}} (m_0 - m_{\rm f}^*) = \frac{1}{n_{\rm f}} (m_0 - m_{\rm c}^*) \quad \longrightarrow m_{\rm c}^* = m_0 - \frac{n_{\rm f}}{n_{\rm c}} (m_0 - m_{\rm f}^*) \ . \tag{148}$$

VII. SUMMARY

We have developed a Lagrangian perturbation framework for the dynamics of mature neutron stars with an inner crust combining an elastic lattice of neutron-rich nuclei and a free neutron component. We paid particular attention to geometric aspects of the problem and the close connection to the convective variational approach often used to derive the equations for multi-fluid systems in general relativity [25]. As the final perturbation equations may be somewhat intimidating, we outlined simplifying assumptions that may apply to problems of astrophysical relevance. This discussion should also help build intuition.

The natural next step will be to apply the result to a specific problem of interest. This may take us in different directions.The formalism lends itself to a range of settings, from crust quakes to pulsar glitches to magnetar seismology and continuous gravitational-wave emission from rotating deformed neutron stars. The last problem is (perhaps) particularly timely given the excitement associated with the breakthrough detection of gravitational waves from neutron star mergers [49] and the ongoing effort to detect gravitational waves from spinning neutron stars [5]. We hope to make progress on this problem in the near future.

Acknowledgments

NA acknowledges support from the STFC via grant numbers ST/M000931/1 and ST/R00045X/1. BH acknowledges support from Polish National Science Centre (NCN) grant SONATA BIS 2015/18/E/ST9/00577.

Appendix: The properties of k_{ab} and η_{ab}

In this Appendix we add some relevant context relating to the description of elastic matter. In particular, we make use of a positive definite, matter-space metric tensor field $k_{AB} = k_{BA}$ to establish the major features of k_{ab} and η_{ab} required for the calculations presented in Sec. IV. The tensor k_{AB} is "fixed" on matter space, in the same sense as n_{ABC}, because it is only a function of its own matter space coordinates X^A. The associated volume form is n_{ABC} in that

$$n_{ABC} = \sqrt{\det{(k_{AB})}}\,[A\ B\ C]_{\mathcal{D}} \ , \tag{149}$$

where

$$[A\ B\ C]_{\mathcal{D}} = 3!\delta^1_{[A}\delta^2_B\delta^3_{C]} = \{\pm 1, 0\} \ , \tag{150}$$

and

$$\det{(k_{AB})} = \frac{1}{3!}\,[A\ B\ C]^{\mathcal{U}}\,[D\ E\ F]^{\mathcal{U}}\,k_{AD}k_{BE}k_{CF} \ , \tag{151}$$

where

$$[A\ B\ C]^{\mathcal{U}} = 3!\delta_1^{[A}\delta_2^B\delta_3^{C]} = \{\pm 1, 0\} \ . \tag{152}$$

Later it will be useful to know

$$[A\ B\ C]^{\mathcal{U}}\,[D\ E\ F]_{\mathcal{D}} = 3!\delta^{[A}_D\delta^B_E\delta^{C]}_F \ . \tag{153}$$

If we let

$$g^{AB} = \psi^A_a\psi^B_b g^{ab} = \psi^A_a\psi^B_b h^{ab} \ , \tag{154}$$

and use Eqs. (5) and (7), then we can show

$$n^2 = -g_{ab} n^a n^b = \frac{1}{3!} \det\left(k_{AB}\right) \det\left(g^{AB}\right) \ , \tag{155}$$

where

$$\det\left(g^{AB}\right) = \frac{1}{3!} \left[A\ B\ C\right]_{\mathcal{D}} \left[D\ E\ F\right]_{\mathcal{D}} g^{AD} g^{BE} g^{CF} \ . \tag{156}$$

Using Eqs. (9) and (40), we can easily establish that the Lagrangian variation of k_{ab} vanishes; namely,

$$\delta k_{ab} = -\mathcal{L}_\xi k_{ab} \quad \Longrightarrow \quad \Delta k_{ab} = 0 \ . \tag{157}$$

