Journal of Nonlinear Mathematical Physics Volume 9, Supplement 2 (2002), 10–23 **Birthday Issue**

The Economy of Complete Symmetry Groups for Linear Higher Dimensional Systems

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Received March, 2002

Abstract

The complete symmetry groups of systems of linear second order ordinary differential equations are considered in the context of the simple harmonic oscillator. One finds that in general the representation of the complete symmetry group is not unique and in the particular case of a four-dimensional system there are two distinct complete symmetry groups. The results for general second order linear systems are indicated in the Conclusion.

1 Introduction

The modern concept of a complete symmetry group was introduced by Krause [9] who defined a complete symmetry group as the group underlying the algebra of the set of Lie symmetries required to specify completely a given differential equation to within, perhaps, arbitrary constant(s) which could be removed by rescaling or translation. To illustrate the concept Krause used the Kepler Problem for which he found it necessary to add nonlocal symmetries to the well-known five Lie point symmetries in order to obtain a complete specification of the equation. These nonlocal symmetries were obtained by an elegant *Ansatz* of the structure of the required symmetries. Nucci [20] demonstrated, with the assistance of her interactive code for the computation of symmetries [18, 19], that these symmetries and more could be calculated by means of the standard methods of the Lie theory if one used the technique of reduction of order [21].

In an interesting development Nucci and Leach [22] demonstrated that a number of problems related to the Kepler Problem, specifically through the possession of a conserved vector of Laplace–Runge–Lenz form, could all be reduced to the one-dimensional simple harmonic oscillator and a succession of first-order equations which reflected conservation laws. In a recent work of Marcelli and Nucci [13] this elaboration of the method of reduction has been successfully extended to other systems such as the Kowalevskaya top.

In a separate direction of development Leach *et al* [11] and Andriopoulos *et al* [2] explored some of the properties of complete symmetry groups. In particular they established three results. Firstly they demonstrated that the representation of the complete symmetry group for an ordinary differential equation need not be unique. Secondly they showed the necessity for the introduction of a requirement of minimality in the specification of the complete symmetry group, in other words algebras of different dimension could completely specify an equation. It was proposed that the expression 'complete symmetry group' be reserved for the group of the algebra of minimum dimension. Thirdly they proved that the dimension of the complete symmetry group of an nth-order linear ordinary differential equation was $n+1$. (There were other results which are not of relevance to this paper.)

In this paper we provide a more formal demonstration of the nonuniqueness of the representation of the complete symmetry group of a differential equation and provide a partial answer to the hitherto unexplored area of the dimension of the complete symmetry group of systems of linear ordinary differential equations. For this discussion we use the simple harmonic oscillator as a vehicle. We conclude with some observations on the structures of the representations of the complete symmetry group of systems of linear second-order ordinary differential equations and directions for future research. In § 2 we show that there are three particular representations of the complete symmetry group for the one-dimensional simple harmonic oscillator and, since all scalar linear second-order equations are equivalent under a point transformation, so all such equations. In § 3 we consider the corresponding problem for the higher dimensional isotropic simple harmonic oscillator and find that the results are quite intriguing. In § 4 we present our observations.

2 Multiple representations of the complete symmetry group of the one-dimensional simple harmonic oscillator

The one-dimensional simple harmonic oscillator is described by the differential equation

$$\ddot{x} + x = 0, \tag{2.1}$$

in which overdot denotes total differentiation with respect to the independent variable t. The equation is chosen for its simplicity, but is representative of all scalar linear second-order ordinary differential equations [23]. Equation (2.1) has the eight Lie point symmetries

$$\begin{aligned} &\Gamma_1 = \partial_t, \qquad \Gamma_2 = x\partial_x, \qquad \Gamma_{3\pm} = \mathrm{e}^{\pm it}\partial_x, \\ &\Gamma_{4\pm} = \mathrm{e}^{\pm 2it}\left(\partial_t \pm ix\partial_x\right), \qquad \Gamma_{5\pm} = x\mathrm{e}^{\pm it}\left(\partial_t \pm ix\partial_x\right), \end{aligned} \tag{2.2}$$

in which Γ_2 is the homogeneity symmetry, $\Gamma_{3\pm}$ are the solution symmetries, $\Gamma_{5\pm}$ are the non-Cartan symmetries and Γ_1 and $\Gamma_{4\pm}$ are the elements of the algebra $sl(2,\mathbb{R})$.

