The edit distance for Reeb graphs of surfaces

Barbara Di Fabio[1,3], Claudia Landi[2,3]

[1]Dipartimento di Matematica, Università di Bologna

[2]Dipartimento di Scienze e Metodi dell'Ingegneria, Università di Modena e Reggio Emilia

[3]ARCES, Università di Bologna

barbara.difabio@unibo.it, claudia.landi@unimore.it

Discrete, Computational and Algebraic Topology
November 10-14, 2014

Outline

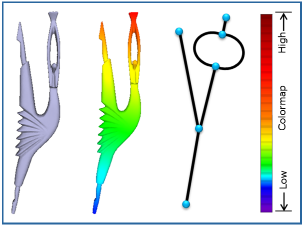
- Background on Reeb graphs
- State-of-the-art in Reeb graphs comparison
- Edit Distance between Reeb graphs of surfaces
  - combinatorial definition;
  - stability property;
  - optimality.
- Relationships with other stable metrics

Background on Reeb graphs

Definition

Let X be a topological space and $f : X \to \mathbb{R}$ a continuous function. For every $p, q \in X$, $p \sim q$ whenever p, q belong to the same connected component of $f^{-1}(f(p))$. The quotient space $X/\sim_f$ is known as the *Reeb graph* associated with f.

[Reeb, 1946]: If $f : \mathscr{M} \to \mathbb{R}$ is a simple Morse function then $R_f = \mathscr{M}/\sim_f$ is a finite simplicial complex of dimension 1.



[Shinagawa-Kunii-Kergosien, 1991]: Surface coding based on Morse theory.

State-of-the-art in Reeb graphs comparison

[Hilaga-Shinagawa-Kohmura-Kunii, 2001]: Similarity between polyhedral models is calculated by comparing Multiresolutional Reeb Graphs constructed based on geodesic distance.

- Define similarity $sim(P,Q)$ between two nodes P,Q weighted on their attributes
- Nodes with maximal similarity are paired according to rules introduced to ensure that topological consistency is preserved when matching nodes.
- The similarity between two MRGs is the sum of all node similarities:

$$SIM(R,S) = \sum_{m \in R, n \in S} sim(\overline{m}, \overline{n})$$

State-of-the-art in Reeb graphs comparison

[Biasotti-Marini-Spagnuolo-Falcidieno, 2006]: Comparison of Extended Reeb Graphs is based on a relaxed version of the notion of best common subgraph.

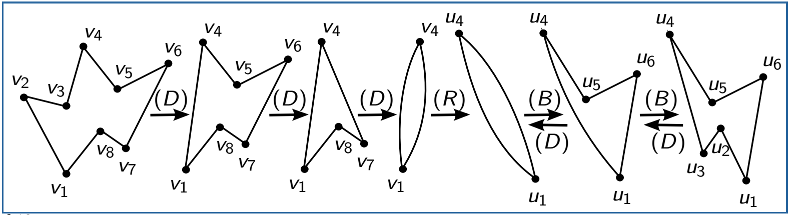
- A distance function d between two nodes v_1 and v_2 involves node and edge attributes.
- The distance measure between two graphs G_1 and G_2 is defined by

$$D(G_1, G_2) = 1 - \sum_{v \in G} \frac{(1 - d(\psi_1(v), \psi_2(v)))}{\max(|G_1|, |G_2|)}$$

  where G is the common sub-graph between G_1 and G_2, and ψ_1 and ψ_2 are the sub-graph isomorphisms from G to G_1 and from G to G_2.
- Heuristics are used to improve quality of the results and computational time

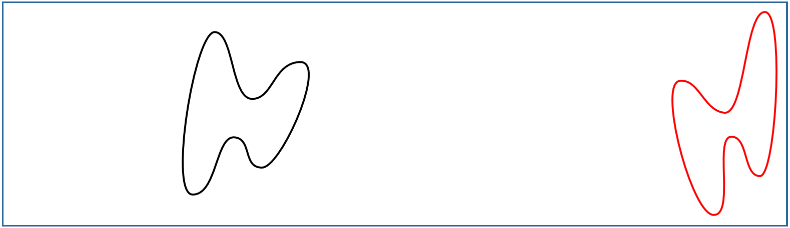
State-of-the-art in Reeb graphs comparison

