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THE EDGE SPAN OF DISTANCE TWO LABELLINGS OF GRAPHS

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Abstract. The radio channel assignment problem can be cast as a graph coloring problem. Vertices correspond to transmitter locations and their labels (colors) to radio channels. The assignment of frequencies to each transmitter (vertex) must avoid interference which depends on the seperation each pair of vertices has. Two levels of interference are assumed in the problem we are concerned. Based on this channel assignment problem, we proposed a graph labelling problem which has two constraints instead of one. We consider the question of finding the minimum edge of this labelling. Several classes of graphs including one that is important to a telecommunication problem have been studied.

1. Introduction

Given a graph $G = (V, E)$, an $L(2,1)$-*labelling* of G is a nonnegative integral function f such that $|f(x) - f(y)| \geq 2$ if $\{x, y\} \in E$ and $|f(x) - f(y)| \geq 1$ whenever the distance between x and y is two in G.

The motivation of studying the $L(2,1)$-labelling comes from the channel assignment problem in radio system [7], where each vertex is taken to be a transmitter location, with the label (color) assigned to it determining the channel on which it transmits. Particularly, the available channels are uniformly spaced in the spectrum justifying integer labellings. The assignment of frequencies to each transmitter (vertex) must avoid interference which depends on the seperation each pair of vertices has. In our problem there are two levels of interference which correspond to two constraints in the labelling.

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There are several articles (see references [1, 3–6, 9–14]) studying the $L(2,1)$-labelling since it was proposed in [12, 6]. In these papers, most people are seeking the smallest m such that there is an $L(2,1)$-labelling f with max f-min f=m, or equivalently max f=m since the minimum label is always 0. This parameter is called the $L(2,1)$-*number* of a graph G (or the $L(2,1)$ *span* of G) and is denoted by $\lambda(G)$. Further, [2, 4] consider generalizations of $L(2,1)$-labellings. This paper focuses on another parameter of the $L(2,1)$-labelling.

Given an $L(2,1)$-labelling f on G, define the $L(2,1)$ *edge span* of f, $\beta(G,f) = \max\{|f(x) - f(y)| : \{x,y\} \in E(G)\}$. The $L(2,1)$ *edge span* of G, $\beta(G)$, is min $\beta(G,f)$, where the minimum runs over all $L(2,1)$-labellings f on G.

In the next section we discuss some results on cycles, trees and complete multipartite graphs. In Section 3 we pay attention to two kinds of graphs, the *triangular lattice* and the *square lattice*, which arise from the design of planar regions for cellular phone networks.

2. Basic Results

It is obvious that $\beta(G) \le \lambda(G)$ for any graph G. If G is a complete graph then $\beta(G) = \lambda(G)$. However, the edge span might far less than the $L(2,1)$-number (the span). To see this, we begin with several well-known graphs. Notice that if H is a subgraph of G then $\beta(H) \le \beta(G)$.

Theorem 2.1. *Let C_n be a cycle of order $n \ge 3$. Then $\beta(C_3) = 4$ and $\beta(C_n) = 3$ for $n \ge 4$.*

Proof. It is easy to see that $\beta(C_3) = 4$. Assume $n \ge 4$. Let $V(C_n) = \{v_0, v_1, \cdots, v_{n-1}\}$, where v_i is adjacent to v_{i+1} for $i = 0, 1, \cdots, n-2$ and v_{n-1} is adjacent to v_0. Since the labels are nonnegative, one of the vertices must have label 0, without loss of generality, say, v_0. By the definition of the $L(2,1)$-labelling, either the label of v_1 or the label of v_{n-1} is at least 3. Thus the $L(2,1)$ edge span of C_n with $n \ge 4$ is greater than or equal to 3.

On the other hand, consider the following labellings:

Case 1. $n = 2k+1$ with $k \ge 2$.

Label $v_0, v_1, \cdots, v_k, v_{k+1}$ with $0, 2, 4, \cdots, 2k, 2k+2$, respectively. Label v_{n-1}, $v_{n-2}, \cdots, v_{k+2}$ with $3, 5, \cdots, 2k-1$, respectively. This labelling is an $L(2,1)$-labelling with edge span 3. Thus $\beta(C_n) \le 3$.