It is known that equation (2.1) possesses the three integrals[1]

$$I_A = \mathrm{e}^{it}\left(x + i\dot{x}\right), \qquad I_B = \mathrm{e}^{-it}\left(x - i\dot{x}\right), \qquad I_C = I_A/I_B, \tag{2.3}$$

[1] I_C, as it is manifestly written, is not an integral independent of I_A and I_B. However, it also possesses the same algebraic properties as I_A and I_B. Generally functions of the elementary integrals do not possess the same algebraic properties [4, 10].

with the Lie point symmetries

$$\begin{array}{lll} A_1 = \mathrm{e}^{it}\partial_x, & A_2 = \partial_t - ix\partial_x, & A_3 = \mathrm{e}^{2it}\left(\partial_t + ix\partial_x\right), \\ B_1 = \mathrm{e}^{-it}\partial_x, & B_2 = \partial_t + ix\partial_x, & B_3 = \mathrm{e}^{-2it}\left(\partial_t - ix\partial_x\right), \\ C_1 = x\partial_x, & C_{2\pm} = x\mathrm{e}^{\pm it}\left(\partial_t \pm ix\partial_x\right) & \end{array} \tag{2.4}$$

and that each set of symmetries is a representation of the group $D \otimes_s 2A_1$, the group of dilations and translations in the plane. One notes that the triplets $\{\Gamma_2, \Gamma_{3\pm}\}$ and $\{\Gamma_2, \Gamma_{5\pm}\}$ also possess the algebra $D \oplus_s T_2$. (The representations of the integrals and the symmetries are adapted from the trigonometric expressions given by Mahomed and Leach [12].) It is an easy exercise to demonstrate that the application of the second extensions of the sets $\{A_i\}$, $\{B_i\}$ or $\{C_i\}$, $i = 1, 3$, to the general second order ordinary differential equation, *videlicet*

$$\ddot{x} = f\left(t, x, \dot{x}\right), \tag{2.5}$$

results in (2.1) being the invariant equation. Thus these are three representations of the complete symmetry group of (2.1) and so, by extension, to all scalar second-order equations related to (2.1) by means of a transformation not generated by one of the elements of the corresponding algebra. We note that these three representations are equivalent since the elements of each representation are transformed into each other by means of point transformations which leave (2.1) invariant [12]. In addition to these representations of $D \otimes_s 2A_1$ with their peculiar relationship to the 'fundamental' first integrals [4] of (2.1) and their ratio there is a fourth representation unrelated to these integrals in addition to the triplets $\{\Gamma_2, \Gamma_{3\pm}\}$ and $\{\Gamma_2, \Gamma_{5\pm}\}$ mentioned above. This is the set of symmetries

$$S_1 = \mathrm{e}^{it}\partial_x, \qquad S_2 = \mathrm{e}^{-it}\partial_x, \qquad S_3 = \partial_t \tag{2.6}$$

which is related to the symmetries above since

$$S_1 = A_1, \qquad S_2 = B_1, \qquad S_3 = \tfrac{1}{2}(A_2 + B_2). \tag{2.7}$$

Note that the transformation is not restricted to being point or contact. The only requirement is that the initial equation and the transformed equation both be scalar second order ordinary differential equations. A by now almost classic example is the equation

$$x\ddot{x} = \dot{x}^2 + \dot{f}(t)x^{n+2} + nf(t)\dot{x}x^{n+1} \tag{2.8}$$

which was touted as an easily integrated equation devoid of symmetry [7] (for general $f(t)$) until it was shown by Abraham-Shrauner *et al* [1] to possess the algebra $sl(3, \mathbb{R})$, the characteristic of linear and linearisable second-order ordinary differential equations, since it was related to that of the 'free particle', *videlicet*