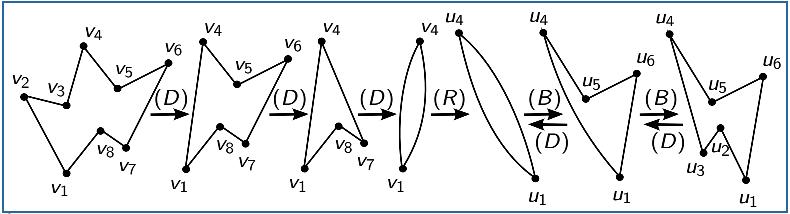
[Di Fabio-L. 2012]: Edit distance for Reeb graphs of curves endowed with simple Morse functions



State-of-the-art in Reeb graphs comparison

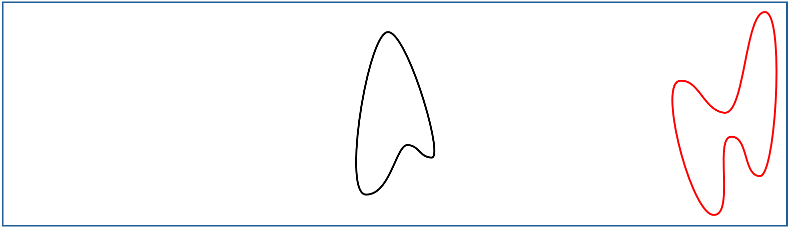
[Di Fabio-L. 2012]: Edit distance for Reeb graphs of curves endowed with simple Morse functions

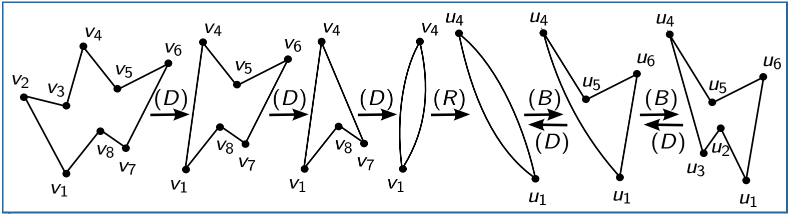




State-of-the-art in Reeb graphs comparison

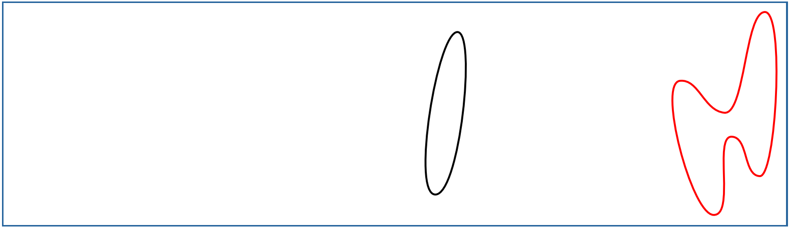
[Di Fabio-L. 2012]: Edit distance for Reeb graphs of curves endowed with simple Morse functions

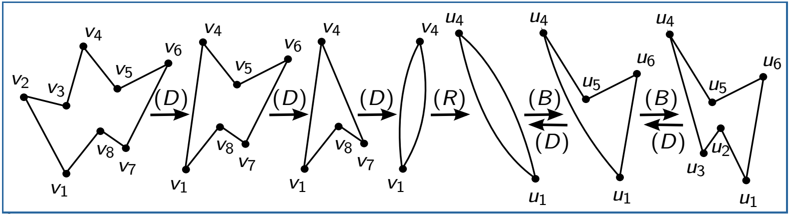




State-of-the-art in Reeb graphs comparison

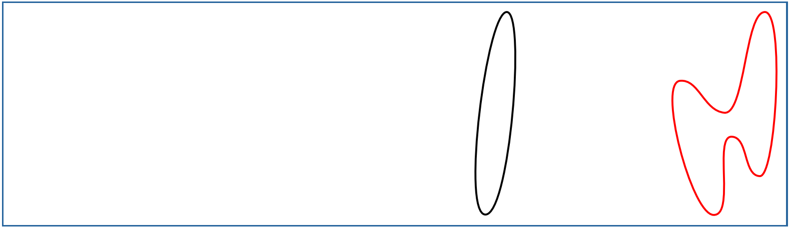
[Di Fabio-L. 2012]: Edit distance for Reeb graphs of curves endowed with simple Morse functions

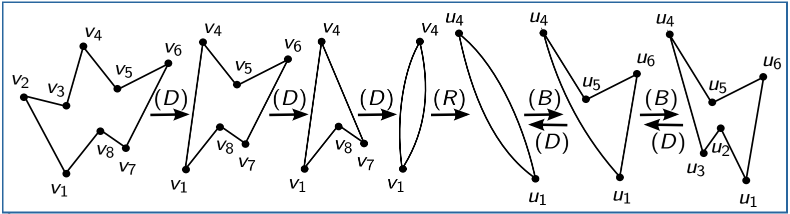




State-of-the-art in Reeb graphs comparison

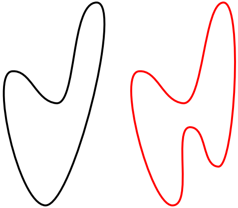
[Di Fabio-L. 2012]: Edit distance for Reeb graphs of curves endowed with simple Morse functions