Case 2. $n = 2k$ with $k \ge 2$.

Label $v_0, v_1, \cdots, v_k$ by $0, 2, 4, \cdots, 2k-2$, respectively and label $v_{n-1}, v_{n-2}, \cdots, v_{k+1}$ by $3, 5, 7, \cdots, 2k+1$, respectively. Again, this labelling is an $L(2,1)$-labelling with edge span 3. Hence $\beta(C_n) \le 3$.

These results assert that the theorem holds. ∎

Theorem 2.2. *Let T be a tree with maximum degree M. Then $\beta(T) = \lceil M/2 \rceil + 1$.*

Proof. Let $m = \lceil M/2 \rceil + 1$. Since the maximum degree of T is M, T contains a $K_{1,M}$ as a subtree. It is easy to see that the $L(2,1)$ edge span of $K_{1,M}$ is greater than or equal to m. So $\beta(T) \ge \beta(K_{1,M}) \ge m$. Next consider the following labelling scheme. It yields the edge span m. The theorem then follows.

Choose an arbitrary vertex of T, call it v_0, and label it by x. (At this moment we can just assume the value of x is large enough so that every label used is nonnegative. This can be done since the number of vertices is finite.) We visit the internal vertices of T by a BFS. Each time a vertex v is visited, v is labeled by some y but its children are unlabeled. As v has at most $M-1$ children, we can use labels in $\{y-2, y+2, \cdots, y-m, y+m\}$ to label these children. The labelling completes when all internal vertices are visited. ∎

Theorem 2.3. *Let $K = K_{n_1, n_2, \cdots, n_k}$ be a complete k-partite graph, where $n_1 \ge n_2 \ge n_3 \ge \cdots \ge n_k$. Then $\beta(K) = \lceil n_1/2 \rceil + n_2 + n_3 + \cdots + n_k + k - 2$.*

Proof. Let $t = \lceil n_1/2 \rceil + n_2 + n_3 + \cdots + n_k + k - 2$. Suppose $V(K) = V_1 \cup V_2 \cup \cdots \cup V_k \cup V_{k+1}$, where $V_1 \cup V_{k+1}$, $V_2, \cdots, V_k$ are partite sets of K, where $|V_1| = \lceil n_1/2 \rceil, |V_{k+1}| = \lfloor n_1/2 \rfloor$ and $|V_i| = n_u,\ i = 2, 3, \cdots, k$. Label the vertices in each V_j with consecutive integers such that the minimum label in V_1 is 0, and the minimum label of V_j is the maximum label of V_{j-1} plus 2. It is straightforward to check that the edge span of this labelling is t. Thus $\beta(K) \le t$.

Suppose f is an $L(2,1)$-labelling with $\beta(K,f) = \beta(K)$. We shall prove that $\beta(K,f) \ge t$. Let $v \in V_r$ and $u \in V_s$ be such that $f(v) = 0$ and $f(u) = p = \max_{w \in V(K)} f(w)$. Note that

$$p \ge \lambda(K) = \sum_{i=1}^{k} n_i + k - 2 = t + \lfloor n_1/2 \rfloor \ge t.$$

If $r \ne s$, then $\beta(K,f) \ge p \ge t$. So, we may assume that $r = s$. In this case, we in fact have that $p \ge t + \lfloor n_1/2 \rfloor + 1$. Let z and y be respectively, the maximum and minimum value of f on all vertices not in V_r.

$$z-y \geq \sum_{i=1}^{k} n_i - n_r + k - 3 = t + \lfloor \frac{n_1}{2} \rfloor - 1 - n_r$$

and so

$$(z-0)+(p-y) \geq 2t + 2\lfloor \frac{n_1}{2} \rfloor - n_r \geq 2t-1.$$

Therefore, either $z-0 \geq t$ or $p-y \geq t$, which implies $\beta(K,f) \geq t$. ∎