$$\frac{\mathrm{d}^2 X}{\mathrm{d}T^2} = 0 \tag{2.9}$$

by means of the nonlocal transformation[2]

$$T = t, \qquad X = \log(-nx^n) - \int_0^t nf(s)x^n(s)\mathrm{d}s. \tag{2.10}$$

Whilst the possession of the algebra $sl(3, \mathbb{R})$ of point symmetries is classed as maximal, the meaning of the expression is not clear when this set of symmetries is embedded in a sea of nonlocal symmetries. Indeed it is sufficient of a conundrum to make one question the meaning of sets of point symmetries such as those with the algebra $sl(3, \mathbb{R})$ [25]. One of the triplets of symmetries which completely specify (2.9) consists of the symmetries

$$A_1 = T\partial_X, \qquad A_2 = T\partial_T, \qquad A_3 = T^2\partial_T + TX\partial_X, \tag{2.11}$$

corresponding, as the notation is intended to suggest, to the A symmetries listed for (2.1) above to which they are related, up to scaling, by means of a point transformation which takes (2.9) to (2.1). Under the transformation (2.10) the symmetries in (2.11) become

$$\begin{aligned}
\bar{A}_1 &= -\frac{x}{n-1}\exp\left[-\int_0^t \frac{x\mathrm{d}s}{n-1}\right]\partial_x, \\
\bar{A}_2 &= x\left[\log(-nx^n) - n\int_0^t fx^n\mathrm{d}t + \exp\left(n\int_0^t fx^n\mathrm{d}t\right)\right. \\
&\quad \left.\times \int_0^t fx^n\left(\log(-nx^n) - n\int_0^s fx^n\mathrm{d}r\right)\exp\left(-n\int_0^s fx^n\mathrm{d}r\right)\mathrm{d}s\right]\partial_x, \\
\bar{A}_3 &= \partial_t + \frac{n}{(n-1)^2}\left\{(n-1)fx^{n+1} - x\exp\left[-\int_0^t \frac{x\mathrm{d}s}{n-1}\right]\right. \\
&\quad \left.\times \int_0^t \left[fx^{n+1}\exp\left(\int_0^s \frac{x\mathrm{d}r}{n-1}\right)\right]\mathrm{d}s\right\}\partial_x
\end{aligned} \tag{2.12}$$

which is a highly nonlocal representation of the algebra, $A_{3,3}$ in the Mubarakzyanov classification [15, 16, 17], of the group of dilations and translations in the plane and yet is the set of symmetries which completely specifies (2.8).

It is an elementary calculation to verify that (2.1) is completely specified by $\{\Gamma_2, \Gamma_{3\pm}\}$ and $\{\Gamma_2, \Gamma_{5\pm}\}$. There are other sets of symmetries of (2.1) which can be used to specify completely (2.1) [2]. In the context of the definition given by Krause [9] such sets would also be described as representations of the complete symmetry group. However, in the more precise definition given by Andriopoulos *et al* as a consequence of the existence of these other sets they are not representations of the complete symmetry group since they are not minimal. One particular set of symmetries of some interest due to one of its subalgebras is

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = x\partial_x, \qquad \Gamma_{4\pm} = \mathrm{e}^{\pm 2it}\left(\partial_t \pm ix\partial_x\right) \tag{2.13}$$

[2]The calculation of the Lie Brackets of nonlocal symmetries is often a nontrivial exercise due to the necessity of a correct identification of variables. The calculation is more than greatly simplified if there exists an obvious transformation to point symmetries for which the calculation of the Lie Brackets is easier if not always nontrivial. For an example see Nucci *et al* [24] in this volume. The preservation of the brackets follows from application of the chain rule.