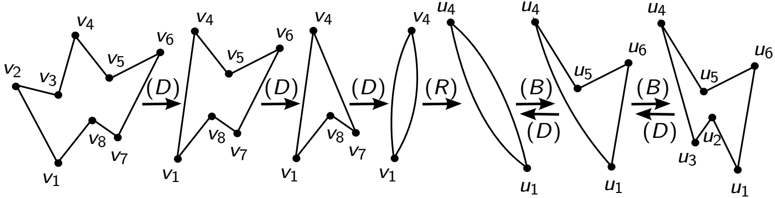




State-of-the-art in Reeb graphs comparison

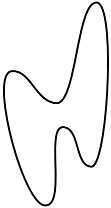
[Di Fabio-L. 2012]: Edit distance for Reeb graphs of curves endowed with simple Morse functions

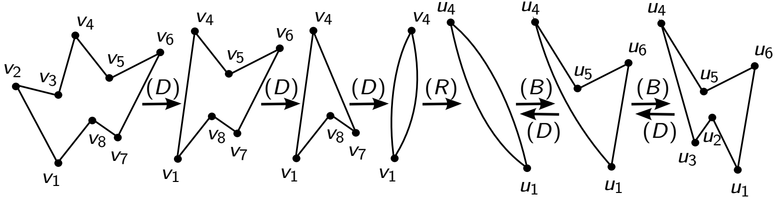




State-of-the-art in Reeb graphs comparison

[Di Fabio-L. 2012]: Edit distance for Reeb graphs of curves endowed with simple Morse functions





State-of-the-art in Reeb graphs comparison

[Bauer-Ge-Wang, 2014]: Functional distorsion distance

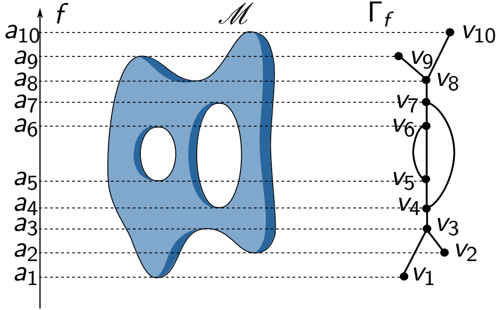
- Compares Reeb graphs R_f and R_g as topological spaces
- measures the minimum distortion in the values of f and g induced by maps $\Phi : R_f \to R_g$ and $\Psi : R_g \to R_f$
- stability property for tame functions on the same space
- more discriminative than the bottleneck distance

Edit distance for Reeb graphs of surfaces

- $\mathscr{M}$ is a connected, closed, orientable, smooth surface of genus $\mathfrak{g}$;
- $f : \mathscr{M} \to \mathbb{R}$ is a simple Morse function;

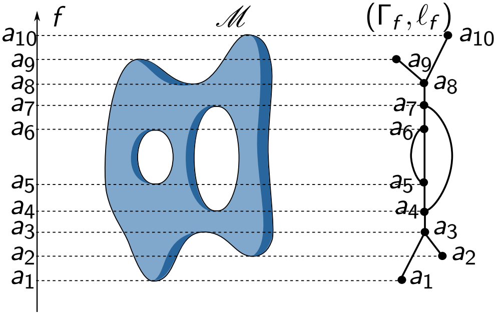
Edit distance for Reeb graphs of surfaces

- $\mathscr{M}$ is a connected, closed, orientable, smooth surface of genus $\mathfrak{g}$;
- $f : \mathscr{M} \to \mathbb{R}$ is a simple Morse function;
- there is a bijective correspondence between critical points of f and vertices of Γ_f.

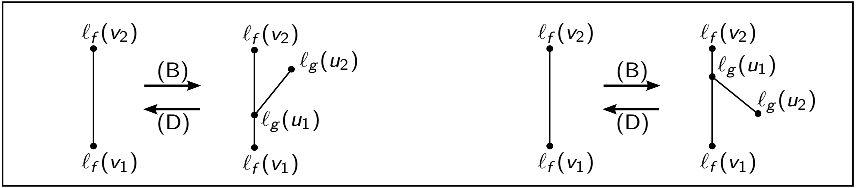


Edit distance for Reeb graphs of surfaces

- $\mathscr{M}$ is a connected, closed, orientable, smooth surface of genus $\mathfrak{g}$;
- $f : \mathscr{M} \to \mathbb{R}$ is a simple Morse function;
- each $v \in V(\Gamma_f)$ is equipped with the value of f at the corresponding critical point.