3. Triangular Lattice and Square Lattice

Define vectors $\epsilon_1 = (1,0)$ and $\epsilon_2 = (1/2, \sqrt{3}/2)$ in the Euclidean plane. Then the *triangular lattice* Λ_Δ is defined by $\Lambda_\Delta = \{i\epsilon_1 + j\epsilon_2 : i,j \in Z\}$ and the *square lattice* $\Lambda_\Delta = \mathbb{Z}^2$, wher $\mathbb{Z}$ is the set of integers. The graphs of Λ_Δ and Λ_Δ, denoted by Δ and $\square$, respectively, are defined by $V(\Delta) = \Lambda_\Delta$, $E(\Delta) = \{uv : u,v \in \Lambda_\Delta, d_E(u,v) = 1\}$, $V(\square) = \Lambda_\square$ and $E(\square) = \{uv : d_E(u,v) = 1\}$, where $d_E(u,v)$ denotes the Euclidean distance between u and v. For brevity, we simply call Δ the triangular lattice and $\square$ the square lattice, with the metric induced by the respective edge sets understood. Notice that both graphs are infinite. See Figures 1 and 2 for $\square$ and Δ, respectively.

In this section we study the $L(2,1)$ edge spans of these two classes of graphs. The triangular lattice is important to the radio engineer, since, if the area of coverage (in the Euclidean plane) of each transmitter is a disk of fixed radius r centered on the transmitter site, then placing those sites at the vertices of a regular triangular lattice (with adjacent sites a distance $r\sqrt{3}$ apart) covers the whole plane with the smallest possible transmitter density (cf. [8]). The square lattice is related to the product of two paths.

Given n,m, denote $\square_{n,m}$ the subgraph of $\square$ induced by $\{(i,j) : 0 \leq i \leq n,\ 0 \leq j \leq m\}$. Notice that $\square_{n,m}$ is isomorphic to the product graph $P_n \times P_m$, where P_n and P_m are paths of length n and m, respectively. Recall that the *product* $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \times V_2$ specified by putting (x_1, y_1) adjacent to (x_2, y_2) if and only if (1) $x_1 = x_2$ and $y_1y_2 \in E_2$ or (2) $y_1 = y_2$ and $x_1x_2 \in E_1$. Without loss of generality, we assume $n \geq m$, since $\square_{n,m}$ is isomorphic to $\square_{m,n}$. For example, the "dark" vertices in Figure 1 induce a $\square_{3,2}$.

Theorem 3.1. $\beta(\square) = \beta(\square_{n,m}) = 3$, $n \geq m \geq 1$.

Proof. Since $\square_{1,1}$ is isomorphic to C_4, by Theorem 2.1, we have $\beta(\square_{1,1}) = \beta(C_4) = 3$. For any $n \geq m \geq 1$, $\square_{n,m}$ contains a $\square_{1,1}$ as a subgraph, hence $\beta(\square) \geq \beta(\square_{n,m}) \geq \beta(\square_{1,1}) = 3$.

On the other hand, define a labelling f on $V(\square)$ by $f(i,j) = 2i+3j$, where $f(i,j)$ stands for $f((i,j))$.

