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The Earth's shape and gravity field: a report of progress from 1958 to 1982

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Thou com'st in such a questionable shape
That I will speak to thee — Shakespeare, *Hamlet*

Whence and what art thou, execrable shape — Milton, *Paradise Lost*

Summary. A major step forward in geophysics during the last 25 years has been the progress in the determination of the Earth's shape and gravity field, from the halting steps of the first satellite orbit analyses to the global solutions expanded in spherical harmonics up to degree 36, and from painstaking gravity surveys on land to the detailed regional geoids derived from altimeter observations. No other geophysical quantity pertaining to lateral variations in the structure of the crust and mantle is now known with a comparable accuracy and spatial resolution. An increasingly acute problem has been to find ways to validate the global results since differences between individual solutions remain substantial. Absolute tests are not available but statistical comparisons produce some useful insight into the status of the recent gravity field models. A number of recent models are evaluated in this paper. A primary conclusion is that the gravity or geoid anomalies are frequently not as well determined as stated by the authors. We estimate, for example, that the root mean square errors of the geoid heights deduced from models by Lerch *et al.* and Gaposchkin are about 3 m and that maximum errors may exceed 10 m in some places. A considerable part of this comes from uncertainties in the low degree harmonics, in particular the degree and order 3 coefficients and more generally the odd degree coefficients, for while the signal-to-noise ratio of these coefficients is high the power in the spectrum is also high. Most tests developed for evaluating the gravity fields are insensitive to the long wavelength components in the spatial spectrum. Future projects call particular attention to improving the high degree part of the geopotential spectrum but thought should also be given to these low degree harmonics. Considerable progress in determining the gravity field can still be made by using data already available.

Introduction

Nineteen fifty-eight saw the launching of the first United States artificial satellite and of the *Geophysical Journal,* two events that were to come together the same year with a paper dealing with the determination of the Earth's gravity field from satellite observations. Prior to these events, knowledge of terrestrial gravity was restricted to surface observations and, apart from the submarine measurements initiated by Vening Meinesz, the observations were restricted to the more readily accessible land areas (see Heiskanen & Vening Meinesz 1958, for a review of the data available at that time). In consequence, the global expansions of the gravity field were restricted to the first few terms of the spherical harmonic expansions (see, for example, Jeffreys 1959, fig. 21). Even these coefficients were poorly known.

The use of artificial satellites as sensors of the Earth's gravity field was anticipated before *Sputnik 1* was launched in 1957 and early analyses of satellite orbit perturbations, particularly of the *Sputnik 2, Explorer 1* and *Vanguard 1* spacecrafts, quickly revolutionized the study of the Earth's gravity field. The first published results appear to have been those of Buchar (1958) and Merson & King-Hele (1958), both published in *Nature,* of Jacchia (1958), published in a *Smithsonian Astrophysical* report, and of Cook (1958), published in the *Geophysical Journal.* Buchar estimated the dynamical flattening C_{20} (see below) on the assumption that C_{40} and higher degree terms are negligible. Merson & King-Hele established that this assumption was unrealistic but, from one satellite orbit alone (*Sputnik 2*), they could only produce a linear relation between C_{20} and C_{40}. Jacchia analysed *Sputnik 2* and *Vanguard 1* orbits to estimate both of these coefficients. Cook combined the Merson & King-Hele results with Jeffreys's analysis of surface gravity in what was the first attempt at combining these two quite distinctly different estimates of the gravitational potential. Results for the odd zonal harmonic C_{30} were first published by O'Keefe, Eckels & Squires (1959) and Kozai (1959). Preliminary results for some of the higher degree zonal harmonics soon followed, notably by King-Hele and Kozai. These early results have been reviewed by King-Hele (1961) (see also this issue).

That the ellipticity of the Earth's equator (defined by the C_{22}, S_{22} terms) could result in a measurable perturbation in satellite motion was recognized already by O'Keefe & Batchelor (1957). Compared to the zonal harmonic perturbations, the longitude-dependent terms will generally cause orbital perturbations that are of short period and it was sometimes argued that these coefficients could not be determined from the analysis of tracking data of satellites (e.g. Cook 1961). Izsak (1961), however, using Baker-Nunn camera observations, derived results for the ellipticity coefficients, as did Kaula (1961a) using the Minitrack interferometry tracking system. In addition to C_{22} and S_{22}, Kaula also obtained estimates for the (4,1), (4,2), (6,1) and (6,2) degree and order coefficients. Newton (1962) analysed Doppler observations of the early navigation satellites and obtained estimates of some of these low degree tesseral harmonics.

It was recognized at a very early stage that these results, however preliminary they may have been, pointed to an Earth that was out of hydrostatic equilibrium (e.g. Cook 1958) and that, when combined with the precession constant, the dynamical flattening provided an estimate of the Earth's moment of inertia (O'Keefe 1959; Henriksen 1960). It was also recognized that the departures from hydrostatic equilibrium implied either a finite strength for the mantle or dynamic processes within it (MacDonald 1962). O'Keefe (1960) argued that convection would not maintain the implied gravity anomalies and that the mantle possessed a finite strength, as already argued by Jeffreys (1959). Kopal (1962), on the other hand, argued for convection in the mantle. Others simply stated that the potential coefficients derived from the satellite orbits were meaningless since that below the crust the Earth, per definition, must be in hydrostatic equilibrium. This point persisted in some circles at least up to 1966 (Ledersteger 1967).

The year 1966 saw two major advances in satellite geodesy. One was the publication of the Smithsonian Astrophysical Observatory's comprehensive gravity field model complete to degree and order 8 (Lundquist & Veis 1966). This solution was based on Baker-Nunn camera observations. It, and the earlier Doppler-based solution by Guier & Newton (1965), provided valuable material for speculating about mantle convection (e.g. Runcorn 1967). The second major advance of 1966 was the publication of Kaula's *Theory of Satellite Geodesy* (Kaula 1966a). His systematic development of close Earth satellite orbital theory, already foreshadowed in Kaula (1961b), quickly became the basis for many subsequent analyses of orbit perturbations.

The 1966 Smithsonian solution was followed some four years later by a much improved model in which the gravitational potential was expanded to degree and order 16 (Gaposchkin & Lambeck 1970, 1971). This solution contained a significant amount of laser range measurements in addition to Baker-Nunn observations. It was also one in which, for the first time, the introduction of surface gravity observations actually led to an improved representation of the satellite motion. Also for the first time, relations between the global gravity field and plate tectonics emerged, with plate margins, particularly converging ones, tending to be associated with positive anomalies while stable plate interiors tended to be associated with negative gravity anomalies (see Kaula 1972, for a discussion of these relations). Further iterations were published by Gaposchkin (1973, 1974, 1977). These solutions have generally been referred to as 'Standard Earth' models and we use the notation SE followed by the year of publication. Independent solutions also appeared from the Goddard Space Flight Center (Smith *et al.* 1976; Lerch *et al.* 1979). These solutions are usually referred to as Goddard Earth Models, or GEM solutions. The Groupe de Recherches de Geodesie Spatiale in France and the Geodetic Research Institute in Munich jointly developed the capability for global potential modelling (Balmino, Reigber & Moynot 1976, 1978). These solutions are referred to by an acronym GRIM whose original meaning appears to have been lost. While these later solutions may not have added much to the geophysical understanding of the gravity field, they have provided valuable and independent verification of the satellite results.

By about 1977 the solutions appeared to be approaching the limits of the methods of analysing perturbation in the satellite motion as deduced from ground-based tracking data, whether camera, electronic or laser. The launch in 1975 of *GEOS 3,* the much-heralded radar altimeter-carrying satellite, represented a next important step in measuring the Earth's gravity field. This was followed in 1978 by the short-lived but productive *SEASAT 1.* Preliminary global solutions incorporating these observations have been published by Gaposchkin (1980) (SE 1980), Lerch *et al.* (1981) (GEM 10B) and Reigber *et al.* (1982) (GRIM 3).

Gravity field representation

It is convenient to expand the gravitational potential $U(r, \phi, \Lambda)$ in spherical harmonics according to

$$U(r, \phi, \Lambda) = \frac{GM}{r}\left\{1 + \sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{R}{r}\right)^{l} P_{lm}(\sin\phi)\,[C_{lm}\cos m\Lambda + S_{lm}\sin m\Lambda]\right\}. \tag{1}$$

In this expansion r, ϕ, Λ are the geocentric distance, latitude and longitude of the point at which this potential is evaluated, referred to a set of inertial axes X_i. If longitude is defined relative to a meridian fixed to the Earth then $\Lambda = \lambda + \theta$, where the sidereal angle θ defines the orientation of the Earth in space. For present purposes the X_1, X_2 axes may be taken to lie in the equatorial plane and X_3 as being parallel to the rotation axis of the Earth. The $P_{lm}(\sin\phi)$ are the Legendre functions of degree l and order m. Different definitions are used in the literature but the two principal ones are the unnormalized functions defined as (e.g. Heiskanen & Moritz 1967)

$$P_{lm}(\sin\phi) = (1-\sin^2\phi)^{m/2}\,\frac{d^m P_{l0}(\sin\phi)}{d(\sin\phi)^m}\,, \tag{2a}$$

with

$$P_{l0}(\sin\phi) = \frac{1}{2^l l!}\,\frac{d^l}{d(\sin\phi)^l}\,(\sin^2\phi-1)^l,$$

or the fully normalized functions defined as

$$\bar{P}_{lm} = N_{lm}P_{lm}, \tag{2b}$$

where N_{lm} is the normalizing factor

$$N_{lm}^2 = (2-\delta_{0m})\,(2l+1)\,(l-m)!/(l+m)! \tag{2c}$$

and where $\delta_{0m}=1$ when $m=0$, otherwise $\delta_{0m}=0$. The fully normalized functions are defined such that

$$\int_\sigma [\bar{P}_{lm}(\sin\phi)\cos m\lambda]^2 d\sigma = \int_\sigma [\bar{P}_{lm}(\sin\phi)\sin m\lambda]^2 d\sigma = 4\pi$$

where the integral is taken over the surface σ of a unit sphere.

The C_{lm}, S_{lm} are the (unnormalized) Stokes coefficients, representing integrals of the internal mass distribution of the planet according to

$$\begin{Bmatrix} C_{lm} \\ S_{lm} \end{Bmatrix} = \frac{1}{MR^l}\,\frac{2(l-m)!}{(l+m)!}\int_M (r')^l P_{lm}(\sin\phi')\begin{Bmatrix}\cos m\lambda' \\ \sin m\lambda'\end{Bmatrix} dM \tag{3a}$$

where the integral is over the mass M of the Earth. In normalized form

$$\bar{C}_{lm} = C_{lm}/N_{lm}. \tag{3b}$$

In equation (1) G is the gravitational constant and R is the mean equatorial radius. Note that the definition of the Stokes coefficients is a function of the last two parameters.

The use of the spherical harmonic expansion of the potential is mathematically convenient in that the so-expanded potential satisfies Laplace's equation outside the Earth and that the expansion converges outside a sphere that encompasses all mass of the Earth (e.g. Heiskanen & Moritz 1967). Furthermore, this representation results in a convenient spectral decomposition of the perturbations in the satellite motion from the Keplerian state (Kaula 1966a).

The sensitivity of the satellite to the gravitational potential depends on three factors. First, the potential decreases with altitude; with contributions from higher degree harmonics attenuating more rapidly than contributions from lower degree harmonics according to the $r^{-1}(R/r)^l$ term in equation (1). Secondly, the Stokes coefficients themselves tend to decrease with degree according to a rule first noted by Kaula, namely that

$$\mathrm{O}\,\{\bar{C}_{lm}, \bar{S}_{lm}\} = A10^{-5}/l^2 \tag{4}$$

with $A \simeq 0.85$ (Lambeck 1976). Taken together, these two factors imply an attenuation of the potential with wavelength and altitude according to

$$\frac{1}{r}\left(\frac{R}{r}\right)^l \frac{1}{l^2}. \tag{5}$$

Thirdly, in a general way, the higher the order m of the harmonics the shorter will be the period and the smaller will be the amplitude of the perturbation (Kaula 1966a; Gaposchkin 1973). (An exception to this is the special case of resonance, discussed below.) The consequence of this attenuation is two-fold. On the one hand, it should be possible to describe the satellite motion with high precision using a finite number of terms in the expansion (1), but, on the other hand, the gravity field can be determined from the perturbation analysis with only a limited resolution. Fig. 1 illustrates schematically these limitations for typical close-earth satellites. In a general way the potential coefficients of degrees and orders that lie outside the shaded area bounded by the curves $l = m$ and AA′ cannot be determined from 1 m accurate tracking data. The curves $l = m$ and CC′ defines the region of coefficients that cannot be detected with 20 cm accurate tracking data. The gain in resolution is substantial, from a complete field of degree 10 for the 1 m data to about degree 18 for the 20 cm data.