Elementary deformations, inverses, and their costs



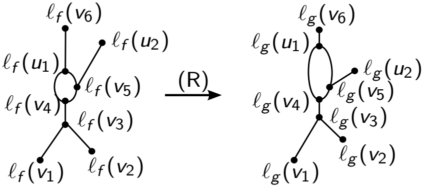
- Birth (B):

$$c(T) = \frac{|\ell_g(u_1) - \ell_g(u_2)|}{2}.$$

- Death (D):

$$c(T) = \frac{|\ell_f(u_1) - \ell_f(u_2)|}{2}.$$

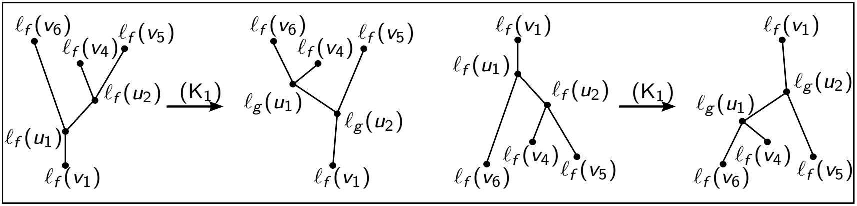
Elementary deformations, inverses, and their costs

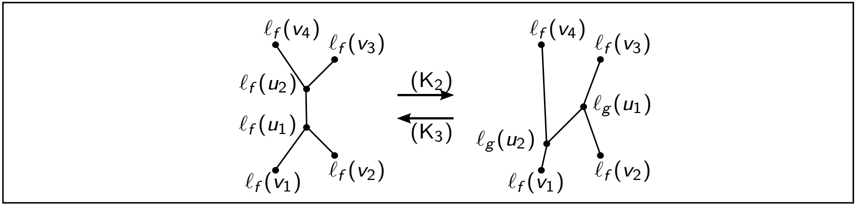


- Relabeling (R):

$$c(T) = \max_{v \in V(\Gamma_f)} |\ell_f(v) - \ell_g(v)|.$$

Elementary deformations, inverses, and their costs





- (K_i), with $i = 1, 2, 3$:

$$c(T) = \max\{|\ell_f(u_1) - \ell_g(u_1)|, |\ell_f(u_2) - \ell_g(u_2)|\}.$$

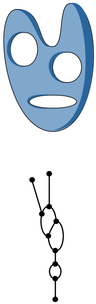
Deformations, inverses, and their costs

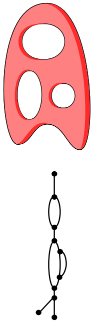
- A deformation of (Γ_f, ℓ_f) is a finite ordered sequence $T = (T_1, T_2, \dots, T_r)$ of elementary deformations such that T_i is an elementary deformation of $T_{i-1}T_{i-2}\cdots T_1(\Gamma_f, \ell_f)$ for every $i = 1, \dots, r$.

- $c(T) = \sum_{i=1}^{r} c(T_i)$.

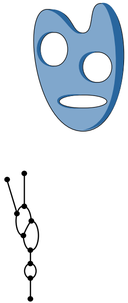
- The inverse deformation of T is $T^{-1} = (T_r^{-1}, \dots, T_1^{-1})$. Clearly, $T^{-1}(\Gamma_g, \ell_g) = T_1^{-1}\cdots T_r^{-1}(\Gamma_g, \ell_g) \simeq (\Gamma_f, \ell_f)$, and $c(T^{-1}) = c(T)$.