selected from the set given in (2.2). The symmetries Γ_1 and $\Gamma_{4\pm}$ constitute a representation of the algebra $sl(2,\mathbb{R})$ and the algebra of the symmetries in (2.13) is $A_1 \oplus sl(2,\mathbb{R})$. The set of symmetries in (2.13) is not a representation of the complete symmetry group since there are four symmetries and not the minimal three. The interest in this particular set of symmetries is that $sl(2,\mathbb{R})$ is the algebra of the complete symmetry group of the Ermakov–Pinney equation [3, 26]

$$\ddot{x} + x = \frac{L^2}{x^3} \tag{2.14}$$

which, apart from its many interesting applications in the area of Ermakov systems, has the structure of the radial equation of a higher dimensional isotropic harmonic oscillator. (In passing we note that in the trigonometric equivalent to Γ_1 and $\Gamma_{4\pm}$ only Γ_1 and one of the trigonometric symmetries are required to specify (2.14) up to the arbitrary value of the rescalable constant L. However, the addition of the second trigonometric symmetry is required to close the algebra. This feature is not exhibited by the exponential form we have adopted in this paper.)

3 Higher dimensional isotropic oscillators

The higher dimensional isotropic simple harmonic oscillator is described by the differential equation

$$\ddot{\boldsymbol{r}} + \boldsymbol{r} = 0. \tag{3.1}$$

In two dimensions (3.1) may be written in cartesian and plane polar coordinates as

$$\begin{aligned} &\ddot{x} + x = 0, \qquad \ddot{r} - r\dot{\theta}^2 + r = 0, \\ &\ddot{y} + y = 0, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{aligned} \tag{3.2}$$

respectively. It is a commonplace that the two-dimensional isotropic simple harmonic oscillator possesses fifteen Lie point symmetries with the algebra $sl(4,\mathbb{R})$. However, our interest is not in the total number of Lie point symmetries but in the minimal number of symmetries required to specify the system (3.2).

The advantage of the description of the system in plane polar coordinates is that the second equation, the angular equation, may be written as a conservation law. Thus we may write the system as

$$\ddot{r} + r = \frac{L^2}{r^3}, \qquad \dot{L} = 0, \tag{3.3}$$

where $L := r^2\dot{\theta}$. The structure of (3.3) is reminiscent of that obtained when one uses the method of reduction of order developed by Nucci [20] except that there has been no change of independent variable. In terms of that method one would use the angular variable, θ, as the fairly obvious new independent variable and L as v_2, *ie* the obvious conserved quantity. The other independent variable is found to be $v_1 = 1/r$. The reduced form of the plane polar form of (3.2) is

$$v_1'' + v_1 = \frac{1}{v_1^3 v_2^2}, \qquad v_2' = 0 \tag{3.4}$$

in which the prime denotes differentiation with respect to the new independent variable, θ.

Krause [9] obtained the complete symmetry group of the Kepler Problem by determining nonlocal symmetries to supplement the Lie point symmetries. Nucci [20] showed that the method of reduction of order led naturally to those same symmetries. Consequently we realise that there are several possible candidates to supply the symmetries required to specify completely the pair of second-order ordinary differential equations which described the two-dimensional isotropic simple harmonic oscillator. We have the original representation, (3.2), with a choice of cartesian or plane polar coordinates. We can look at the conserved form, (3.3), of the original system written in plane polar coordinates. Alternatively we could consider the reduced system (3.4). We note that there is very little difference between (3.3) and (3.4). The structure is the same in both cases. Only the independent variable is different.

We commence with the system (3.4) which has the Lie point symmetries

$$R_1 = \partial_\theta, \qquad R_{2\pm} = \mathrm{e}^{\pm 2i\theta}\left(\partial_\theta \pm iv_1\partial_{v_1}\right), \qquad R_4 = v_1\partial_{v_1} - 2v_2\partial_{v_2}. \tag{3.5}$$

To relate a symmetry of the original system, (3.2), (in plane polar coordinates) to one of (3.4) we have the equivalence

$$\tau\partial_t + \eta\partial_r + \zeta\partial_\theta \longrightarrow \zeta\partial_\theta + \Omega\partial_{v_1} + \Sigma\partial_{v_2}, \tag{3.6}$$

where

$$\Omega = -\frac{\eta}{r^2} \qquad \text{and} \qquad \Sigma = 2\eta r\dot\theta + r^2\left(\dot\zeta - \dot\theta\dot\tau\right). \tag{3.7}$$