By 1977 the accuracy of laser ranging data had improved to about 20 cm (e.g. Pearlman *et al.* 1977) for the Smithsonian instrumentation, and to about 10 cm (e.g. Vonbun 1977) for the Goddard lasers. Most of the data collected in earlier years had a precision of only 1 m or worse. Thus an appropriate average precision estimate for all the laser observations at that time may be about 50 cm and the satellite should be sensitive to a field expanded up to about degree 14 (curves BB′ of Fig. 1). Equally important to precision is that there are a number of satellites, in different orbits, that are sensitive to a particular group of potential coefficients. This arises from the fact that there are groups of coefficients that introduce perturbations of equal period. To separate them it is desirable to track satellites with different orbital characteristics, particularly with different inclinations. In recent years precise tracking has been restricted to only a few of the laser reflector tracking satellites. Most recent observations are restricted to three satellites of which one, *LAGEOS*, on a very high altitude orbit, is sensitive only to harmonics of relatively low degree ($\lesssim 8$). Referring to

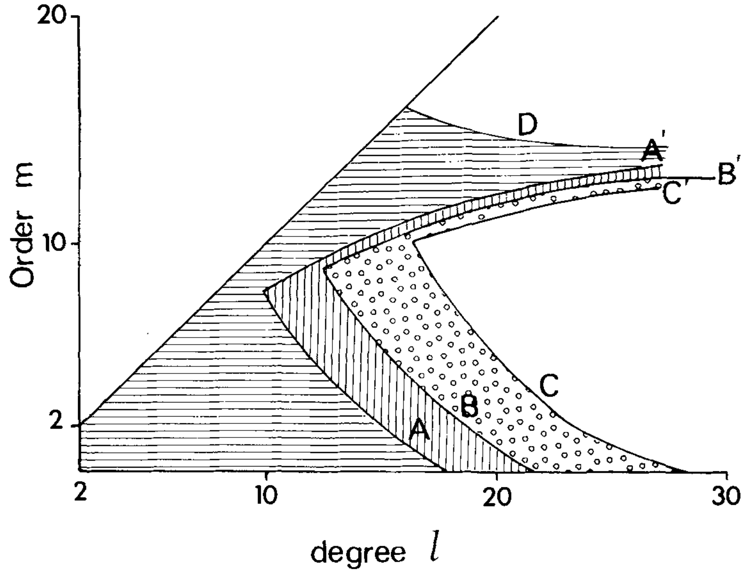

Figure 1. Schematic representation of the non-zonal Stokes coefficients that can be determined from satellite orbit perturbation analyses. With tracking data precise to 1 m, coefficients lying in the shaded region to the left of the curves AA′ can be determined. With 50 cm tracking data this boundary is shifted to BB′, and with 20 cm tracking data the limits have shifted to CC′. The coefficients between the curve D and A′, B′, C′ are determined from resonance analyses.

Fig. 1, only two satellite orbits are available in such recent solutions as that of Lerch *et al.* (1981) from which the harmonics in the region between the curves BB′ and CC′ can be determined. Solutions based on satellite observations alone may, therefore, exhibit strong correlations between the higher degree and order coefficients.

As a result of these limitations, attention has been focused on alternative methods of determining the gravity field, by radar altimeter measurements of the shape of the geoid over oceans, by measuring the differences in potential between neighbouring low altitude satellites, or by measuring the gradient of gravity in a low altitude spacecraft. These alternatives have been discussed by the National Research Council (1979) and Lambeck (1979). In these approaches the spherical harmonic representations of the potential quickly become unwieldy and some alternatives have been explored by Kaula (1983).

Intercomparisons: a statistical evaluation

A principal difficulty in interpreting the gravity field models published over the years is the inability to make objective evaluations of the solutions. Surface gravity has often been used as a standard for evaluating the purely satellite solutions (e.g. Kaula 1966b; Lambeck 1971) but the quality of this data set is itself questionable when treated globally. More recently, altimeter data have been used for comparison purposes, but these tests, as used by Lerch *et al.* (1979) and Reigber *et al.* (1982), are generally insensitive to the very long wavelength components of the gravitational potential or geoid heights. The precision with which the satellite motion can be described is also used as a criterion for evaluating gravity field models but it must be recalled that what is sought is a set of geophysically meaningful parameters that describes this field, not necessarily a set of parameters that defines the motion of a particular satellite or group of satellites. Klokocnik & Pospisilova (1981) discuss one such test in which the effect of a linear combination of selected harmonics on the satellite motion is evaluated. The real difficulty with applying these tests to final iterations is that the data are rarely independent since optimum solutions will incorporate surface gravity, radar altimetry and satellite observations in an iterative process. These tests are important for establishing relative weights of the constituent data sets. An intercomparison of different solutions, by estimating what information is common to the various models, is more straightforward, although this does not provide an absolute test of the quality of the solution.

The normalized discrete power spectrum of the gravitational potential defined by equation (1) is given as (e.g. Kaula 1967)

$$V_l^2(U) = \sum_m (\bar{C}_{lm}^2 + \bar{S}_{lm}^2) = \sum_{i=1}^{2} \bar{C}_{ilm}^2 \tag{6}$$

where $\bar{C}_{1lm} \equiv \bar{C}_{lm}$ and $\bar{C}_{2lm} \equiv \bar{S}_{lm}$. Two independent estimates U_1 and U_2 of the potential, each expanded according to (1) up to the same degree l^* and order m^*, can be written as

$$U_1 = U_0 + \epsilon_1; \qquad U_2 = U_0 + \epsilon_2 \tag{7}$$

where U_0 is the true contribution to the total potential defined either by all harmonics with $l \leqslant l^*$, $m \leqslant m^*$ or by a subset of potential coefficients. The ϵ_1 and ϵ_2 are the errors in the estimates of the U_1 and U_2. If the expected value of a quantity x is denoted by $\mathrm{E}(x)$ and the mean value of this quantity by $\langle x \rangle$, then with the assumption that ϵ_1, ϵ_2 and U_0 are uncorrelated

$$\mathrm{E}\,\{(U_1 - U_2)^2\} = \langle (U_1 - U_2)^2 \rangle = \langle U_1^2 \rangle - 2\langle U_1 U_2 \rangle + \langle U_2^2 \rangle$$

$$= \sum_i \sum_m \bar{C}_{(1)ilm}^2 + \sum_i \sum_m \bar{C}_{(2)ilm}^2 - 2 \sum_{im} \bar{C}_{(1)ilm} \bar{C}_{(2)ilm}, \tag{8a}$$

where $C_{(1)ilm}$ are the Stokes coefficients corresponding to the potential U_1 and the $C_{(2)ilm}$ correspond to U_2. Also

$$\mathrm{E}\{U_0^2\} = \langle U_1 U_2 \rangle = \sum_{im} \bar{C}_{(1)ilm} \bar{C}_{(2)ilm} . \tag{8b}$$

With (8a) the error spectra are

$$\mathrm{E}\{\epsilon_1^2\} = \langle U_1^2 \rangle - \langle U_1 U_2 \rangle = \langle U_1^2 \rangle - \langle U_0^2 \rangle, \tag{8c}$$

$$\mathrm{E}\{\epsilon_2^2\} = \langle U_2^2 \rangle - \langle U_1 U_2 \rangle = \langle U_2^2 \rangle - \langle U_0^2 \rangle. \tag{8d}$$

The first of these relations (8a) provides a measure of the agreement between the two estimates of the potential and

$$\langle (U_1 - U_2)^2 \rangle = \mathrm{E}\{\epsilon_1^2\} + \mathrm{E}\{\epsilon_2^2\} .$$

The second quantity (8b) is a measure of the amount of information common to the two estimates of the potential, or a measure of the true power in the geopotential spectrum. These comparisons can be carried out for the total field up to a fixed degree or, alternatively, for each degree separately although, in the latter case, the estimates for the lower degree terms become unreliable because of the small sample sizes.

Results

PRE-ALTIMETER SOLUTIONS

By 1978 the three principal solutions of the global gravity field were by Gaposchkin (1977), Lerch *et al.* (1979) and Balmino *et al.* (1978). The Gaposchkin solution, referred to here as SE 1977, is based mainly on an analytical description of the satellite motion (see also Gaposchkin 1973). In the other two solutions the equations of motion are solved directly by numerical integration (e.g. Balmino 1975; Gaposchkin 1979). The virtues of the former approach are its elegance, that it leads to a deeper understanding of the orbital mechanics, and that long orbital arcs (about 30 days or more) can be analysed without excessive computer requirements. On the negative side, for high precision orbits the theory becomes complex due to the need to take into account the interaction between C_{20} and the tesseral harmonics. The virtues of the latter approach are its potentially high accuracy and its convenience. Orbital arc lengths are generally limited to about one week, making the approach less convenient for analysing long period perturbations.

The only satellite observations used by Gaposchkin in the SE 1977 solution are laser ranges to nine satellites tracked from 14 stations. Essentially the same data are used in the other two solutions. Balmino *et al.*, in their GRIM2 solution, used the less precise camera data as well. Lerch *et al.*, in producing their GEM9 and 10 models, used laser, camera, Doppler and other electronic data, some of which is less precise than the laser data by as much as three orders of magnitude and which, because orbital arc lengths are short, must add little to the solution. In this last-mentioned solution a total of 840 000 observations to 30 satellites have been used. This includes about 250 000 laser range observations to the same nine satellites used by Gaposchkin and Balmino *et al.* Two solutions are given by Lerch *et al.* One, GEM9, is based on satellite tracking data only and the second, GEM10, includes surface gravity. The SE 1977 and GRIM2 solutions represent iterations upon earlier models in which all the satellite orbits have been recomputed using the best available force models and gravity field parameters. In general the GEM series of models are not true iterations in this sense. Rather, the successive models represent increasing amounts of observation equations, referenced to a variety of force models and physical parameters, without the orbits being recomputed with the improved parameters of the previous iteration. In consequence, while the solutions may converge it may be that they do not converge on the true potential.

The three solutions differ in their treatment of both zonal and resonant harmonics. The latter are discussed separately below. Gaposchkin adopts a set of zonal coefficients derived separately from long-arc analyses (Gaposchkin 1973) and keeps the resulting values fixed in the solution for the longitude dependent terms. Balmino *et al.* include the zonal harmonic observation equations of Kozai (1969), Gaposchkin (1973) and Cazenave *et al.* (1972) in their inversion for all the Stokes coefficients. Lerch *et al.* apparently solve for all harmonics simultaneously without introducing observation equations based on long-arc analyses. Since the orbital arcs analysed are restricted to 5 or 7 days, whereas the zonal harmonic perturbations are of much longer period, the coefficients are determined mainly from the short period perturbations that they introduce into the satellite motion. The less precise electronic data mentioned above would contribute to the determination of the zonal harmonics if long orbital arcs were analysed.

Because of the inability to solve for the higher degree and order terms, surface gravity data have been incorporated in all three solutions. A very similar basic data set of $1^\circ \times 1^\circ$ mean values has been used throughout. Minor differences exist in the manner in which these data have been combined into larger ($5^\circ \times 5^\circ$ or 550×550 km) area means but this is unlikely to be important. More significant is that in all three solutions little attempt has been made to critically evaluate the data: the fact that area means exist for all $1^\circ \times 1^\circ$ blocks over Asia, for example, is a consequence of prediction rather than of the availability of measurement. As reported to one of us, observed area means over the Soviet Union point to gross errors in the predicted values used in these compilations and to significant discrepancies between the observed values and the global models discussed here.

Gaposchkin solved for the harmonics up to degree and order 24 without assuming any *a priori* information on the power spectrum of the potential. Balmino *et al.* determine the coefficients to degree and order 30 while Lerch *et al.* include coefficients to degree and order 22. Because the satellites are insensitive to many of the coefficients in these expansions, both Balmino *et al.* and Lerch *et al.* have imposed constraints to control the inversion of otherwise ill-conditioned matrices. This process assumes that the potential coefficients decay according to some *a priori* rule, in this case Kaula's rule (4). The imposed condition equations are of the form

$$(\bar{C}_{lm}, \bar{S}_{lm}) = 0 \pm A 10^{-5}/l^2 \tag{9}$$

Balmino *et al.* impose these constraints on the lower degree harmonics ($l \leq 10$) while Lerch *et al.* impose these constraints on the non-resonant harmonics with $l \geq 12$. That this condition is required for the lower degree harmonics is perhaps symptomatic of insufficient precise tracking data to permit a separation of the coefficients. That the need is seen to introduce the constraint for the high degree harmonics confirms the earlier conclusion that the satellite tracking data alone are inadequate, and that the surface gravity data are also insufficient. One consequence of introducing the constraint (9) is that the resulting geopotential spectrum may not reflect the true nature of the gravity field since the rule (4) is not a law. While there are simple physical explanations for it (e.g. Lambeck 1976; Kaula 1977), there are no reasons why the actual spectrum should not depeart from it at any wavenumber.