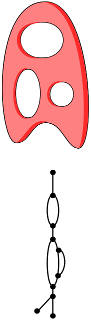
Connecting Reeb graphs by deformations



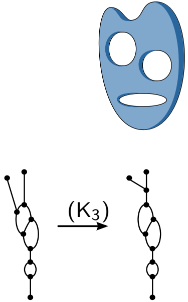


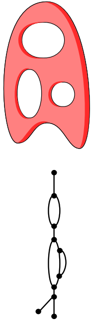
Connecting Reeb graphs by deformations





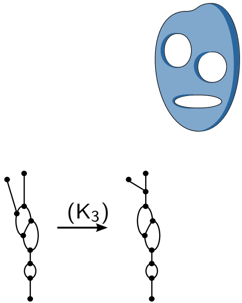
Connecting Reeb graphs by deformations

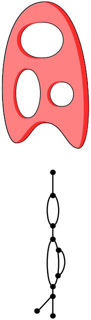




$T_1(\Gamma_f, \ell_f)$

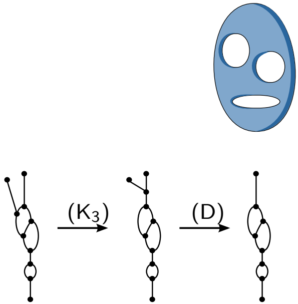
Connecting Reeb graphs by deformations

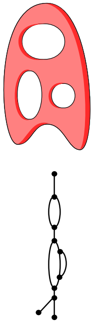




$T_1(\Gamma_f, \ell_f)$

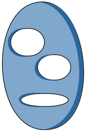
Connecting Reeb graphs by deformations

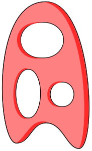


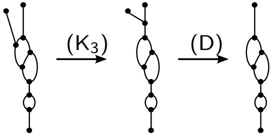


$T_2 T_1(\Gamma_f, \ell_f)$

Connecting Reeb graphs by deformations

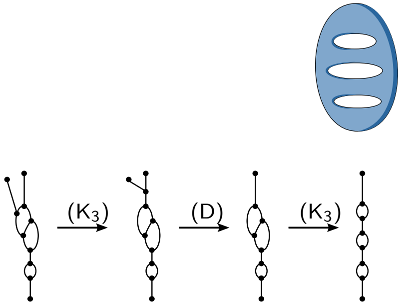