In the case of R_1 it is obvious that $\eta = 0$ since $\Omega = 0$. From the second of (3.7) we have

$$r^2\dot\theta\dot\tau = 0. \tag{3.8}$$

Consequently in R_1 there is implied the symmetry ∂_t as well as ∂_θ. We treat the remaining symmetries similarly. Corresponding to the set of symmetries in (3.5) we obtain for (3.2)

$$\bar R_1 = \partial_\theta, \qquad \bar R_{2\pm} = \mathrm{e}^{\pm 2i\theta}\left(\partial_\theta \mp ir\partial_r\right), \qquad \bar R_4 = r\partial_r, \qquad \bar R_5 = \partial_t, \tag{3.9}$$

in which a minus sign has been removed from $\bar R_4$.

The corresponding analysis of (3.3) produces the set of symmetries

$$T_1 = \partial_t, \qquad T_{2\pm} = \mathrm{e}^{\pm 2it}\left(\partial_t \pm ir\partial_r\right), \qquad T_4 = r\partial_r, \qquad T_5 = \partial_\theta. \tag{3.10}$$

The roles of t and θ have simply been interchanged. The algebra in each case is the same.

An analysis of the actions of the symmetries represented in (3.9) and in (3.10) on the system

$$\begin{aligned} \ddot r &= f(t, r, \theta, \dot r, \dot\theta), \\ \ddot\theta &= g(t, r, \theta, \dot r, \dot\theta) \end{aligned} \tag{3.11}$$

shows that in both cases the number of symmetries required to recover the plane polar form for the two-dimensional isotropic oscillator is not five but four, *ie* the symmetry lost in the reduction of order to the system (3.4) is not necessary to specify completely the polar form of (3.2). The interesting part about this is that the algebra of the symmetries remains the same, *ie* $A_1 \oplus sl(2,\mathbb{R})$, as we observed in the case of using $sl(2,\mathbb{R})$ as the starting point for the specification of (2.1).

The 15 Lie point symmetries of the cartesian form of (3.2) are a duplication, with the exception of ∂_t, of those listed in (2.2) with y in place of x. The obvious generalisation of the triplets $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$ would be the quintets with an additional two y-based symmetries. The algebra would be $A_1 \oplus_s \{2A_1 \oplus 2A_1\}$. It should be quite obvious that these five Lie point symmetries would specify completely the cartesian form of (3.2) since the process is simply additive.

Thus we have

Proposition 1. *The set of symmetries*

$$\Gamma_1 = \sum_{i=1}^{n} x_i \partial_{x_i}, \qquad \Gamma_{i\pm} = \mathrm{e}^{\pm it}\partial_{x_i}, \qquad i = 1, n \tag{3.12}$$

is sufficient to specify completely the set of equations

$$\ddot{x}_i + x_i = 0, \qquad i = 1, n. \tag{3.13}$$

Proof. Consider the general equation

$$\ddot{x}_j = f_j(t, x, \dot{x}). \tag{3.14}$$

The action of $\Gamma_{i+}^{[2]}$ on (3.14) is

$$-\delta_{ij} = \frac{\partial f_j}{\partial x_i} + i\frac{\partial f_j}{\partial \dot{x}_i}, \qquad i = 1, n \tag{3.15}$$

and gives

$$f_j = -x_j + F_{j1}(t, x + i\dot{x}) \tag{3.16}$$

so that now (3.14) is

$$\ddot{x}_j = -x_j + F_{j1}(t, u), \qquad u_i = x_i + i\dot{x}_i. \tag{3.17}$$

The action of $\Gamma_{i-}^{[2]}$ on (3.17) is

$$-\delta_{ij} = -\delta_{ij} + 2\frac{\partial F_{j1}}{\partial u_i}, \qquad i = 1, n \tag{3.18}$$

from which it follows that

$$F_{j1} = F_{j2}(t) \tag{3.19}$$

and the equation is now

$$\ddot{x}_j = -x_j + F_{j2}(t). \tag{3.20}$$

The action of the homogeneity symmetry Γ_1 forces $F_{j2}(t) = 0$, $j = 1, n$. Hence the proposition follows. ■