Fig. 2 illustrates the power spectra for the three solutions, SE 1977, GEM10 and GRIM2. The zonal harmonics have been excluded in these spectra calculations. Power is comparable in all three solutions up to about degree 13, but at higher degrees the Goddard model (GEM10) yields significantly lower power estimates. This is a direct consequence of the

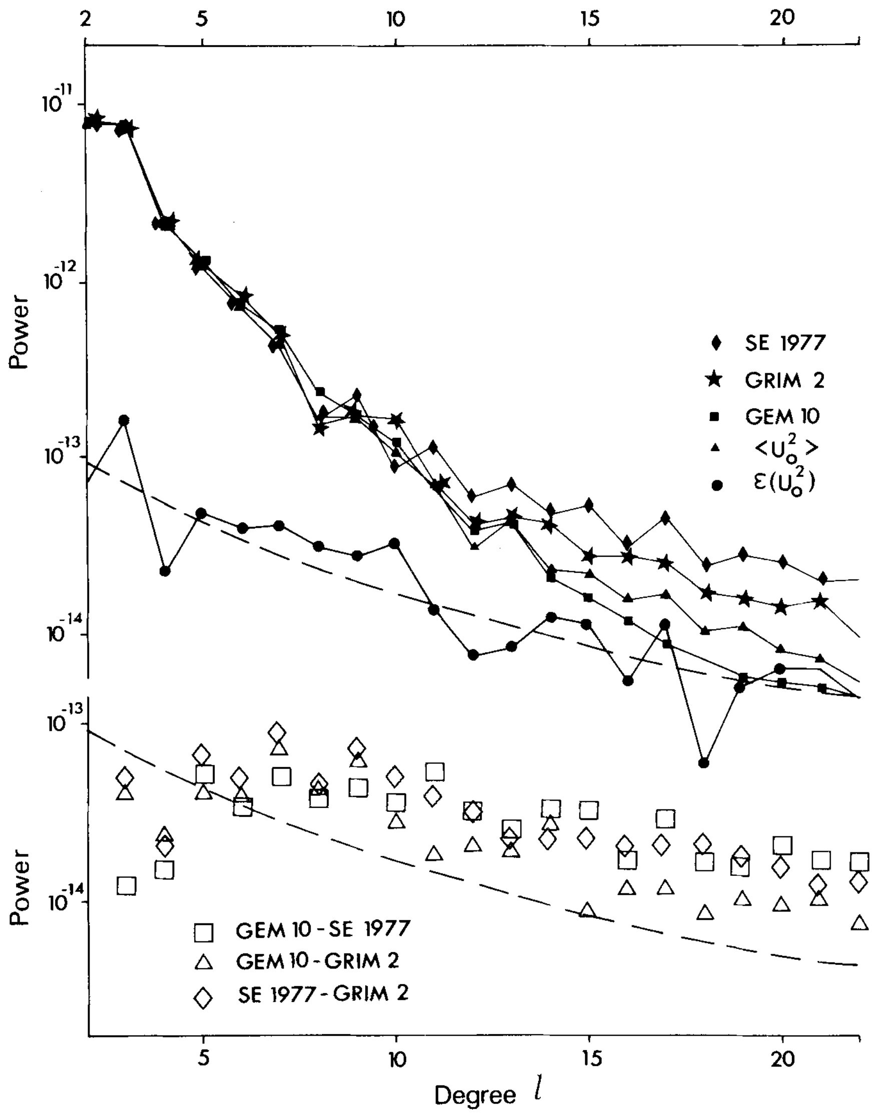

Figure 2. Power spectrum of the normalized gravitational potential $V_l^2(U)$ as estimated from three solutions which do not include altimetry data. Also indicated is the 'best' estimate $\langle \tilde{U}_0^2 \rangle$ of this power and the uncertainty estimate $\epsilon(U_0^2)$ of this quantity. The dashed curve represents the average error spectrum. Difference spectra, defined by (8a) are also indicated.

constraint (9) imposed in that solution. Comparisons of pairs of solutions provides estimates of the 'true' power in the spectrum according to the relation (8b). Fig. 2 illustrates the 'best' estimate of this spectrum $\langle \tilde{U}_0^2 \rangle$ namely the mean of the three, not now independent, estimates $\langle U_0^2 \rangle$. The obvious assumption made here is that all three solutions are of comparable accuracy. The uncertainty estimates of this mean spectrum are also indicated. This quantity $\epsilon(U_0^2)$ is defined as

$$\epsilon(U_0^2) = \left\{ \sum_{i=1}^{3} [\langle U_0^2 \rangle_i - \langle U_0^2 \rangle]^2 \right\}^{1/2} \tag{10}$$

where the index $i = 1, 2, 3$ refers to the three possible estimates of $\langle U_0^2 \rangle$. At higher degrees these estimates approach or equal the signal. The mean spectrum remains strongly biased by the Goddard model. Also illustrated in Fig. 2 are the difference spectra defined by (8a). These may be considered as estimates of the upper limits to the individual error spectra. Already at degree 12 the difference spectra approach the power estimate of the signal, $\langle \tilde{U}_0^2 \rangle$. Estimates of the error spectra, defined by the expressions (8c) and (8d) with $\langle \tilde{U}_0^2 \rangle$ for the estimate of the true power, are not very satisfactory since they are frequently negative. For the low degrees this is mainly a consequence of small sample sizes but it may also reflect departures from the conditions of independence assumed above. This will be in part due to the very similar data sets and force models used in all solutions and in part due to the introduction of the constraint (9) in the GRIM and GEM models.

The total power in the error spectrum of the potential U_i is given by

$$\sum_l \mathrm{E}(\epsilon_l^2)$$

and the root mean square geoid height error ϵ_N is

$$\epsilon_N = \left[R^2 \sum_l \mathrm{E}(\epsilon_l^2)\right]^{1/2}. \tag{11a}$$

An upper limit to ϵ_N is given by

$$\epsilon_N|_{\max} = \left[R^2 \sum_l |\mathrm{E}(\epsilon_l^2)|\right]^{1/2}. \tag{11b}$$

Total power

$$\sum_l \langle U_l^2 \rangle$$

in the three solutions is comparable (Table 1), for while the GEM10 solution contains least power at high degrees this represents only a small contribution to the total power. At low degrees ($l \lesssim 10$), the power estimates are least in the GRIM2 model (Table 2) and this also points to the influence of the condition (9). Within the common wavenumber range of $2 \lesssim l \lesssim 22$, the accuracy of the geoid heights as defined by (11a) ranges from 2.3 to 4.7 m according to the particular solution. Upper limits of this quantity, defined by (11b), are of the order 5–6 m (Table 1). A check on these error spectra follows from a comparison with the quantity $\epsilon(U_0^2)$ defined by (10). In general $\epsilon(U_0^2)$ is approximately equal to the mean of the three error estimates at all degrees except at low degrees where better agreement is found if $\epsilon(U_0^2)$ is compared with

$$\left[\sum_i |\epsilon_i^2|\right] /3.$$

That much of the contribution to the error estimates comes from the low degree harmonics, particularly $l = 2$ and 3, is perhaps the most important point to emerge from these comparisons: for while the signal-to-noise ratio of these harmonics is high these harmonics remain inadequately known, in an absolute sense. Table 3 summarizes the degree 3 coefficients: major discrepancies between the solutions occur for the sine terms. Why these differences occur is not obvious to us although one reason may be the neglect of the coupling terms between the zonal and tesseral harmonics in the analytical solution of Gaposchkin. These results do explain why these gravity field models do not describe the

Table 1. Comparisons of total power in the potential spectrum $\Sigma_l V_l^2(U)$ for three gravity models. The summation is for all non-zonal harmonics with $2 \leqslant l \leqslant 22$. $\Sigma\langle \tilde{U}_0^2 \rangle$ is the average of the three estimates of the true power in the geopotential. The power in the error spectra are defined by equations (11a) and (11b). All power estimates are to be multiplied by $(GMR^{-1}10^{-6})^2$. The rms geoid height errors are in metres.

	SE 1977	GRIM2	GEM10
$\sum_l V_l^2(U)$	21.82	21.49	21.40
$\sum_l \langle \tilde{U}_0^2 \rangle$		21.24 ± 0.56	
$\sum_l \mathrm{E}\{\epsilon^2\}$	0.55	0.24	0.13
$\sum_l \lvert \mathrm{E}\{\epsilon^2\} \rvert$	0.83	0.61	0.51
ϵ_N	4.7m	3.1	2.3
$\epsilon_N \rvert_{max}$	5.8	5.0	4.6

Table 2. Power estimates of the potential and error spectra for three models. All power estimates are to be multiplied by $(GMR^{-1}\,10^{-6})^2$. The rms geoid height error estimates (in metres) are defined according to equations (11).

l	$\langle \tilde{U}_0^2 \rangle$	SE 1977		GRIM2		GEM10	
		$V_l^2(U)$	$\mathrm{E}(\epsilon^2)$	$V_l^2(U)$	$\mathrm{E}(\epsilon^2)$	$V_l^2(U)$	$\mathrm{E}(\epsilon^2)$
2	7.932	7.870	−0.062	8.051	0.119	7.882	−0.050
3	7.961	8.239	0.279	7.847	−0.114	7.850	−0.111
4	2.193	2.222	0.032	2.169	−0.024	2.217	0.024
5	1.259	1.283	0.024	1.225	0.034	1.348	0.085
6	0.755	0.719	−0.036	0.833	0.079	0.774	0.019
7	0.464	0.447	−0.017	0.506	0.042	0.549	0.085
8	0.168	0.170	0.002	0.155	−0.013	0.241	0.073
9	0.168	0.236	0.068	0.176	0.008	0.181	0.015
10	0.108	0.086	−0.022	0.172	0.064	0.123	0.015
Σ	21.008	21.272	0.268	21.134	0.195	21.165	0.155
Σ| |			0.542		0.497		0.477
ϵ_N(m)			3.3		2.8		2.5
$\epsilon_N \rvert_{max}$(m)			4.7		4.5		4.4

Table 3. Third degree Stokes coefficients ($\times 10^6$) from different geopotential solutions.

C_{lm}	GEM10	SE 1977	GRIM2	GEML2	GEM10B	SE 1980
C_{30}	0.958	0.960	0.961	0.958	0.959	0.960
C_{31}	2.028	2.049	1.962	2.029	2.031	2.038
S_{31}	0.252	0.277	0.155	0.250	0.253	0.275
C_{32}	0.893	0.918	0.864	0.903	0.894	0.884
S_{32}	−0.623	−0.681	−0.545	−0.616	−0.621	−0.627
C_{33}	0.700	0.665	0.712	0.722	0.713	0.671
S_{33}	1.412	1.489	1.557	1.414	1.419	1.496

motion of the *LAGEOS* satellite as well as they ought to. This point must have been recognized by Gaposchkin for he determined a new solution by adding laser observations of *LAGEOS* to the SE 1977 solution (Gaposchkin & Mendes 1977). (Neither details nor coefficients of this solution have been published). Lerch, Klosko & Patel (1982) have also added *LAGEOS* observations to the GEM9 solution to obtain an improved field denoted by GEML2. Clearly this solution is strongly correlated with the GEM10 solution and this is reflected in the results in Table 3. We return to this solution below.

Fig. 3 illustrates the difference between the GEM10 and SE 1977 solution where both are expanded to include terms up to degree and order 5. A significant part of this comes from the 3,3 coefficients. That the largest discrepancies occur over oceanic areas could reflect a problem associated with the surface data since gravity data in most of these regions are

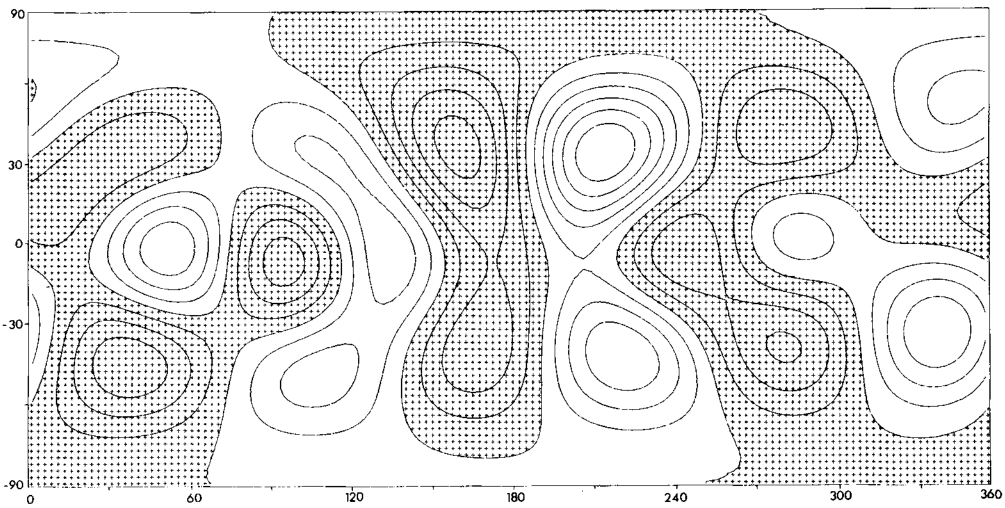

Figure 3. Difference in the geoid solutions GEM10 and SE 1977 with both solutions truncated so as to include all harmonics with l, $m \leq 5$. Contour interval is 1 m, shaded areas are negative.