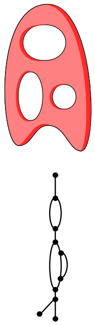




$T_2 T_1(\Gamma_f, \ell_f)$

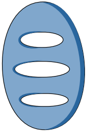
Connecting Reeb graphs by deformations

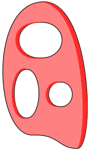


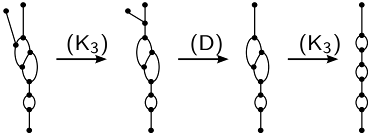


$T_3\,T_2\,T_1(\Gamma_f, \ell_f)$

Connecting Reeb graphs by deformations

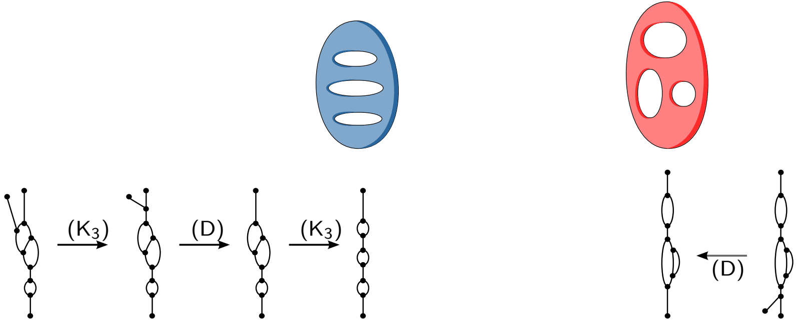






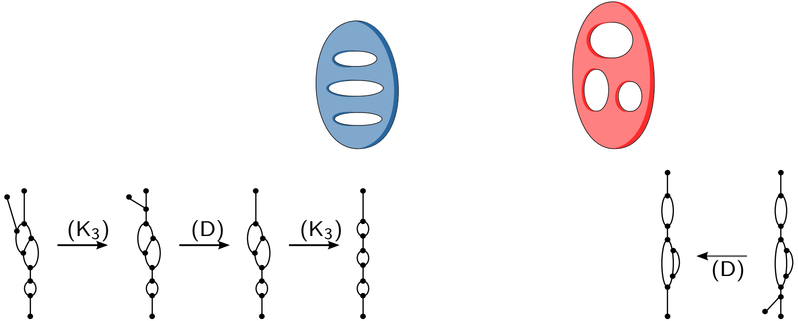
$T_3 T_2 T_1(\Gamma_f, \ell_f)$

Connecting Reeb graphs by deformations



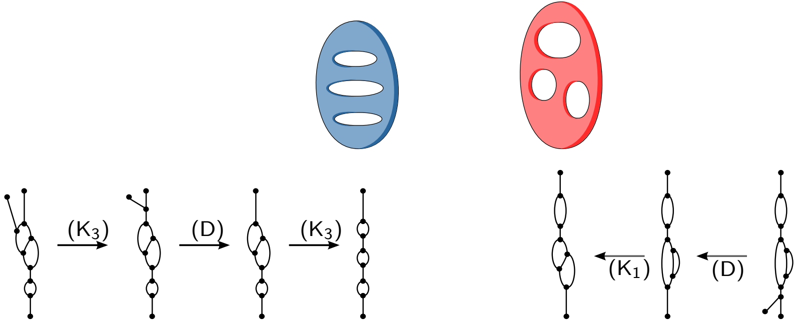
$T_3\,T_2\,T_1(\Gamma_f, \ell_f)$

Connecting Reeb graphs by deformations



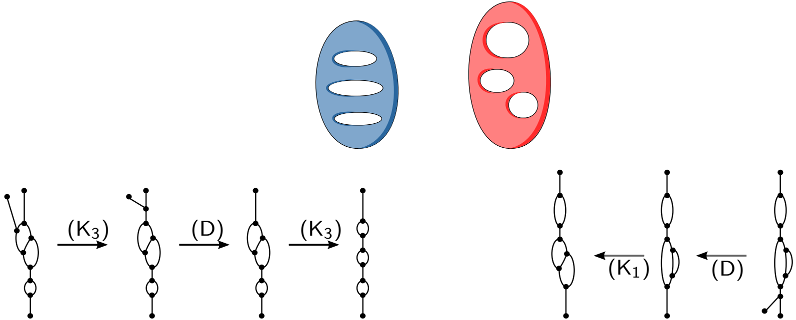
$T_3\, T_2\, T_1(\Gamma_f, \ell_f)$

Connecting Reeb graphs by deformations



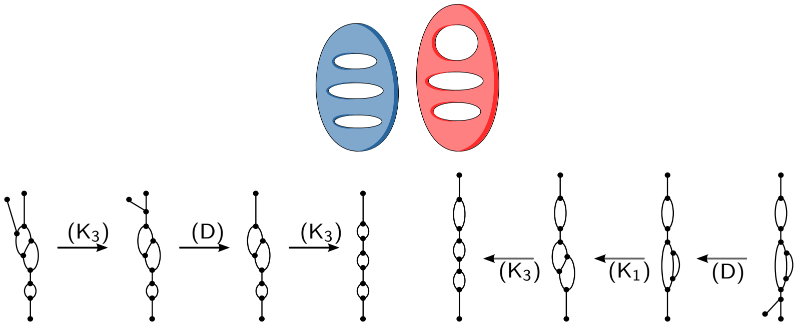
$T_3 T_2 T_1(\Gamma_f, \ell_f)$