Corollary. *The set of symmetries*

$$Z_1 = x_i\partial_{x_i},$$
$$Z_{i\pm} = x_i e^{\pm it}\left(\partial_t \pm ix_i\partial_{x_i}\right), \qquad i = 1, n \tag{3.21}$$

is sufficient to specify completely the system (3.13).

Naturally one wonders if the $(2n+1)$-dimensional algebra $\{\Gamma_1, \Gamma_{i\pm},\ i = 1, n\}$ (equally $\{Z_1, Z_{i\pm},\ i = 1, n\}$) is the algebra of minimal dimension required completely to specify the system (3.13). In Proposition I the n homogeneity symmetries are conflated into one. Can the same be envisaged for the two sets of solution symmetries? We examine the possibility in two dimensions. Consider the actions of

$$\Gamma_1 = x\partial_x + y\partial_y, \qquad \Gamma_{2\pm} = e^{\pm it}\left(\partial_x + \partial_y\right) \tag{3.22}$$

on the system

$$\ddot{x} = f(t, x, \dot{x}, y, \dot{y}), \qquad \ddot{y} = g(t, x, \dot{x}, y, \dot{y}). \tag{3.23}$$

The action of $\Gamma_{2+}^{[2]}$ on (3.23) gives

$$\ddot{x} = -x + F_1(t, x + i\dot{x}, y + i\dot{y}, x - y),$$
$$\ddot{y} = -y + G_1(t, x + i\dot{x}, y + i\dot{y}, x - y). \tag{3.24}$$

The action of $\Gamma_{2-}^{[2]}$ on (3.24) gives

$$\ddot{x} = -x + F_2(t, u, w,)$$
$$\ddot{y} = -y + G_2(t, u, w), \tag{3.25}$$

where $u = (x + i\dot{x})/(y + i\dot{y})$ and $w = x - y$. The action of $\Gamma_1^{[2]}$ on (3.25) gives

$$F_2 = w\frac{\partial F_2}{\partial w}, \qquad G_2 = w\frac{\partial G_2}{\partial w} \tag{3.26}$$

so that (3.25) is reduced to

$$\ddot{x} = -x + F_3(t, u)w, \qquad \ddot{y} = -y + G_3(t, u)w. \tag{3.27}$$

Evidently the three symmetries are not sufficient to reduce (3.23) to the two-dimensional oscillator. If we add the operator

$$\Gamma_4 = x\partial_y - y\partial_x \tag{3.28}$$

to the other three, we find that the action of $\Gamma_4^{[2]}$ on (3.27) gives

$$G_3(t, u)w = w\left(1 + w^2\right)\frac{\partial F_3}{\partial u} + (w + 2y)\,F_3,$$
$$F_3(t, u)w = -w\left(1 + w^2\right)\frac{\partial G_3}{\partial u} - (w + 2y)\,G_3 \tag{3.29}$$

from which it follows that $F_3 = 0 = G_3$.

This suggests

Proposition 2. *The system of equations*

$$\ddot{x}_i + x_i = 0, \qquad i = 1, n \tag{3.30}$$

is completely specified by the $\frac{1}{2}(n^2 - n + 6)$ *operators*

$$\Gamma_1 = \sum_{i=1}^{n} x_i \partial_{x_i},$$
$$\Gamma_{2\pm} = \mathrm{e}^{\pm it} \sum_{i=1}^{n} \partial_{x_i},$$
$$\Gamma_{4ij} = x_i \partial_{x_j} - x_j \partial_{x_i}, \qquad i, j = 1, n, \quad i \neq j. \tag{3.31}$$

Corollary. *The system (3.30) is completely specified by similar combinations based upon the triplets* $\{A\}$, $\{B\}$, $\{C\}$ *and* $\{Z\}$.