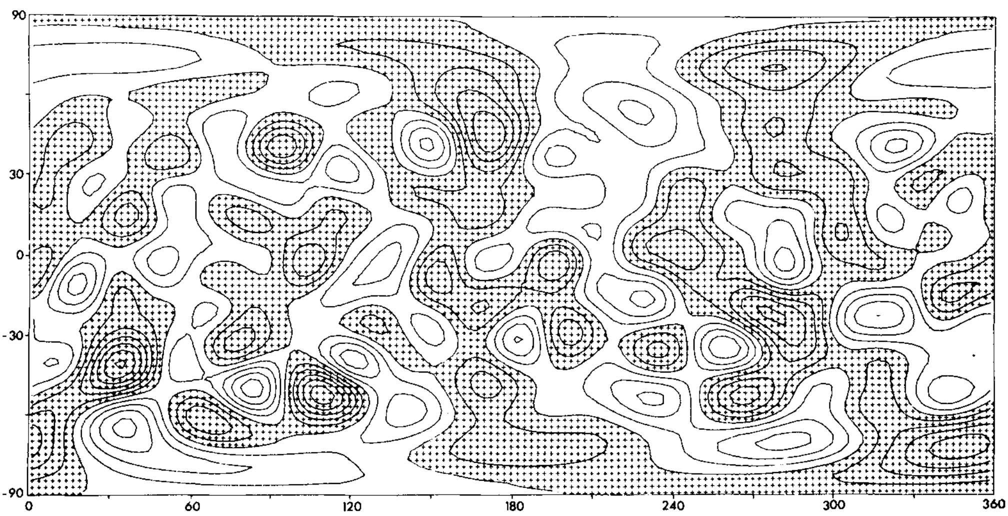

Figure 4. Same as Fig. 3 but with l, $m \leq 16$. Contour interval is 3 m.

sparse. However, it could be anticipated that these data should not contribute significantly to the low degree terms. The dipole-like discrepancy over the Indian Ocean reflects a difference in the location of the negative anomaly south of India at 70° east longitude.

Of the solutions considered, none appear to be very reliable at high degrees (see Fig. 2). An appropriate point at which to truncate the expansions is perhaps at about degree 16 or 18 where the difference power spectrum, defined by (8a), begins to exceed the signal. Neither the satellite nor surface data contain useful information beyond this degree. Fig. 4 illustrates the difference in geoid heights for the SE 1977 and GEM10 models, with both solutions truncated at degree 16. Some of the discrepancies now exceed 18 m, with many of the larger differences occurring over the unsurveyed southern oceans, Asia, Africa and Antarctica.

If a conclusion is to be drawn at all from these solutions it is that the Gaposchkin solutions do not meet the requirements of a Standard Earth model, that the Lerch *et al.* models may not be as good as their names imply and that the Balmino *et al.* solution is not as severe as implied by its acronym.

ALTIMETER SOLUTIONS

Solutions which incorporate *GEOS 3* altimetry observations in addition to the above data sets have been published by Gaposchkin (1980) (SE 1980), by Lerch *et al.* (1981) and by Reigber *et al.* (1982) (GRIM3). Gaposchkin computed 1° x 1° area altimeter mean geoid heights, using the satellite ephemeris provided with the altimeter data. 550 x 550 km area-means were then computed and for each value an observation equation was established which equated the mean height of the area with a geoid expansion in terms of the Stokes coefficients. Uncertainties in the mean heights arise from ephemeris errors, altimeter measurement errors, 'oceanographic noise' and inadequate sampling. The observation equations were combined with the solution reported by Gaposchkin & Mendes (1977) based on laser range data to 10 satellites and surface gravity. The inversion was for all non-zonal harmonics to degree and order 30. It represents a departure from earlier practice in that it does not appear to be a true iteration since the *GEOS 3* orbits to which the altimeter data have been referenced, have not been adjusted.

Power in the SE 1980 model is reduced from that in the earlier iteration SE 1977 for most of the degrees common to the two. At low degrees, the difference is substantial (compare Tables 2 and 4) but because of the intervening *LAGEOS* solution it is not possible to draw conclusions about the influence of the altimetry data on the low degree harmonics. A comparison of SE 1980 and GEML2 (Lerch *et al.* 1982), both truncated at degree 5, is

Table 4. Power estimates of the potential in two recent solutions containing *LAGEOS* observations. The error estimates of the power $E(\epsilon_1^2)$ $E(\epsilon_2^2)$ refer to the SE 1980 and GEML2 solutions respectively. All power estimates are to be multiplied by $(GMR^{-1}\,10^{-6})^2$. The rms geoid height errors are in metres.

l	$V_l^2(U_1)$ SE 1980	$V_l^2(U_2)$ GEML2	$\langle U_0^2 \rangle$	$E(\epsilon_1^2)$	$E(\epsilon_2^2)$
2	7.791	7.899	7.845	−0.054	0.054
3	8.093	7.893	7.988	0.105	−0.095
4	2.269	2.226	2.246	0.023	−0.020
5	1.207	1.349	1.263	−0.056	0.086
Σ	19.360	19.367	19.342	0.018	0.025
$\Sigma\vert\ \vert$				0.238	0.255
ϵ_N(m)				0.86	1.01
$\epsilon_N\vert_{max}$(m)				3.12	3.23

summarized in Table 4. Both solutions contain *LAGEOS* orbits. Power in the harmonics 2–5 is very comparable. Agreement between them is better than SE 1977 and GEM10. Nevertheless, the maximum geoid height errors are still important, about 3 m for both solutions, but rms estimates according to (11a) are better, about 1 m for GEML2 and 0.9 for SE 1980 for the (5,5) truncated geoids. Because of the small sample sizes these low degree error estimates remain unreliable. A major part of this discrepancy again comes from the (3,3) coefficients.

Reigber *et al.* have used a similar approach to Gaposchkin except that they do not use the altimeter data directly; instead, they introduce area-mean gravity anomalies deduced from the altimeter and surface gravity observations. The altimeter data entering into this calculation follow from a set of mean geoid heights that are similar to those used by Gaposchkin. The solution may suffer, therefore, from the same limitation of introducing error from the original *GEOS 3* reference orbits. The rationale for first computing gravity anomalies is not evident and may introduce a number of unnecessary uncertainties. First, this process decreases the signal-to-noise level at high frequencies since the calculation is essentially one of differentiation of the geoid heights. Secondly, and more important, the computation of gravity anomalies over the oceans from altimetry data, requires a knowledge of gravity on land. Unsatisfactory gravity data, as they are for many areas, therefore contaminate the results and unpredictable errors may occur, particularly in the vicinity of continental margins. Finally, the surface gravity data enter into the solution a second time when the altimeter derived gravity anomalies over the oceans are combined with the surface gravity measurements over both land and sea. The GRIM3 solution is complete to degree and order 36. Comparisons of this model with the other two indicates that its precision at low degrees is comparable to that of SE1980 and that the GEM10B and GEML2 models may be somewhat superior to the other two. Some problems apparently occur in this solution from the introduction of Doppler data (B. Moynot 1982, private communication) and until this is clarified we do not consider the solution further. A new iteration, GRIM4, is in preparation.

Lerch *et al.* (1981) present two altimeter solutions, GEM10B and 10C. In 10B a subset of altimeter data has been selected for each observation for which an equation has been established that relates the observation to: (1) the sea surface geometry and, (2) the Stokes coefficients required to describe the satellite motion. That is, the observation equation is of the form

$$h_{\mathrm{obs}} \simeq r(K_j, \beta_n, C_{il'm'}) - R(C_{ilm}) \tag{12}$$

where the first function defines the position of the spacecraft in terms of orbital elements K_j, surface force parameters β_n, and those Stokes functions required to define the satellite motion with an accuracy that is commensurate with that of the altimeter height observation h_{obs}. The second function describes the ocean surface to a degree and order that is commensurate with the accuracy of the altimeter data. In general one would expect that $l > l'$, $m > m'$. The introduction of the above observation equation into the GEM solutions appears to have been carried out in two steps although the documentation is not clear on this point. In the first instance the altimeter data and *GEOS 3* laser tracking observations have been used to establish improved orbital elements K_j using a fixed gravity field, apparently GEM9. In the second step, equations of the form (12) are introduced – together with all the other observation equations – and treating the orbital elements as known quantities. This process leaves the distinct impression, but we stand to be corrected, that the final solution will be at least partly constrained by the GEM9 results. In particular, it suggests that the 10B solution retains the influence of the condition (9), even though it has now not been explicitly introduced. The solution is complete to degree and order 36. In a second solution, GEM10C, Lerch *et al.* introduce a global set of $1^\circ \times 1^\circ$ altimeter observed geoid heights, apparently using the orbital ephemeris provided with the altimeter data. They also replaced the 5° x 5° gravity anomaly data set by a 1° x 1° set. In the resulting inversion the GEM10B coefficients were held fixed and harmonics of degree 36–180 were estimated. This assumes that the GEM10B harmonics are error-free, an unlikely situation. It also assumes that gravity is sufficiently well known over land to compute 1° x 1° area means. But if their data set includes value for all 1° squares, this is largely a result of prediction. We do not consider this solution further.

Comparisons of the GEM10 and 10B models, using the above formalism, clearly violates the fundamental assumptions of independence. Nevertheless, some comments can be usefully made. One is that the power in 10B exceeds that in GEM10 for $l \gtrsim 12$. This points to the decreased influence of the constraint (9) imposed in GEM10. Considerable differences in power occur at lower degrees between the two GEM models, a difference to which the troublesome third-degree terms contribute a substantial part. This may point to either numerical instabilities in the inversion or to 'oceanographic noise' in the latter solution. We cannot comment on the first except that in the GRIM2 model, using comparable laser tracking data, Balmino *et al.* found it necessary to constrain the low degrees by the condition (9). We can, however, test for the second interpretation. We write for the geoid heights N (equivalent to the dimensionless form of the potential as used in equation 6)

$$N_{10\mathrm{B}} = N_{10} + \delta N_{\mathrm{oceanic}} \tag{13a}$$

where N_{10} and $N_{10\mathrm{B}}$ refer to geoid heights based on the two GEM models and where $\delta N_{\mathrm{oceanic}}$ is the sea surface topography arising from the departure of the mean sea surface from the geoid. Then

$$R^{-2}\langle N_{10\mathrm{B}}^2\rangle \equiv \langle U_{10\mathrm{B}}^2\rangle \geqslant \langle U_{10}^2\rangle \tag{13b}$$

since there should be no correlation between the two quantities on the right-hand side of (13a). This is indeed observed for the harmonics from 3 to 7 and suggests that the 10B

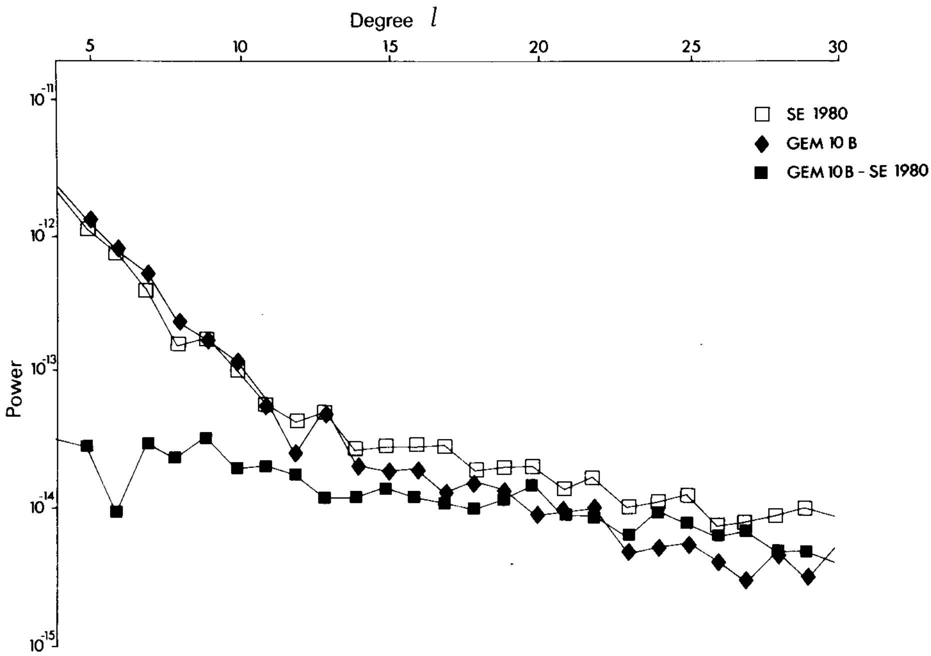

Figure 5. Power spectrum of the gravitational potential according to the solutions GEM10B and SE 1980 both of which contain some *GEOS 3* altimeter observations. The difference spectrum is also indicated.

Table 5. Comparison of total power of the gravitational potential in the two solutions including altimetry data for non-zonal harmonics with $2 < l \leqslant 30$. These estimates and error spectra are to be multiplied by $(GMR^{-1}\,10^{-6})^2$. The rms geoid height errors are in metres.

	SE 1980		GEM10B
$\sum_l V_l^2(U)$	21.399		21.544
$\sum_l \langle U_0^2 \rangle$		21.290	
$\sum_l E(\epsilon^2)$	0.109		0.254
ϵ_N(m)	2.11		3.22

spectrum contains some oceanographic information. However, for $l = 3$, using the coefficients in Table 3, $\delta N_{\text{oceanic}} \simeq 1.4$ m. This is too large to be attributed to sea surface topography (Mather *et al.* 1978). Similar conclusions are reached for other low ($l \leqslant 10$) degree harmonics. For others $\langle U_{10B}^2 \rangle < \langle U_{10}^2 \rangle$ and this also points to noise in the solutions. A comparison of the SE 1977 and SE 1980 models yields the same conclusions.