Connecting Reeb graphs by deformations



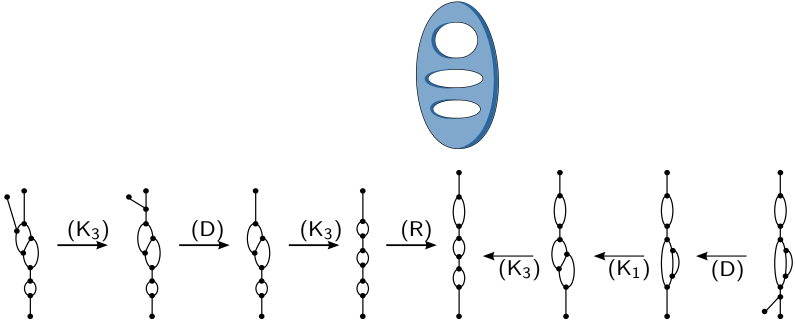
$T_3 T_2 T_1(\Gamma_f, \ell_f)$

Connecting Reeb graphs by deformations



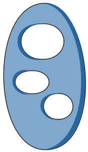
$T_3 T_2 T_1(\Gamma_f, \ell_f)$

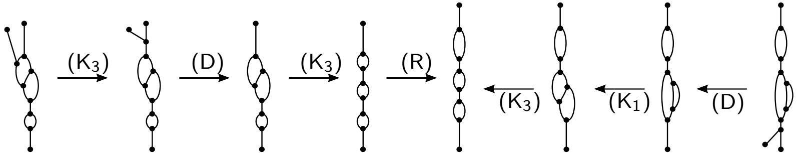
Connecting Reeb graphs by deformations



$T_4\, T_3\, T_2\, T_1(\Gamma_f, \ell_f)$

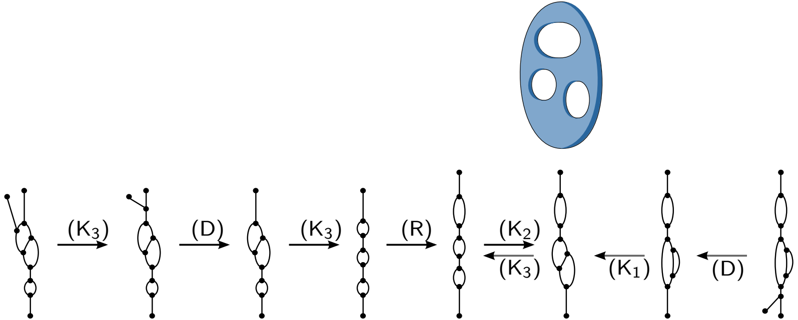
Connecting Reeb graphs by deformations





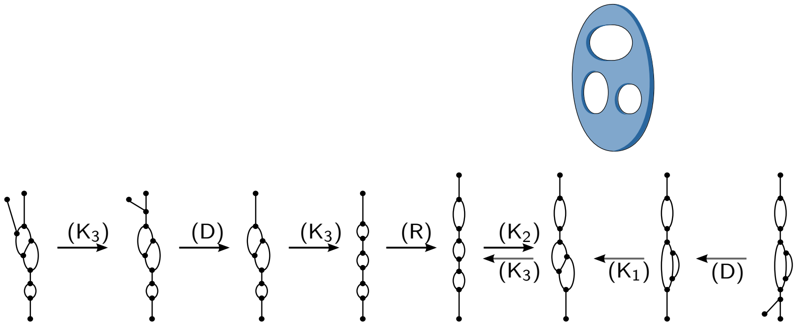
$T_4 T_3 T_2 T_1(\Gamma_f, \ell_f)$

Connecting Reeb graphs by deformations



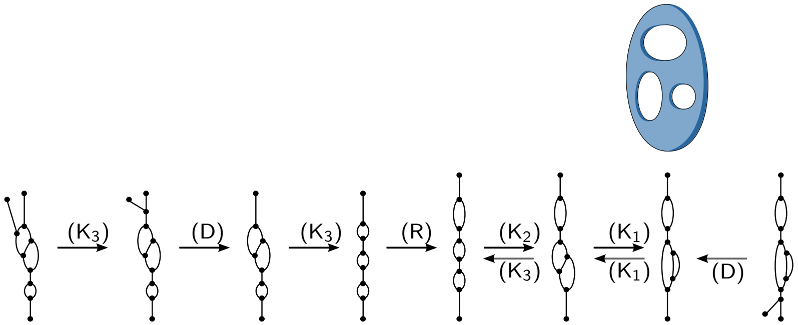
$T_5\, T_4\, T_3\, T_2\, T_1(\Gamma_f, \ell_f)$

Connecting Reeb graphs by deformations



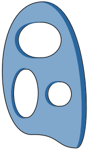
$T_5\, T_4\, T_3\, T_2\, T_1(\Gamma_f, \ell_f)$