Another set of symmetries which specifies the cartesian form of the two-dimensional isotropic oscillator is

$$\Phi_{1\alpha} = x\partial_x, \qquad \Phi_{1\beta} = y\partial_y, \qquad \Phi_{2\pm} = \mathrm{e}^{\pm it}\left(\partial_x + \partial_y\right), \tag{3.32}$$

which can be easily verified in a manner similar to the one presented above for the symmetries in (3.22). This obviously suggests

Proposition 3. *The set of symmetries*

$$\Phi_i = x_i \partial_{x_i}, \qquad i = 1, n,$$
$$\Phi_{2\pm} = \mathrm{e}^{\pm it} \sum_{i=1}^{n} \partial_{x_i} \tag{3.33}$$

completely specifies the set of equations

$$\ddot{x}_i + x_i = 0, \qquad i = 1, n. \tag{3.34}$$

Proof. All we have to do is apply the second extensions of the symmetries (3.33) on the general equation

$$\ddot{x}_j = f_j(t, x_j, \dot{x}_j)$$

in turn until we recover (3.34).

Using Φ_i we restrict f_j to be

$$f_j = x_j F_j\left(t, u_k\right), \qquad j = 1, n,$$

where $u_k = \dot{x}_k / x_k$, and on introducing $\Phi_{2\pm}$ we obtain

$$-1 = F_j + x_j \frac{\partial F_j}{\partial u_k}\left(\frac{-u_k \pm i}{x_k}\right) \tag{3.35}$$

by summation on k and not on j.

Due to the presence of x_j/x_k in (3.35) for $k \neq j$ (3.35) yields

$$\frac{\partial F_j}{\partial u_k} = 0$$

and for $k = j$ (3.35) yields

$$-1 = F_j + \frac{\partial F_j}{\partial u_j}(-u_j \pm i). \tag{3.36}$$

Addition and subtraction of the two equations in (3.36) yields the system

$$-1 = F_j - u_j \frac{\partial F_j}{\partial u_j},$$
$$0 = \frac{\partial F_j}{\partial u_j},$$

which simply implies that $F_j = -1$. Thus

$$\ddot{x}_i = -x_i$$

and the proposition follows. ■

We have yet to generalise the results obtained after treating the two-dimensional oscillator in plane polar coordinates, an important case since the number of symmetries there was four instead of the expected five. Thus we have

Proposition 4. *The set of symmetries*

$$\Psi_1 = \partial_t,$$
$$\Psi_\pm = \mathrm{e}^{\pm 2it}\left(\partial_t \pm i \sum_{i=1}^{n} x_i \partial_{x_i}\right),$$
$$\Psi_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i}, \tag{3.37}$$

with the algebra $sl(2,\mathbb{R}) \oplus so(n)$ of $\frac{1}{2}(n^2-n+6)$ elements, is sufficient to specify completely the n-dimensional isotropic oscillator.

It is amusing to note that, when $n = 3$, the algebra above is the same as that for the equation of motion of the classical monopole [14].

We have yet to resolve the question of the complete symmetry group in the minimal sense of Andriopoulos *et al.* The number of operators in Proposition 1 is $2n+1$, the number in Proposition 2 and 4 is $\frac{1}{2}(n^2 - n + 6)$ and in Proposition 3 is $n + 2$. The operators in Proposition 3 are minimal if $n + 2 < \frac{1}{2}(n^2 - n + 6)$ and $n + 2 < 2n + 1$ *ie* $n > 2$ and $n > 1$ respectively. On the other hand the operators in Proposition 1 are fewer than those in Propositions 2 or 4 if $2n + 1 < \frac{1}{2}(n^2 - n + 6)$ *ie* $n > 4$. We observe that in any case the minimal number of operators required to specify completely (3.30) is given by those in Proposition 3. However, the operators in Proposition 3 do not form a closed algebra! Thus we need to add more symmetries in order to close the algebra and consequently lose minimality! Furthermore this is also the case for the symmetries of Proposition 2.