The statistical comparison of the two 'independent' solutions, SE 1980 and GEM10B, is illustrated in Fig. 5. Total power in the common wavenumber range $2 < l \leqslant 30$, excluding zonal harmonics, is comparable (Table 5). This is despite the observation that for $l \gtrsim 12$ the power in the Gaposchkin solution exceeds that in the GEM10B model. The error spectrum, estimated according to (8c) leads to comparable total power for the two solutions, corresponding to rms geoid height errors of about 2 m. As before, much of this comes from the lower degree terms particularly the odd degree harmonics. The estimate $\langle U_0^2 \rangle$ from these two solutions is consistently greater for low degrees than $\langle \tilde{U}_0^2 \rangle$ deduced from the pre-altimeter solutions. This can only be partly attributed to the influence of the constraint (9) introduced into the GRIM2 model. The difference in power for $2 \leqslant l \leqslant 8$ between these two estimates of the 'true' power, corresponds to an rms geoid height difference of 1.5 m. This is greater than can be readily attributed to oceanographic noise.

Fig. 6 illustrates the difference in geoid height between the GEM10B and SE 1980 solutions, with both truncated at 5,5. The general pattern is the same as illustrated in Fig. 3

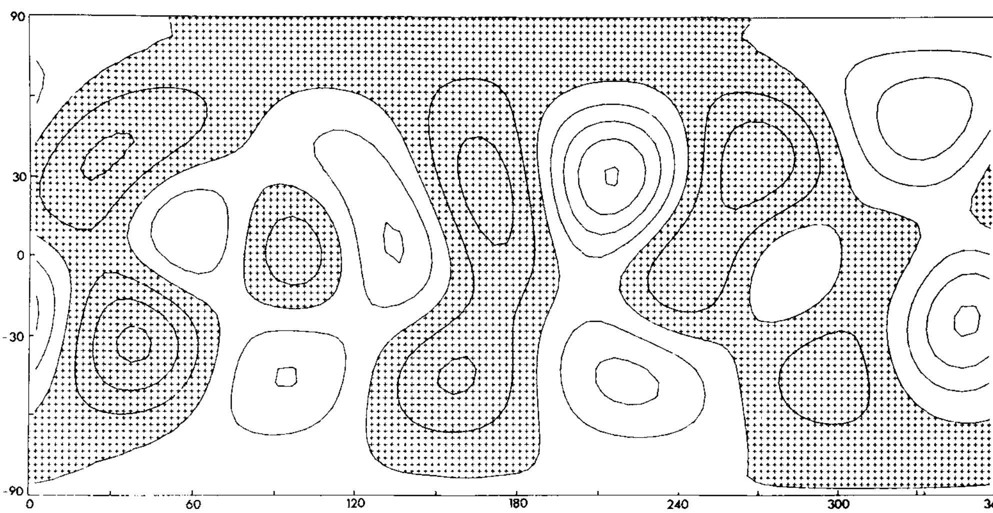

Figure 6. Difference in the geoid solutions GEM10B and SE 1980 with both solutions truncated to include harmonics with l, $m \leqslant 5$. Contour interval is 1 m, shaded areas are negative.

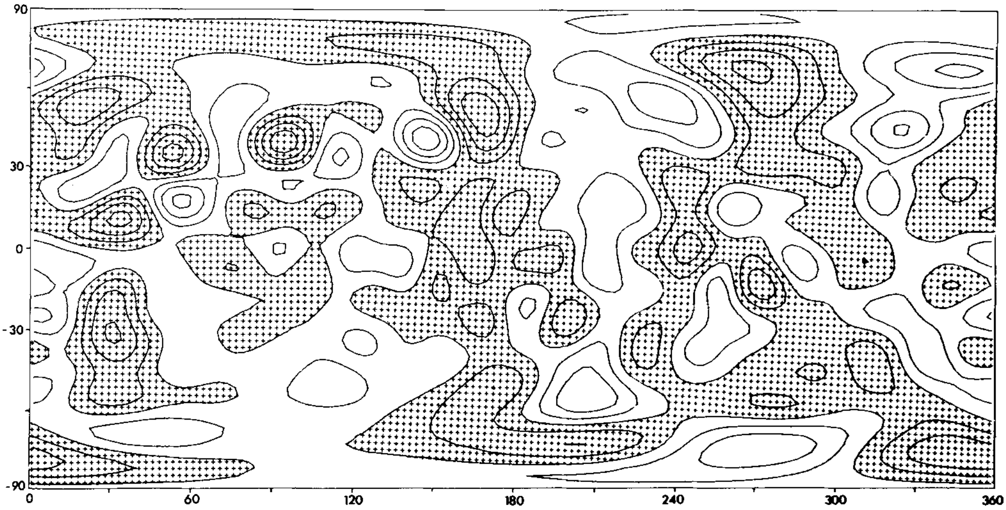

Figure 7. Same as Fig. 6 but with l, $m \leqslant 16$. Contour interval is 3 m.

except that the amplitudes of the major discrepancies are reduced (from an rms of 5 to 3 m). That is, either the influence of the altimeter observations is already seen in these long wavelengths or the difference is a result of the *LAGEOS* data introduced into the Smithsonian solution.

At higher degrees, the GEM10B spectrum exhibits a substantial increase in power at $l = 13$ compared with that at $l = 12$ and for $l > 13$. A similar tendency is seen in the GEM10 solution. In the latter case this could be attributed to the influence of the constraint (9) which was applied for degrees $l > 12$. Fig. 7 illustrates the differences for the two fields (GEM10B and SE 1980) truncated at degree 16. This can be compared directly with Fig. 4 illustrating the same result for the pre-altimeter models. The introduction of the altimeter data has removed many of the earlier discrepancies over the southern oceans, but the large discrepancies over Asia and Africa remain. Not insignificant differences also remain over the oceans: for example in the central Pacific. There is also a tendency for the discrepancies to increase above latitudes of about 60°, where there is no *GEOS 3* altimetry coverage. These results point to differences in relative weighting of surface gravity, altimetry and orbit perturbation data. This is a well-known, but still largely unresolved, problem in gravity field modelling (e.g. Gaposchkin & Lambeck 1971). The usual approach is one of trial and error. Gaposchkin (1980) discusses this in some detail and has experimented with various combinations of weights. Lerch *et al.* (1981) are less clear in their discussion of the problem.

RESONANCES

In some situations a satellite's motion can resonate with certain harmonics in the Earth's gravity field. This occurs when the mean motion (n) of the satellite is an integer fraction of the Earth's rate of rotation ($\dot{\theta}$), or when

$$\dot{\gamma} \equiv (l - 2p + q)n - m\dot{\theta} \approx 0. \tag{14}$$

Here $p = 0, 1, \ldots l$ and $q = 0, \pm 1, \pm 2, \ldots$ are integers arising from the expansion of the potential into Keplerian coordinates (Kaula 1966a). Mathematically, the resonance arises because $\dot{\gamma}$ enters into the divisor of the perturbation equations describing the satellite's departures from Keplerian motion. Physically, the resonance arises when successive ground tracks of the satellite are separated by an interval in longitude that is equal to the wavelength of the geopotential harmonic. After a number of revolutions of the satellite in its orbit, the ground track sequence repeats itself exactly, and the satellite's motion is perturbed in an identical manner, enhancing the earlier perturbation.

The existence of these resonance conditions, particularly for geostationary orbits, has long been recognized (e.g. Sehnal 1960; Groves 1960; Cook 1960). For geostationary satellites the resonance occurs primarily when $lmpq$ = 2200 or 210(−1). But resonances also occur for the coefficients $lmpq$ = 4210, 421(−1), 6220, 622(−1) etc. The satellite may therefore resonate to many harmonics of the same order m. Because the magnitude of the perturbation is proportional to the factor (5), and for geostationary orbits $R/r \simeq 6.6$, these higher degree resonances will generally be unimportant. For a typical satellite orbiting the planet nearly 13 times a day, the resonance occurs from $lmpq$ = 13,13,6,0; 15,13,7,0; 17,13,8,0 etc., with the frequency of the perturbation being given approximately by (14). Observations of the resonances then provide a linear relation between the Stokes coefficients of degree and order 13,13; 15,13; 17,13 etc. with coefficients that are orbit dependent (e.g. Yionoulis 1966). The contributions from the higher degrees (15,17 etc.) decay, but slowly. Secondary resonances occur for $lmpq$ = 14,13,6,−1; 16,13,7,−1, etc. but these will only be significant for eccentric orbits. By selecting satellite orbits for which the condition (14) is nearly met, many of the higher degree and order harmonics in the gravitational potential can be determined, provided that there are sufficient satellites in different orbits to separate all coefficients contributing to the resonances. It is these resonance harmonics that permit the coefficients near m = 13,14 in Fig. 1 to be determined. The resonances discussed are for the combination of indices such that $l-2p+q=1$ and $n=m\dot{\theta}$. In general, resonances occur when $l-2p+q=j$ or when $n=jm\dot{\theta}$ where j is an integer. Of these only the $j=1$ (the above case) and $j=2$ resonances have been observed (e.g. Anderle & Smith 1968). Thus for a satellite making 14 revolutions day^{-1} the resonant harmonics will be of order 28 ($j=3$ resonances were reported by G. Balmino and C. Reigber in 1976, and may be significant in the orbital motion of the satellite *Starlette*).

No general theory yet exists for handling the resonances of close Earth satellites although theories have been developed for resonances due to a single potential coefficient or subset of coefficients (e.g. Morando 1962; Allan 1973). Complete theories are complicated by the fact that the orbit may be resonant with many harmonics of different orders, as well as by the need to take into account interactions with other perturbations mainly due to the zonal harmonics and air drag. When $\dot{\gamma}$ (equation 14) becomes small but not zero, linear theories remain largely adequate and, for these reasons, near-resonant cases have been extensively used in studies of the Earth's gravity field (e.g. Gaposchkin 1973; Reigber & Balmino 1975; Wagner 1974).

The higher the order m the greater must be the mean motion or the lower the orbital altitude for resonance to occur. For the low altitude orbits the number of Stokes coefficients that contribute to the resonances becomes large and a considerable number of satellites in different orbits of similar resonance characteristics need to be analysed (e.g. King-Hele & Walker 1982a). But low altitude satellites suffer appreciably from drag forces and for this reason such orbits are usually ignored in gravity field analysis. However, the drag also draws the satellite through the resonance configuration, maximising this gravitational perturbation. This has been used to great effect by King-Hele and co-authors to determine 14th- and 15th-order harmonics (King-Hele, Walker & Gooding 1979; King-Hele & Walker 1982a; see also Wagner & Klosko 1975; Klokočník 1975). Preliminary analysis of 16th-order harmonics have also been carried out (Walker 1982).

The importance of the resonance harmonics is that they permit estimates of otherwise inaccessible harmonics of high degree and order. They therefore provide a measure of the physical behaviour of the geopotential at high wavenumbers, something that is useful in statistical interpretations of the gravity field. In the present context the resonance harmonics are also of considerable value in that they provide an independent test of those gravitational potential solutions that are primarily based on non-resonant, orbit analyses, surface gravity and altimetry data.

In the solutions by Gaposchkin, any resonance conditions are treated simultaneously with the shorter period perturbations. This is a convenient way to proceed since Gaposchkin's analytical approach to describing the orbital motions enable him to include the longer period perturbations in his analyses. Problems of coupling between the resonance and zonal coefficients may, however, restrict the potential accuracy of the analytical solutions. In the GRIM models the observation equations include any terms arising from near resonance. These models also include the observation equations determined by King-Hele, Reigber & Balmino and Klokočník & Kostelecky. This strengthens considerably the solution but these resonance results cannot now be used for independent testing purposes. Lerch *et al.* do not state explicitly how the resonances are treated in their solutions. Many of the satellite orbits used in their work are only in shallow resonance (where $\dot{\gamma}$ in equation 14 is not particularly small) and the resonance periods are of the order of a few days, comparable in duration to the computed 5 or 7 day orbital arcs. Hence these cases can be treated similarly to the non-resonant terms by assuring that the observation equations include the appropriate additional terms. For other satellites the resonance period is considerably greater than the orbital arclengths and it is not clear how these are treated in the recent Goddard solutions. Most of the resonances included in the Smithsonian and Goddard solutions are for orders 9–14.

A solution for 14th-order harmonics is by King-Hele *et al.* (1979) who estimated the coefficients l, m from 14,14 to 22,14. These harmonics can be compared with the global solutions in the same statistical manner as before. In this case U_1 and U_2 (equation 7) refer to the two estimates of the potential defined by the above 14th-order resonance harmonics according to: (1) the resonance study (U_1), and (2) the global solution (U_2). Table 6 summarizes the results for the comparison of the King-Hele *et al.* solution with GEM10B. The estimate of the power in the error spectrum of U_1 is significantly less than that corresponding to U_2, as would be anticipated in view of the special nature of the resonance phenomenon and of its analysis. Signal-to-noise ratios are high for both solutions, reflecting the good agreement between the two. On the assumption that the power in the error spectra is independent of degree throughout the range in question, the total power can be estimated and compared with that obtained from the intercomparisons of the global solutions. The agreement between the two error estimates so obtained for GEM10B is most satisfactory (Table 7).