Finally, since $u^a \psi^A_a = 0$, and k_{AB} is a function of X^A, we have

$$\mathcal{L}_u k_{AB} = u^a \psi^C_a \frac{\partial k_{AB}}{\partial X^C} = 0 \ , \tag{158}$$

and therefore

$$\begin{aligned} \mathcal{L}_u k_{ab} = k_{AB} \mathcal{L}_u \psi^A_a \psi^B_b & \\ = k_{AB} \left[u^c \frac{\partial}{\partial x^c} \left(\psi^A_a \psi^B_b\right) + \psi^A_c \psi^B_b \frac{\partial u^c}{\partial x^a} + \psi^A_a \psi^B_c \frac{\partial u^c}{\partial x^b} \right] & \\ = k_{AB} u^c \left[\frac{\partial^2 X^A}{\partial x^c \partial x^a} \psi^B_b + \psi^A_a \frac{\partial^2 X^B}{\partial x^c \partial x^b} - \frac{\partial^2 X^A}{\partial x^a \partial x^c} \psi^B_b - \psi^A_a \frac{\partial^2 X^B}{\partial x^b \partial x^c} \right] = 0 \ . & \end{aligned} \tag{159}$$

In fact, one can prove that the Lagrangian variation of <u>all</u> such tensors vanishes (see Carter and Quintana [19]).

Because k_{ab} is flowline orthogonal, computing its determinant requires some work. The key problem is that even though k_{ab} carries full spacetime indices, it is degenerate, effectively meaning its full spacetime determinant is zero. So, let us consider the problem from the perspective of a local frame which follows the u^a worldline. Clearly, $u^i = 0$ in this frame and we can choose the local time to match the proper time so that $u^0 = 1$. Finally, if we now consider a point on the worldline, we can arrange for the metric to be the flat-space metric in Minkowski coordinates. This means several things:

$$u^a \psi^A_a = \psi^A_0 = 0 \ , \tag{160}$$

$$u^a k_{ab} = u^0 k_{0b} + u^i k_{ib} = k_{0b} = 0 \ , \tag{161}$$

$$g^{AB} = \psi^A_1 \psi^B_1 + \psi^A_2 \psi^B_2 + \psi^A_2 \psi^B_2 \ , \tag{162}$$

since $g^{11} = g^{22} = g^{33} = 1$ and the off-diagonal components are zero. Therefore, the only non-zero components of k_{ab} are the k_{ij} and the determinant of k_{ab}, to be denoted k, is obtained from

$$\begin{aligned} k = & \frac{1}{3!}\left(3!\delta_1^{[i}\delta_2^{j}\delta_3^{k]}\right)\left(3!\delta_1^{[l}\delta_2^{m}\delta_3^{n]}\right) k_{il}k_{jm}k_{kn} \\ & = \frac{1}{3!}\left(3!\delta_1^{[i}\delta_2^{j}\delta_3^{k]}\right)\left(3!\delta_1^{[l}\delta_2^{m}\delta_3^{n]}\right) \psi_i^{[A}\psi_j^{B}\psi_k^{C]}\psi_l^{[D}\psi_m^{E}\psi_n^{F]} k_{AD}k_{BE}k_{CF} \\ & = \frac{1}{3!}\left(\psi_1^{[G}\psi_2^{H}\psi_3^{I]}\delta_G^{[A}\delta_H^{B}\delta_I^{C]}\right)\left(\psi_1^{[J}\psi_2^{K}\psi_3^{L]}\delta_J^{[D}\delta_K^{E}\delta_L^{F]}\right) k_{AD}k_{BE}k_{CF} \\ = & \frac{1}{3!}\left(\frac{1}{3!}\left[G\ H\ I\right]_{\mathcal{D}}\left[J\ K\ L\right]_{\mathcal{D}} g^{GJ}g^{HK}g^{IL}\right)\left(\frac{1}{3!}\left[A\ B\ C\right]^{\mathcal{U}}\left[D\ E\ F\right]^{\mathcal{U}} k_{AD}k_{BE}k_{CF}\right) \\ & = \frac{1}{3!}\det\left(k_{AB}\right)\det\left(g^{AB}\right) , \end{aligned} \quad (163)$$

where we have used Eq. (153). Upon comparing with Eq. (155) we see $k = n^2$.