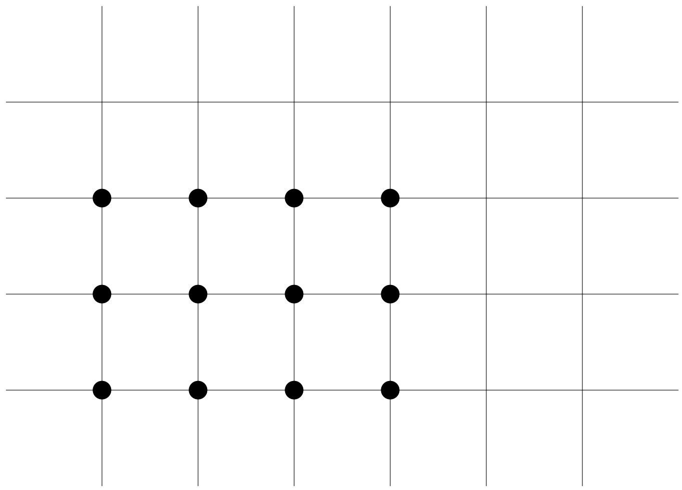

Figure 1. The square lattice $\square$

We know whenever (i_1, j_1) is adjacent to (i_2, j_2), either $|i_1 - i_2| = 1$ and $j_1 = j_2$, or $i_1 = i_2$ and $|j_1 - j_2| = 1$. Thus if (i_1, j_1) and (i_2, j_2) are adjacent then $|f(i_1, j_1) - f(i_2, j_2)|$ is either 2 or 3. If the distance between (i_1, j_1) and (i_2, j_2) is two, then $|i_1 - i_2| = 1 = |j_1 - j_2|$, $|i_1 - i_2| = 2$ and $j_1 = j_2$, or $i_1 = i_2$ and $|j_1 - j_2| = 2$. In each case $|f(i_1, j_1) - f(i_2, j_2)|$ is never 0. Therefore, we find that f is an $L(2,1)$-labelling with edge span 3. This proves the theorem. ∎

In the proof above we observe that the largest label we used is $2n + 3m$ which depends on the order of $\square_{n,m}$. We like to ask "Is there an optimal labelling with the maximum label independent of n and m?" It is unknown for us at this moment. However we do have a "near" optimal solution to the problem.

We call a labelling f a *d-$L(2,1)$-labelling* of a graph G, if f is an $L(2,1)$-labelling using labels less than or equal to d. Notice that such a d-$L(2,1)$-labelling exists provided $d \geq \lambda(G)$. The $L(2,1)$ edge span of f is denoted by $\beta(G, d, f)$. The edge span $\beta(G, d) = \min \beta(G, d, f)$, where the minimum runs over all d-$L(2,1)$-labellings of G. Thus $\beta(G) = \min\{\beta(G, d) : \text{for all possible } d\}$.

[14] has proved that $\lambda(P_n \times P_m) = 6$ for $n \geq m \geq 2$. Translating this into our language, we have $\lambda(\square_{n,m}) = 6$ for $n \geq m \geq 2$. Next we investigate $\beta(\square, d)$ when $d = \lambda(\square_{n,m}) = 6$.

Theorem 3.2. *$\beta(\square_{n,m}, 6) = 5$ for $n \geq m \geq 2$ and $\beta(\square, 6) = 5$.*

Proof. Define the labelling f from $V(\square)$ to $\{0, 1, \cdots, 6\}$ by $f(i, j) = (2i + 3j) \pmod 7$.

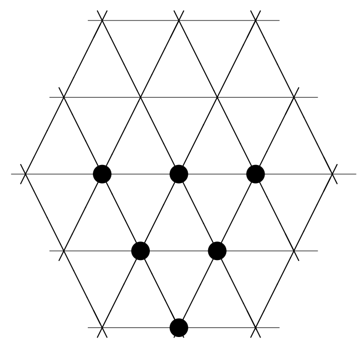

Figure 2. The triangular lattice Δ

As in Theorem 3.1, we can verify that f is a 6-$L(2,1)$-labelling with edge span 5. Therefore $\beta(\Box, 6) \leq 5$. On the other hand, we can check that $5 \leq \beta(\Box_{2,2}, 6) \leq \beta(\Box_{n,m}, 6) \leq \beta(\Box)$. The first assertion is proved. Since the somain of f above can be extended to the set $V(\Box)$ so that the argument is still correct, the second assertion is obvious. ■

In order to investigate $\beta(\Delta)$, we first study the following subgraph of Δ. Let Δ_m be the subgraph induced by $\{(i,j) : -m \leq i \leq 0,\ 0 \leq j \leq m$ and $0 \leq i+j \leq m$ for $m \geq 2\}$ in Δ. For example, "dark" vertices in Figure 2 induce a Δ_2.

Theorem 3.3. $\beta(\Delta_m) = 5$, $m \geq 2$.