A similar comparison can be carried out for the 15th-order harmonics which have been analysed most thoroughly by King-Hele & Walker (1982a). Their preferred solution is for degrees 15–23. The results of the comparison with GEM10B are given in Tables 6 and 7 and

Table 6. Comparison of power in the spectra of the resonance coefficients according to King-Hele and co-authors $\langle U_1^2 \rangle$ and to the GEM10B solution $\langle U_2^2 \rangle$. n is the number of coefficients determined in the resonance analyses. All power estimates are to be multiplied by $(GMR^{-1})^2\,10^{-15}$.

Order	n	$\langle U_1^2 \rangle$	$\langle U_2^2 \rangle$	$\langle U_0^2 \rangle$	$E\{\epsilon_1^2\}$	$E\{\epsilon_2^2\}$
14	18	6.14	6.67	5.63	0.51	1.04
15	20	6.31	6.56	6.03	0.29	0.53

Table 7. Comparison of total power in the error spectra from: (1) degrees 14–22 as determined from the comparisons of the 14th-order resonance analyses of King-Hele $E\{\epsilon_1^2\}$, and GEM10B $E\{\epsilon_2^2\}$ and the comparable estimate of the latter $E\{\epsilon_3^2\}$ as determined from the comparison of 10B with SE 1980; (2) degrees 15–24, 15th-order resonances; (3) degrees 14–24, 14th- and 15th-order resonances. All power estimates are to be multiplied by $(GMR^{-1})^2\,10^{-15}$.

Order		$E\{\epsilon_1^2\}$	$E\{\epsilon_2^2\}$	$E\{\epsilon_3^2\}$
1	14	9.2	18.7	21.0
2	15	5.6	10.3	17.7
3	14 + 15	8.8	17.2	23.1

they lead to similar conclusions to those reached for the 14th-order resonances. In particular, the estimate of the GEM10B power spectrum based on the intercomparison with the Gaposchkin (1980) solution appears to be realistic. Similar tests carried out between these resonance studies and the SE 1980 model confirms King-Hele's conclusion, that these higher degree and order terms are less well determined than the corresponding coefficients in GEM10B. Nevertheless, the resulting SE 1980 error estimates agree with those derived from the comparison of this model with 10B.

Resonances with 29th-, 30th- and 31st-order harmonics have been analysed by King-Hele & Walker (1982b) but individual coefficients were only obtained for the 30th order and degrees 30, 32, 34 and 36. These compare favourably with the GEM10B results.

Discussion

From the above comparisons it is not obvious as to which solution is the most accurate or reliable and at what point the available geoid or gravity expansions become dominated by noise, or at what degree they should be truncated before geophysical interpretations are attempted. What is more obvious is that not insignificant discrepancies may occur in all solutions. This alone should lead to caution in interpreting any one result, particularly when it is noted that many of the discrepancies occur over geologically interesting areas.

Perhaps the single most important point to emerge is that the solutions do not model the geoid with the accuracies claimed by some of the authors. In terms of signal-to-noise ratios, the low degree harmonics are well established. Yet geoid height differences between the solutions GEM10B and SE 1980 are in excess of 3 m when both fields are truncated at a lowly degree and order of 5 (see Fig. 6). The GEML2 model of Lerch *et al.* (1983) gives an improved result for the low degree harmonics when it is compared with SE 1980 (Table 4). Yet the discrepancies are such that we have little confidence in the statement by these authors that this model represents the geoid with 8 cm precision in the wavenumber range $2 \lesssim l \lesssim 4$. Their statement of precision is based mainly on a comparison with other GEM models. But good agreement with 10B is not a measure of accuracy. Neither is the criterion, that GEML2 predicts the motion of a geostationary satellite, entirely adequate since such a satellite is relatively insensitive to harmonics other than 2,2. The inadequacy of the low degree harmonics in describing the satellite motion has also been emphasized by Klokocnik & Pospisilova (1981).

These cautionary conclusions reached about these geoid height accuracies means that the present solutions may not be adequate for exploring the long wavelength and low frequency characteristics of the sea surface topography (the difference between the geoid and the physical ocean surface). This is confirmed by the work of Marsh & Martin (1982) who found it necessary to further modify the GEM models in order to obtain what they consider to be

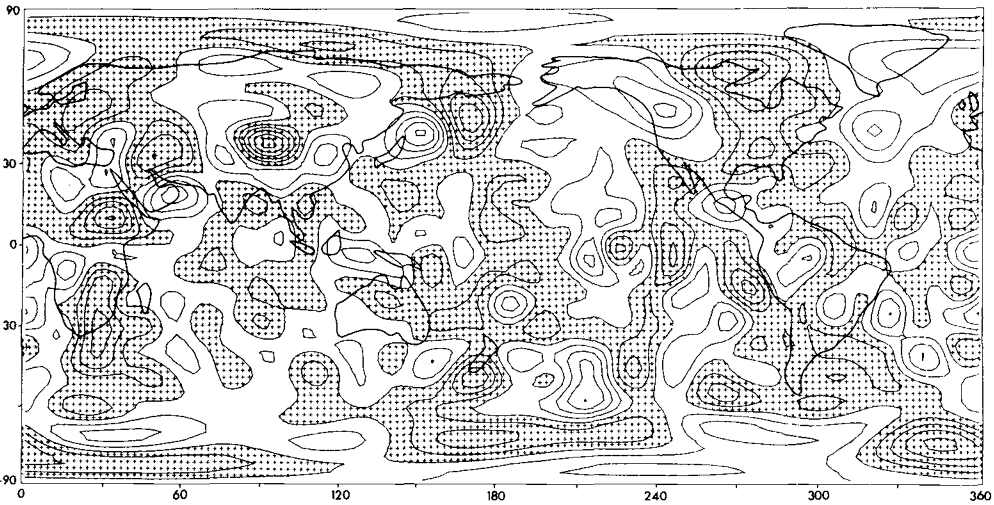

Figure 8. Same as Fig. 7 but with l, $m \leqslant 22$.

satisfactory sea surface topography. The rms geoid height difference between this adjusted model and GEM10B is 20 cm for $2 \leqslant l \leqslant 5$ and 55 cm for $2 \leqslant l \leqslant 10$. The statistical comparison of non-zonal solutions with and without altimeter data suggests that the power of the sea surface topography is of the order 1.5–2 m compared with an 'expected' power of the order of 20 cm. Possibly a more realistic estimate of the precision of the GEM10B and GEML2 geoids is about 50 cm for $l \leqslant 5$. What is required to improve the low degree field, so as to produce geoids of sub-decimetre accuracy, is one or a number of *LAGEOS* type satellites in different inclination orbits, satellites that can be tracked with great precision and which are predominantly sensitive to the lower degree harmonics.

At higher degrees, the signal-to-noise ratio of the coefficients increases, yet the power in the potential decreases. Hence the rms error of the geoid heights for the total field is dominated by the uncertainties in the low degree terms. This appears to be the reason for the claims by Lerch *et al.* that their GEM10 and GEM10B models have an accuracy of about 1.5 m. The tests they employ are generally insensitive to the low degree harmonics: surface gravity, short-arc altimeter passes and satellite-to-satellite tracking data used to evaluate these solutions do not provide good control on the low degree terms. Nor do the resonance tests performed by Lerch *et al.* We conclude, from the statistical comparison with the SE 1980 solution and the resonance solutions of King-Hele *et al.*, that a more appropriate rms geoid height error is about 2 m for the GEM10B model, and about 3 m for the SE 1980 model.

In both models, noise estimates approach the signal at degree $l \geqslant 22$. Individual discrepancies between the two solutions exceed 10 m in several instances for expansions truncated at this degree (Fig. 8). When truncated at degree 16 the maximum difference, in excess of 12 m, occurs over the Tibetan plateau. This difference – as is the one north-east of Japan – is mainly a consequence of a displacement of the geoid height anomaly in one solution relative to the other; with both solutions modelling in a quite similar manner the series of geoid undulations over Tibet and Sinkiang but the locations of the highs and lows differing by nearly 500 km. In the solution truncated at degree 22 (Fig. 8), the majority of the larger discrepancies occur over: (1) continents for which surface gravity data are poor, i.e. Africa, Antarctica and Asia, (2) at latitudes above about $\pm 60°$, in the regions unsampled by the *GEOS 3* altimeter data, and (3) in oceanic regions where the *GEOS 3* data used in the two solutions were relatively sparse, mainly the central and southern Pacific and the north-west

Figure 9. Geoid heights according to the Gaposchkin SE 1980 solution truncated to include harmonics with l, $m \leqslant 22$. Contour interval is 3 m. Shaded areas are negative.

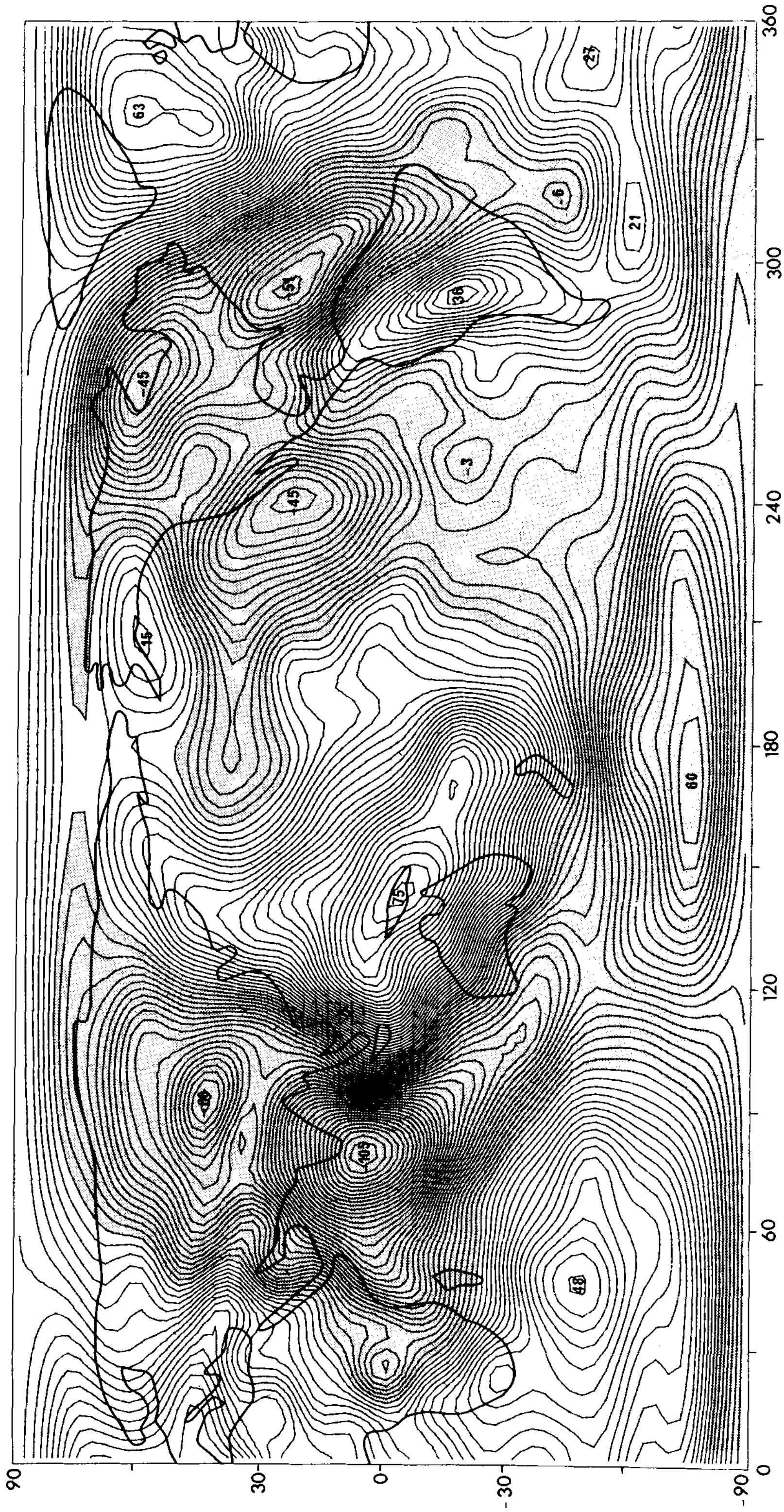

Figure 10. Same as Fig. 9 but according to the GEM10B solution of Lerch *et al.*

Pacific (see fig. 2 of Gaposchkin 1980, and fig. 1 of Lerch *et al.* 1981). (Note that neither solution used the complete *GEOS* data set now available.)

Figs 9 and 10 illustrate the two geoid solutions truncated at degree and order 22. These two figures should be viewed together with their difference (Fig. 8), particularly in geophysical interpretations. King-Hele (1975) has described the geoid as '... the western hemisphere has been taken over by a goat which is in deep discussion with a man from the east whose cranium dominates Asia'. With the longitude convention adopted here the two individuals are no longer on speaking terms and have turned their backs on each other. Elsewhere King-Hele referred to the high-brow as Popeye!