Karlovini and Samuelsson [15] introduce the matter space tensor η_{AB} to quantify the so-called *unsheared* state. Its defining characteristic is that it is the inverse to g^{AB} but only for the unsheared state (when the energy density $\epsilon = \check{\epsilon}$):

$$g^{AC}\eta_{CB} = \delta^A_B \quad , \quad \epsilon = \check{\epsilon} \ . \quad (164)$$

If we introduce

$$\epsilon^{ABC} = \psi^A_a\psi^B_b\psi^C_c u_d \epsilon^{dabc} \ , \quad (165)$$

then from Eq. (7) we can infer

$$n_{ABC} = n\epsilon_{ABC} \ , \quad (166)$$

where

$$\epsilon^{ABC}\epsilon_{DEG} = 3!\delta^{[A}_D\delta^B_E\delta^{C]}_G \ , \quad (167)$$

and

$$\epsilon^{ABC}\epsilon^{DEF}\eta_{AD}\eta_{BE}\eta_{CF} = 3! \ ; \quad (168)$$

in other words,

$$\epsilon_{ABC} = \sqrt{\det\left(\eta_{AB}\right)}\left[A\ B\ C\right]_{\mathcal{D}} \ . \quad (169)$$

This tensor is useful because it allows a straightforward way to model conformal elastic deformations; namely, if f is the conformal factor, we let

$$k_{AB} = f\eta_{AB} \quad \Longrightarrow \quad \det\left(k_{AB}\right) = f^3\det\left(\eta_{AB}\right) \ . \quad (170)$$

But,

$$n_{ABC} = \sqrt{\det\left(k_{AB}\right)}\,[A\ B\ C]_{\mathcal{D}} = n\epsilon_{ABC} = n\sqrt{\det\left(\eta_{AB}\right)}\,[A\ B\ C]_{\mathcal{D}}\ , \tag{171}$$

therefore, $f = n^{2/3}$.

---

[1] D. Viganò, N. Rea, J. A. Pons, R. Perna, D. N. Aguilera, and J. A. Miralles, Mon. Not. R. Astron. Soc. **434**, 123 (2013).

[2] A. L. Watts and T. E. Strohmayer, Astrophys. J. **637**, L117 (2006).

[3] L. Samuelsson and N. Andersson, Mon. Not. R. Astro. Soc. **374**, 256 (2007).

[4] L. Samuelsson and N. Andersson, Classical and Quantum Gravity **26**, 155016 (2009).

[5] B. P. Abbott, et al., Astrophys. J. **839**, 12 (2017).

[6] C. J. Horowitz and K. Kadau, Physical Review Letters **102**, 191102 (2009).

[7] N. K. Johnson-McDaniel and B. J. Owen, Phys. Rev. D **88**, 044004 (2013).

[8] B. Haskell and A. Melatos, International Journal of Modern Physics D **24**, 1530008 (2015).

[9] S. Seveso, P. M. Pizzochero, F. Grill, and B. Haskell, Mon. Not. R. Ast. Soc. **455**, 3952 (2016).

[10] J. L. Friedman, Communications in Mathematical Physics **62**, 247 (1978).

[11] N. Andersson, Class. Quant. Grav. **20**, 105 (2003).

[12] C. J. Pethick, N. Chamel, and S. Reddy, (2010), arXiv:1009.2303.

[13] V. Cirigliano, S. Reddy, and R. Sharma, Phys. Rev. C **84**, 045809 (2011).

[14] B. Carter, N. Chamel, and P. Haensel, Int. J. Mod. Phys. **D15**, 777 (2006).

[15] M. Karlovini and L. Samuelsson, Class. Quant. Grav. **20**, 3613 (2003).

[16] M. Karlovini, L. Samuelsson, and M. Zarroug, Class. Quant. Grav. **21**, 1559 (2004).