All the above considerations yield that the complete symmetry group of (3.30) is given by the operators in Proposition 1 since they have no problems of closure under the operation of taking the Lie Bracket. The symmetries concerned are the $2n$ solution symmetries and the diagonal homogeneity symmetry. If $n = 2, 3$, the complete symmetry group is given by the operators in Proposition 4. If $n = 4$, the same number of operators is given in both cases. We have observed before that the representation of the complete symmetry group is not unique. Here we have an instance in which the group itself is not unique.

4 Comments

In this paper we have reported on the number of symmetries and their algebras required to specify completely the second order system of ordinary differential equations describing the simple harmonic oscillator. We have seen in the case of the one-dimensional system that there are four representations of the complete symmetry group. The first of these is based upon the homogeneity and solutions symmetries. The other three representations are the three-dimensional algebras associated with the linear and phase integrals of the system. In all cases the algebra is $D \oplus_s T_2$, the representation of dilations and translations in the plane. Representations of other algebras may also specify the equation completely, but they do not satisfy the requirement of minimality introduced by Andriopoulos *et al* [2]. For this reason we reject algebras such as $A_1 \oplus sl(2, \mathbb{R})$.

In our considerations of higher order simple harmonic oscillators the obvious route to generalisation of the result for the one-dimensional oscillator was simply to add two solution symmetries for each dimension added to the system. This led to the $(2n + 1)$-dimensional algebra $A_1 \oplus_s 2nA_1$, which may be interpreted as a representation of the group of dilations and translations in the $(2n)$-plane, *ie* $D \otimes_s T_{2n}$. In the case of the isotropic simple harmonic oscillator there is another way to add symmetries to describe the system. This is to conflate the solution symmetries as well as the homogeneity symmetry, thus reducing the $2n+1$ symmetries to three only, and to allow for the elements of the rotational symmetry group represented by the generators of $so(n)$. In the case that $n = 1$ the number of elements in each is the same since there is no rotation group. For $n = 2, 3$ the fourth representation is more economical in terms of the number of symmetries and so it becomes the complete symmetry group for these two cases. For $n > 4$ the first representation is more economical and so is the complete symmetry group. In the case that $n = 4$ both algebras have the same number of elements and we have the interesting situation in which there are two groups of equal validity for the complete symmetry group. This was not an expected outcome.

The question arises, naturally one would hope, as to the situation with anisotropic linear systems. In the case of a system of n second order linear equations [5, 6, 8] it is known that there are four classes of Lie point symmetries which have been designated as the b, c, d and e symmetries. With an increase in anisotropy the numbers in some classes fall away and even classes fall away. The first casualty is the b class, the non-Cartan symmetries, which require that the system be both isotropic and diagonal. In a similar fashion the c class is reduced from three to one for an autonomous system and removed completely for a general nonautonomous system. The number of d symmetries decreases from a maximum number of n^2+n+2 to a minimum of $3n+1$. These symmetries have the nature of rescaling and their decline in number is indicative of an increasing anisotropy of the system. What remains unchanged is the number of solution symmetries, $2n$, and the diagonal rescaling symmetry. Consequently the first representation — that given in Proposition 1 — of the complete symmetry group remains unchanged for all of these linear systems. With the breaking of the rotational symmetry the $so(n)$ algebra is lost. Obviously some subset of the full rotational symmetry algebra could be retained and this could have an effect upon which form the complete symmetry group of a given system would take. In general a linear system which has a rotationally invariant subspace of two or three dimensions would be expected to have a complete symmetry group which is a hybrid the two representations. In the case that the rotationally invariant subspace is of four dimensions there will be two distinct complete symmetry groups.

Acknowledgements

PGLL expresses his appreciation of the hospitality of GEODYSYC, Department of Mathematics, University of the Aegean, during the period in which this work was undertaken and acknowledges the support of the National Research Foundation of South Africa and the University of Natal. AK acknowledges the support of the Municipality of Samos through its award of a scholarship.

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