Proof. Notice that $V(\Delta) = \{i\epsilon_1 + j\epsilon_2 : i, j \in Z\}$. For convenience we use (i,j) to represent a vertex $v = i\epsilon_1 + j\epsilon_2$ in Δ.

First assume $m \geq 3$. Define $f : V(\Delta) \to \{0, 1, \cdots, 5m\}$ by $f(i,j) = -3i + 2j$. Let (i_1, j_1) and (i_2, j_2) be two adjacent vertices in Δ. Then $i_1 = i_2$ and $|j_1 - j_2| = 1$, $j_1 = j_2$ and $|i_1 - i_2| = 1$, or $(i_1 - i_2)(j_1 - j_2) = -1$. For each case, $|f(i_1, j_1) - f(i_2, j_2)| \geq 2$. (The difference is either 2 or 3.)

We see that f is an $L(2,1)$-labelling with edge span 5. On the other hand, it suffices to find a subgraph of Δ_m such that the edge span is 5. This will provide the lower bound on $\beta(\Delta_m)$ and hence the theorem follows. Let W be the subgraph induced by $\{(1,0), (2,0), (0,1), (1,1), (2,1), (0,2), (1,2)\}$. It is isomorphic to the subgraph induced by the "dark" vertices in Figure 3. We can show easily that W is what we need.

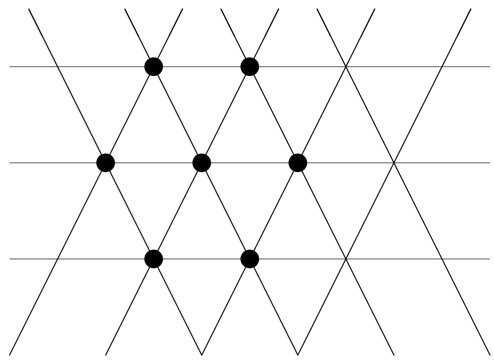

Figure 3. W

The case of $m = 2$ can be verified directly. ■

Again we are interested in seeking a number d (not dependent on m) such that $\beta(\Delta_m) = \beta(\Delta, d) = 5$ for $m \geq 2$. The following result gives us a solution closed to the best one.

Proposition 3.4. $\lambda(\Delta) = 8$.

Proof. We use the same coordinate as above. Define $f : V \to \{0, 1, 2, \cdots 8\}$ by $f(i, j) = -3i + 2j \bmod 9$. The verification for f being an $L(2, 1)$-labelling is the same as in the proof of Theorem 3.3. The maximum label of f is 8. Thus we have $\lambda(\Delta) \leq 8$.

Let H be the subgraph induced by $\{(1, 0), (2, 0), (3, 0), (0, 1), (1, 1), (2, 1), (3, 1), (0, 2), (1, 2), (2, 2)\}$ in Δ. It is isomorphic to the subgraph induced by the "dark" vertices in Figure 4. We can verify easily that $\lambda(H) = 8$. So $\lambda(\Delta) \geq \lambda(H) = 8$. Now we can conclude that $\lambda(\Delta) = 8$. ■

We also can see that the $L(2, 1)$-labelling f defined above is a labelling with edge span 7. Further by considering the $L(2, 1)$ edge span of H above, we have $\beta(\Delta, 8) \geq 7$. Hence we have the following theorem.

Theorem 3.5. $\beta(\Delta, 8) = 7$.

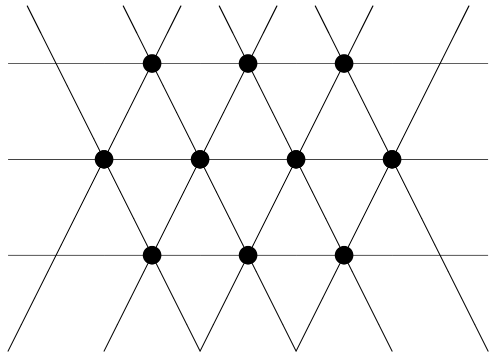

Figure 4. H

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