[17] M. Karlovini and L. Samuelsson, Class. Quant. Grav. **21**, 4531 (2004).

[18] M. Karlovini and L. Samuelsson, Class. Quant. Grav. **24**, 3171 (2007).

[19] B. Carter and H. Quintana, Proceedings of the Royal Society of London Series A **331**, 57 (1972).

[20] B. Carter and L. Samuelsson, Class. Quant. Grav. **23**, 5367 (2006).

[21] B. Carter, Commun. Math. Phys. **30**, 261 (1973).

[22] B. Carter, N. Chamel, and P. Haensel, Nucl. Phys. A **759**, 441 (2005).

[23] B. Carter, N. Chamel, and P. Haensel, International Journal of Modern Physics D **15**, 777 (2006), arXiv:astro-ph/0408083.

[24] B. Carter, in *Relativistic Fluid Dynamics (Noto, 1987)*, edited by A. Anile and M. Choquet-Bruhat (Springer-Verlag, Heidelberg, Germany, 1989), vol. 1385 of *Lecture Notes in Mathematics*, pp. 1–64.

[25] N. Andersson and G. L. Comer, Living Reviews in Relativity **10**, 1 (2007).

[26] N. Andersson and G. L. Comer, Class. Quant. Grav. **32**, 075008 (2015).

[27] L. E. Reichl, *A Modern Course in Statistical Physics, 2nd Edition* (Wiley-VCH, 1998).

[28] Y. Kojima, Phys. Rev. D **46**, 4289 (1992).

[29] G. Birkhoff and S. Lane, *A Survey of Modern Algebra* (Macmillan, 1965).

[30] B. L. Schumaker and K. S. Thorne, Mon. Not. R. Ast. Soc. **203**, 457 (1983).

[31] A. J. Penner, N. Andersson, D. I. Jones, L. Samuelsson, and I. Hawke, Astrophys. J. Lett. **749**, L36 (2012).

[32] D. Tsang, J. S. Read, T. Hinderer, A. L. Piro, and R. Bondarescu, Physical Review Letters **108**, 011102 (2012).

[33] A. F. Andreev and E. P. Bashkin, Sov. Phys. JETP **42**, 164 (1975).

[34] M. Borumand, R. Joynt, and W. Kluzniak, Phys. Rev. C **54**, 2745 (1996).

[35] G. L. Comer and R. Joynt, Phys. Rev. D **68**, 023002 (2003).

[36] N. Chamel and P. Haensel, Living Reviews in Relativity **11**, 10 (2008).

[37] N. Chamel, Phys. Rev. C **85**, 035801 (2012).

[38] N. Martin and M. Urban, Phys. Rev. C. **94**, 065801 (2016).

[39] G. Watanabe and C. J. Pethick, Phys. Rev. Lett. **119**, 062701 (2017).

[40] N. Chamel, in *Neutron Stars and Pulsars: Challenges and Opportunities after 80 years*, edited by J. van Leeuwen (2013), vol. 291 of *IAU Symposium*, pp. 73–76.

[41] N. Andersson, B. Haskell, and L. Samuelsson, MNRAS **416**, 118 (2011).

[42] G. L. Comer, D. Langlois, and L. M. Lin, Phys. Rev. D **60**, 104025 (1999).

[43] N. Andersson, G. L. Comer, and D. Langlois, Phys. Rev. D **66**, 104002 (2002).

[44] N. Andersson, G. L. Comer, and K. Grosart, **355**, 918 (2004).

[45] G. L. Comer, Phys. Rev. D **69**, 123009 (2004).

[46] A. Reisenegger and P. Goldreich, Ap. J. **395**, 240 (1992).

[47] R. Prix, G. L. Comer, and N. Andersson, Astron. Astrophys. **381**, 178 (2002).

[48] N. Andersson, K. Glampedakis, W. C. G. Ho, and C. M. Espinoza, Physical Review Letters **109**, 241103 (2012).

[49] B. P. Abbott, et al., Physical Review Letters **119**, 161101 (2017).