What should be done to improve the state of the present knowledge of the Earth's gravity field? A shopping list for future projects is easy to provide and would include additional *LAGEOS*-type satellites for the low degree field and low altitude satellite-to-satellite tracking experiments for the high degree field. The fulfilment of the order is of course a different matter. But even without these additional satellites, considerable progress should be possible from a re-examination of the available data. In doing this, some general points to be considered would include the following.

(1) The introduction of additional laser tracking data of *LAGEOS* and *STARLETTE,* the low altitude equivalent of the former. Most of these data are already available. The former will help strengthen the solution at low degrees while the latter will also help in strengthening the solution for intermediate degrees. The addition of further Doppler data may also be beneficial for improving the low degree part of the solution, for while these data are less precise than the laser observations they are usually more continuous.

(2) A careful scrutiny of the surface gravity data and the elimination of the unreliable data for the predominantly unsurveyed areas.

(3) The incorporation of satellite-to-satellite tracking data, mainly from the *GEOS 3/ ATS 6* experiment. Analyses of these data shows that they contain considerable gravitational information. More significant, such tracking data are available over the continents of South America, and southern Asia (Kahn, Klosko & Wells 1981) where surface gravity data are inadequate.

(4) The incorporation of a more complete set of *GEOS 3* altimeter observations and the introduction of *SEASAT* data. Improved methods of incorporating these data should be developed.

(5) The incorporation of more recent observation equations for zonal (e.g. King-Hele, Brookes & Cook 1981) and resonance (e.g. King-Hele *et al.* 1979 and King-Hele & Walker 1982a) harmonics as constraints in the general inversions.

Possibly it is out of place to suggest specific improvements that could be made to individual solutions, particularly when the available documentation may not be sufficiently complete to form a basis for a considered critique. Nevertheless some points should be made about the GEM models. One is that the successive solutions do not appear to represent true iterations. Therefore it remains unclear to what extent any one solution is influenced by earlier ones. We would like to see a solution in which all orbits have been re-computed using the best available force models, consistent sets of parameters and the most up-to-date versions of the computer programs. Secondly, we would like to see a more careful consideration of what data should be included. We do not think, for example, that the addition of Minitrack data contributes to the overall solution in short-arc analyses. Thirdly, we would like to see a more complete discussion of the relative weighting considerations that must have gone into the inversions. In making these comments we do not wish to degrade the work done during the last twenty-five years in reaching the present knowledge of the gravity field. What we wish to emphasize is that even better solutions can be achieved with the data available at present. Examples of this are seen when we turn to the altimetry data alone. Figs 11 and 12 illustrates two examples of regional geoids computed from *GEOS 3* and *SEASAT* altimetry data. The south-west Pacific geoid (Fig. 11) is discussed in detail elsewhere. The geoid accuracy is of the order of ±50 cm for the shorter wavelengths

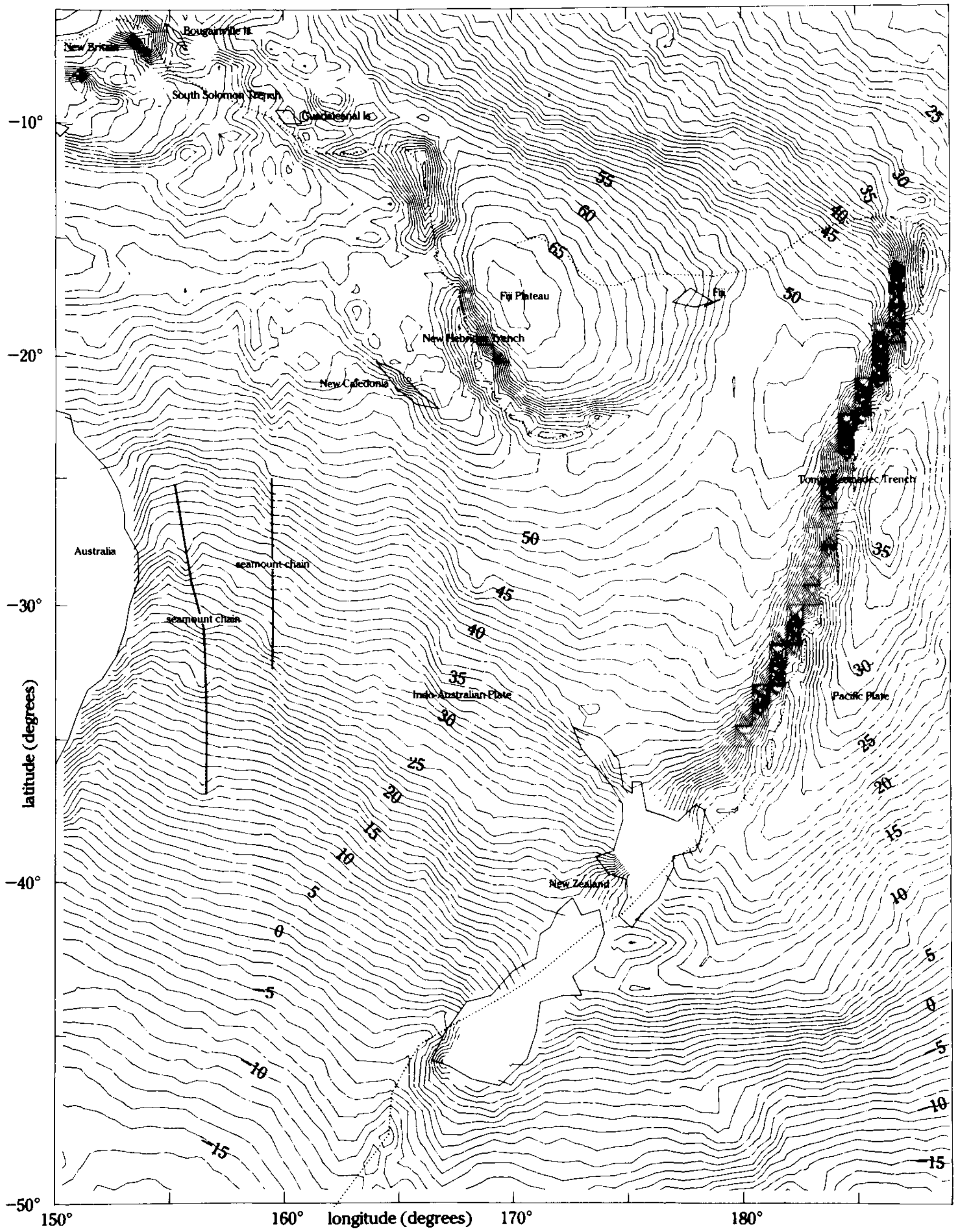

Figure 11. Regional geoid for the south-west Pacific determined from all available *GEOS 3* and *SEASAT* data. Contour interval is 1 m.

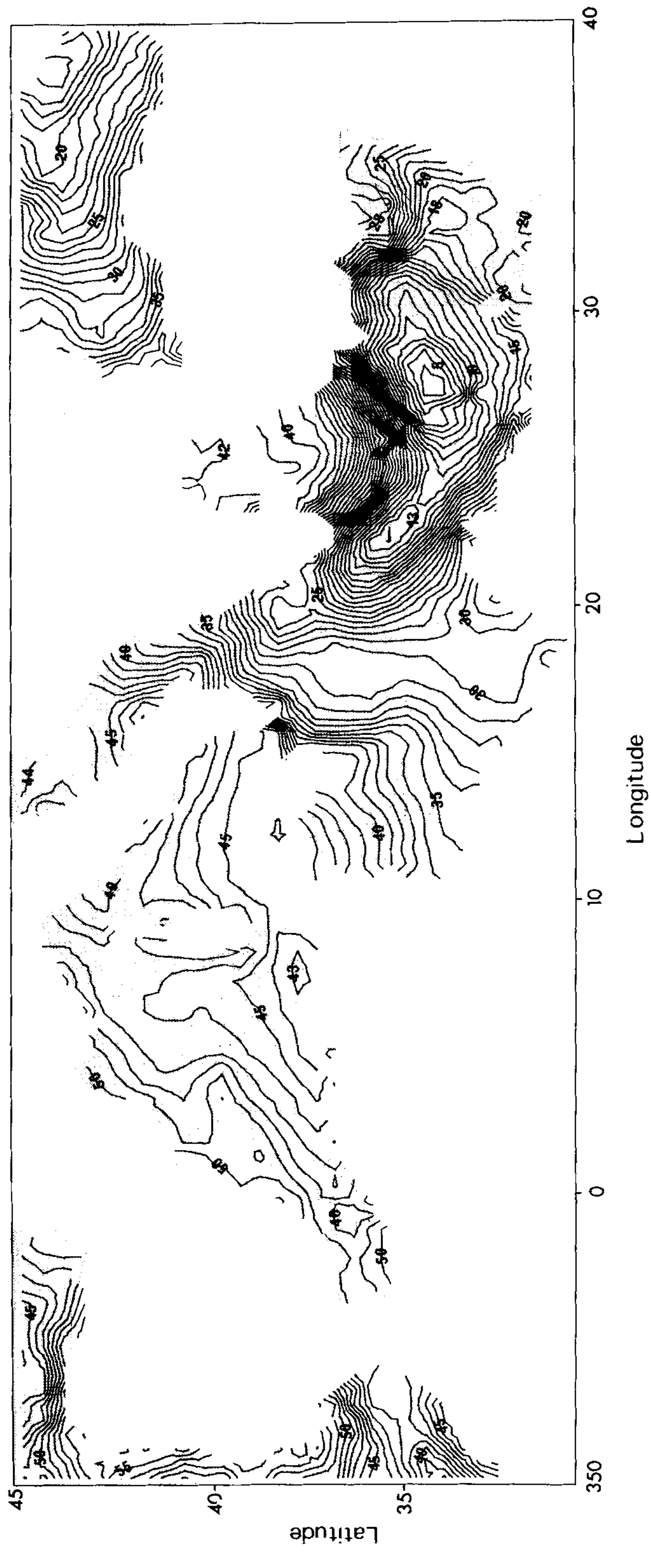

Figure 12. Same as Fig. 11 but for the Mediterranean.

(300 km $\leqslant \lambda \leqslant$ 2000 km) although, if we consider the results of Fig. 8, there may be some uncertainty in the magnitude of the overall south to north geoid slope. For the Mediterranean (Fig. 12) there may be a comparable long wavelength uncertainty but the shorter wavelength information should also be accurate to about ±50 cm. There are no other geophysical observations, not even bathymetry (Lambeck & Coleman 1982a, b), that are better known than this. These regional geoids are proving to be of very considerable value in studies of specific geophysical problems, whether they be ocean ridges (Sandwell & Schubert 1980), transform faults (Cazenave, Lago & Dominh 1982), seamounts (Lambeck 1981) or convection in general (McKenzie *et al.* 1980).

References

Allan, R. R., 1973. Satellite resonance with longitude-dependent gravity – III. Inclination changes for close satellites, *Planet. Space Sci.*, **21**, 205–225.

Anderle, R. J. & Smith, S. J., 1968. Observation of twenty-seventh and twenty-eighth order gravity coefficients based on Doppler observations, *J. Astronaut. Sci.*, **15**, 1–4.

Balmino, G., 1975. Numerical methods of orbital dynamics, in *Satellite Dynamics*, pp. 50–97, ed. Giacaglia, G. E. O., Springer-Verlag, Berlin.

Balmino, G., Reigber, C. & Moynot, B., 1976. A geopotential model determined from recent satellite observing campaigns (GRIM 1), *Manuscripta Geodaetica*, **1**(1), 41–69.

Balmino, G., Reigber, C. & Moynot, B., 1978. Le modele de potential gravitationnel terrestre, GRIM 2. Determination, evaluation, *Annales Geophysique*, **34**(2), 55–78.

Buchar, E., 1958. Motion of the nodal line of the second Russian earth satellite, and flattening of the Earth, *Nature*, **182**, 198–199.

Cazenave, A., Forestier, F., Nouel, F. & Peiplu, J. L., 1972. Improvement of zonal harmonics by the use of observations of low inclination satellites dial, SAS and people, in *The Use of Artificial Satellites for Geodesy*, 145–150, eds Henriken, S. W., Mancini, A. & Chovitz, B. H., *Geophys Monogr.*, **15**, American Geophysical Union, Washington, DC.

Cazenave, A., Lago, B. & Dominh, K., 1982. Geoid anomalies over the north-east Pacific fracture zones from satellite altimeter data, *Geophys. J. R. astr. Soc.*, **69**, 15–32.

Cook, A. H., 1958. Determination of the Earth's gravitational potential from observations of *Sputnik 2* (1957), *Geophys. J. R. astr. Soc.*, **1**(4), 341–345.

Cook, A. H., 1960. Resonant orbits of artificial satellites, *Space Research*, p. 476, North-Holland, Amsterdam.

Cook, A. H., 1961. Resonant orbits of artificial satellites and longitude terms in the Earth's external gravitational potential, *Geophys. J. R. astr. Soc.*, **4**, 53–69.

Gaposchkin, E. M. (ed.), 1973. 1973 Smithsonian standard earth (III), *Spec. Rep. Smithson. Astrophys. Obs.*, **353**, 388 pp.

Gaposchkin, E. M., 1974. Earth's gravity field to the eighteenth degree and geocentric coordinates for 104 stations from satellite and terrestrial data, *J. geophys. Res.*, **79**, 5377–5411.

Gaposchkin, E. M., 1977. Gravity-field determination using laser observations, *Phil. Trans. R. Soc. A*, **284**, 515–527.

Gaposchkin, E. M., 1979. Recent advances in analytical satellite theory, in *Applications of Geodesy to Geodynamics, Proc. 9th GEOP Conf.*, ed. Mueller, I. I., *Rep. Dep. Geodetic Sci.*, **280**, 197–206, Ohio State University, Columbus.

Gaposchkin, E. M., 1980. Global gravity field to degree and order 30 from *GEOS 3* satellite altimetry and other data, *J. geophys. Res.*, **85**(B12), 7221–7234.

Gaposchkin, E. M. & Lambeck, K., 1970. 1969 Smithsonian standard earth (II), *Spec. Rep. Smithson. Astrophys. Obs.*, **315**, 93 pp.

Gaposchkin, E. M. & Lambeck, K., 1971. The earth's gravity field to sixteenth degree and station coordinates from satellite and terrestrial data, *J. geophys. Res.*, **76**, 4855–4883.

Gaposchkin, E. M. & Mendes, G. M., 1977. Geopotential model to degree and order 30 from satellite and terrestrial data, *Trans. Am. geophys. Un.*, **58**(6), 371.

Groves, G. V., 1960. Motion of a satellite in the earth's gravitational field, *Proc. R. Soc. A*, **254**, 48–65.

Guier, W. H. & Newton, R. R., 1965. The earth's gravity field as deduced from the doppler tracking of five satellites, *J. geophys. Res.*, **70**, 4613–4626.

Heiskanen, W. H. & Moritz, H., 1967. *Physical Geodesy*, Freeman, San Francisco, 364 pp.

Heiskanen, W. H. & Vening, Meinesz, F. A., 1958. *The Earth and its Gravity Field,* McGraw-Hill, New York.

Henriksen, S. W., 1960. The hydrostatic flattening of the Earth, *Annals of the IGY,* **12**(1), 197–198.

Izsak, I. G., 1961. A determination of the ellipticity of the Earth's equator from the motion of two satellites, *Space Research*, pp. 352–359, North-Holland, Amsterdam.

Jacchia, L. G., 1958. The Earth's gravitational potential as derived from satellites 1957.1 and 1958.2, *Spec. Rep. Smithson. Astrophys. Obs.,* **19,** 1–5.

Jeffreys, H., 1959. *The Earth,* 4th edn, Cambridge University Press.

Kahn, W. D., Klosko, S. M. & Wells, W. T., 1981. Mean gravity anomalies from a combination of *Apollo/ATS-6* and *GEOS-3/ATS-6* SST tracking campaigns, *NASA Tech. Mem. 82120,* Goddard Space Flight Center, Greenbelt, 47 pp.

Kaula, W. M., 1961a. Analysis of satellite observations for longitudinal variations of the gravitational field, *Space Research,* p. 360, North-Holland, Amsterdam.

Kaula, W. M., 1961b. Analysis of gravitational and geometric aspects of geodetic utilization of satellites. *Geophys. J. R. astr. Soc.,* **5,** 104–133.

Kaula, W. M., 1966a. *Theory of Satellite Geodesy,* Blaisdell, Waltham, Massachusetts, 124 pp.

Kaula, W. M., 1966b. Tests and combinations of satellite determinations of the gravity fields with gravimetry, *J. geophys. Res.,* **71,** 5303–5314.

Kaula, W. M., 1967. Theory of statistical analysis of data distributed over a sphere, *Rev. Geophys.,* **5,** 83–107.

Kaula, W. M., 1972. Global gravity and tectonics, in *The Nature of the Solid Earth,* pp. 385–405, eds Robinson, E. C., Hays, J. F. & Knopoff, L., McGraw-Hill, New York.

Kaula, W. M., 1977. Geophysical inferences from statistical analysis of the gravity field, in *The Changing World of Geodetic Science,* ed. Uotila, U. A., *Rep. Dep. Geodetic Sci.,* **250,** 119–141, Ohio State University, Columbus.

Kaula, W. M., 1983. Inferences of variations in the gravity field from satellite-to-satellite range-rate, *J. geophys. Res.,* submitted.

King-Hele, D. G., 1961. The Earth's gravitational potential, deduced from the orbits of artificial satellites, *Geophys. J. R. astr. Soc.,* **4,** 3–16.

King-Hele, D. G., 1975. The Bakerian lecture, 1974: a view of the Earth and air, *Phil. Trans. R. Soc.,* **278,** 67–109.

King-Hele, D. G., Brookes, C. J. & Cook, G. E., 1981. Odd zonal harmonics in the geopotential, from analysis of 28 satellite orbits, *Geophys. J. R. astr. Soc.,* **64,** 3–30.

King-Hele, D. G. & Walker, D. M. C., 1982a. Evaluation of 15th-order harmonics in the geopotential from analysis of resonant orbits, *Proc. R. Soc. A,* **379,** 247–288.

King-Hele, D. G. & Walker, D. M. C., 1982b. Geopotential harmonics of order 29, 30 and 31 from analysis of resonance orbits, *Planet. Space Sci.,* **30,** 411–425.

King-Hele, D. G., Walker, D. M. C. & Gooding, R. H., 1979. Evaluation of 14th-order harmonics in the geopotential, *Planet. Space Sci.,* **27,** 1–18.

Klokočník, J., 1975. Resonant coefficients of the Earth's gravity field derived from the inclination of interkosmos satelliates, *Bull. astr. Inst. Czech.,* **26,** (5), 293–300.

Klokocnik, J. & Pospisilova, L., 1981. Intercomparisons of Earth models by means of lumped coefficients, *Planet. Space Sci.,* **29,** 653–671.

Kopal, Z., 1962. The effects of convection in the mantle on the gravitational field of the Earth, in *The Use of Artificial Satellites for Geodesy,* pp. 400–408, ed. Veis, G., North Holland, Amsterdam.

Kozai, Y., 1959. On the effects of the Sun and the Moon upon the motion of a close Earth satellite, *Spec. Rep. Smithson. Astrophys. Obs.,* **22,** 7–10.

Kozai, Y., 1969. Revised values for coefficients of zonal spherical harmonics in the geopotential, *Spec. Rep. Smithson. Astrophys. Obs.,* **295,** 17 pp.

Lambeck, K., 1971. Comparison of surface gravity data with satellite data, *Bull. Geod.,* **100,** 203–219.

Lambeck, K., 1976. Lateral density anomalies in the mantle, *J. geophys. Res.,* **81,** 6333–6340.

Lambeck, K., 1979. Methods and geophysical applications of satellite geodesy, *Rep. Prog. Phys.,* **42,** 547–628.

Lambeck, K., 1981. Lithospheric response to volcanic loading in the Southern Cook Islands, *Earth planet. Sci. Lett.,* **55,** 482–496.

Lambeck, K. & Coleman, R., 1982a. Verification of bathymetric charts from satellite altimetry data in the region of the Cook Islands, *N. Z. J. Sci.,* **25,** 183–194.

Lambeck, K. & Coleman, R., 1982b. A search for seamounts in the Southern Cook and Austral region, *Geophys. Res. Lett.,* **9,** 389–392.

Ledersteger, K., 1967. Critical notes on the mass-functions derived from orbital perturbations of artificial satellites, in *The Use of Artificial Satellites for Geodesy,* II, 163–168, ed. Veis. G., National Technical University, Greece.

Lerch, F. J., Klosko, S. M., Laubscher, R. E. & Wagner, C. A., 1979. Gravity model improvement using *Geos 3* (GEM9 and 10), *J. geophys. Res.,* **84**(B8), 3897–3916.

Lerch, F. J., Klosko, S. M. & Patel, G. B., 1982. A refined gravity model from *Lageos* (GEM-L2), *Geophys. Res. Lett.,* **9**, 1263–1266.

Lerch, F. J., Putney, B. H., Wagner, C. A. & Klosko, S. M., 1981. Goddard earth models for oceanographic applications (GEM10B and 10C), *Mar. Geod.,* **5**(2), 145–187.

Lundquist, C. A. & Veis, G. (eds), 1966. Geodetic parameters for a 1966 Smithsonian Institution standard earth, Vol. I, *Spec. Rep. Smithson. Astrophys. Obs.,* **200**, 231 pp.

MacDonald, G. J. F., 1962. Geophysical implications of satellite geodesy, in *The Use of Artificial Satellites for Geodesy,* pp. 389–399, ed. Veis, G., North-Holland, Amsterdam.

Marsh, J. G. & Martin, T. V., 1982. The *SEASAT* altimeter mean sea surface model, *J. geophys. Res.,* **87**, 3269–3280.

Mather, R. S., Lerch, F. J., Rizos, C., Masters, E. G. & Hirsh, B., 1978. Determination of some dominant parameters of the global dynamic sea surface topography from *GEOS 3* altimetry, *NASA Techn. Memo 79558,* Goddard Space Flight Center, Greenbelt, 39pp.

McKenzie, D., Watts, A., Parsons, B. & Roufosse, M., 1980. Planform of mantle convection beneath the Pacific ocean, *Nature,* **288**, 442–446.

Merson, R. H. & King-Hele, D. G., 1958. Use of artificial satellites to explore the Earth's gravitational field: results from *Sputnik 2, Nature,* **182**, 640–641.

Morando, B., 1962. Recherches sur les orbites de resonance, in *The Use of Artificial Satellites for Geodesy,* pp. 42–51, ed. Veis, G., North-Holland, Amsterdam.

National Research Council, 1979. *Applications of a Dedicated Gravitational Satellite Mission,* National Academy of Sciences, Washington, DC, 53 pp.

Newton, R. R., 1962. Non-zonal harmonics deduced from the motion of Transit 4A, in *The Use of Artificial Satellites for Geodesy,* pp. 323–328, ed. Veis, G., North-Holland, Amsterdam.

O'Keefe, J. A., 1959. Zonal harmonics of the Earth's gravitational field and the basic hypothesis of geodesy, *J. geophys. Res.,* **64**, 2389–2392.

O'Keefe, J. A., 1960. Determination of the Earth's gravitational field, *Space Research,* pp. 448–457, North-Holland, Amsterdam.

O'Keefe, J. A. & Batchelor, C. D., 1957. Perturbations of a close satellite by the equatorial ellipticity of the Earth, *Astr. J.,* **62**, 183–185.

O'Keefe, J. A., Eckels, A. & Squires, R. K., 1959. The gravitational field of the Earth, *Astr. J.,* **64**, 245–253.

Pearlman, M. R., Lanham, N. W., Lehr, C. G. & Wohn, J., 1977. Smithsonian Astrophysical Observatory laser tracking systems, *Phil. Trans. R. Soc. A,* **284**, 431–442.

Reigber, Ch. & Balmino, G., 1975. Even and odd degree 13th-order harmonics from analysis of stable near-resonant satellite orbits, *Bull. Groupe Rech. Geod. Spatiale, No. 15.*

Reigber, C., Balmino, G., Moynot, B. & Muller, H., 1982. GRIM3 earth model evaluation, paper presented at *Fourth Annual Conf. on NASA Geodynamics Program,* Washington, DC, 1982 January 26–29.

Runcorn, S. K., 1967. Flow in the mantle inferred from the low degree harmonics of the geopotential, *Geophys. J.,* **14**, 375–384.

Sandwell, D. & Schubert, G., 1980. Geoid height versus age for symmetric spreading ridges, *J. geophys. Res.,* **85**(B12), 7235–7241.

Sehnal, L., 1960. The perturbations of the orbit of the stationary satellite of the Earth, *Bull. Astr. Inst. Czech. 11.*

Smith, D. E., Lerch, F. J., Marsh, J. G., Wagner, C. A., Kolenkiewicz, R. & Khan, M. A., 1976. Contributions to the national geodetic satellite program by Goddard Space Flight Center, *J. geophys. Res.,* **81**(5), 1006–1026.

Vonbun, F. O., 1977. Goddard laser systems and their accuracies, *Phil. Trans. R. Soc. A,* **284**, 443–450.

Wagner, C. A., 1974. Eleventh-order geopotential resonance on the orbit of *VANGUARD 3, J. geophys. Res.,* **79**, 3335–3341.

Wagner, C. A. & Klosko, S. M., 1975. 15th order resonance terms using the decaying orbit of *TETR-3, Planet. Space Sci.,* **23**, 541–549.

Walker, D. M. C., 1982. The last 14 days of *Skylab 1:* orbit determination and analysis, *Tech. Rep. 82067,* Royal Aircraft Establishment, Farborough, 44 pp.

Yionoulis, S. M., 1966. Determination of coefficients associated with the geo-potential harmonics of order thirteen, *J. geophys. Res.,* **71**, 1768.

Note added in proof

This paper cannot be considered as a complete review since much of the information required for a thorough investigation of published gravity models remains unavailable. This is particularly true for the latest GEM models and for GRIM 3. Perhaps, by pointing the bone at aspects of various solutions, this information will be forthcoming in the open literature. The initial response to this paper by some of the protagonists of the various models has already given us more insight into their work. But we do not think that this invalidates the basic conclusion; that the various models are not as good as they are said to be. If they were, the differences between them should not be so great as they are (Figs 7 and 8).