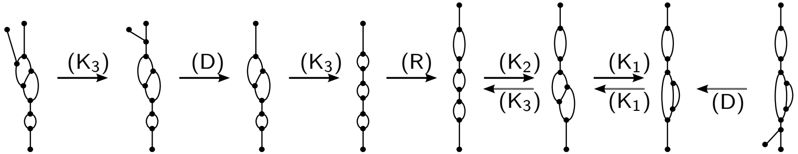
Connecting Reeb graphs by deformations



$T_6\,T_5\,T_4\,T_3\,T_2\,T_1(\Gamma_f, \ell_f)$

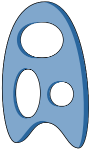
Connecting Reeb graphs by deformations

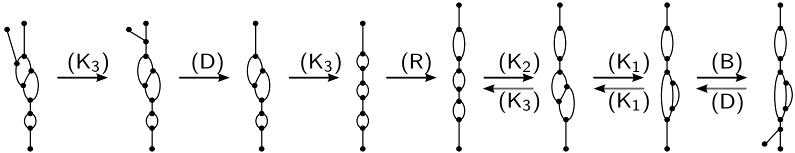




$T_6\, T_5\, T_4\, T_3\, T_2\, T_1(\Gamma_f, \ell_f)$

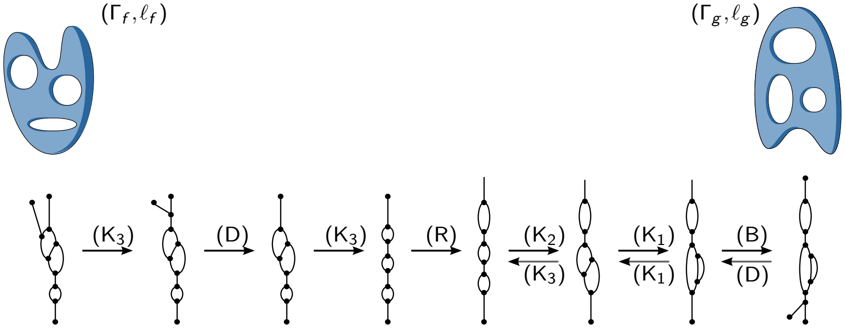
Connecting Reeb graphs by deformations





$$T_7\,T_6\,T_5\,T_4\,T_3\,T_2\,T_1(\Gamma_f,\ell_f) = (\Gamma_g,\ell_g)$$

Connecting Reeb graphs by deformations



$\Rightarrow\ T = (T_1, \ldots, T_7) \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$

The edit distance

Definition

For every two labeled Reeb graphs (Γ_f, ℓ_f) and (Γ_g, ℓ_g), we set

$$d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = \inf_{T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))} c(T).$$

The edit distance

Definition

For every two labeled Reeb graphs (Γ_f, ℓ_f) and (Γ_g, ℓ_g), we set

$$d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = \inf_{T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))} c(T).$$

Definition

$(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, if there exists an edge-preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ such that $\ell_f(v) = \ell_g(\Phi(v))$ for all $v \in V(\Gamma_f)$.

Theorem

d is a pseudo-metric on isomorphism classes of labeled Reeb graphs.

Stability property

Theorem

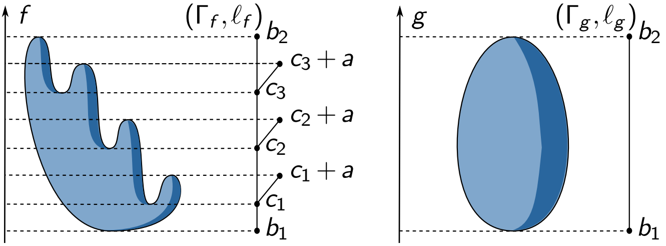
$d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \|f - g\|_\infty$.

Stability property

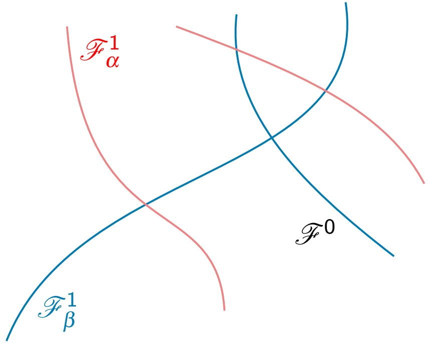
Theorem

$d((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \leq \|f-g\|_\infty.$

An example in which $d((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \leq \frac{a}{2}$:



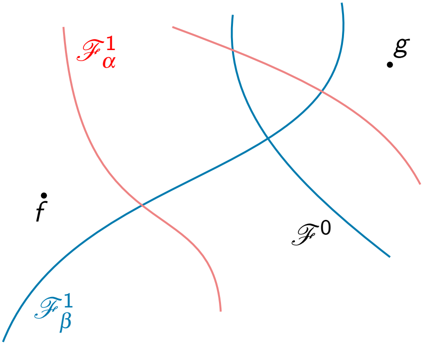
Stability property (sketch of the proof)



Let $\mathscr{F} = C^\infty(\mathscr{M}, \mathbb{R}) = \mathscr{F}^0 \cup \mathscr{F}^1_\alpha \cup \mathscr{F}^1_\beta \cup \ldots$, with

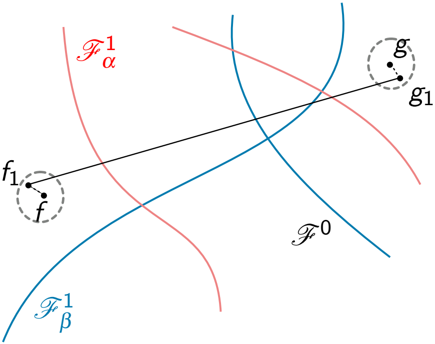
- $\mathscr{F}^0 =$ simple Morse functions;
- $\mathscr{F}^1_\alpha =$ simple functions with exactly one degenerate critical point;
- $\mathscr{F}^1_\beta =$ Morse functions with exactly one complicate point.

Stability property (sketch of the proof)



Let $f, g \in \mathscr{F}^0$. We want to find the relationship between $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ and $\|f - g\|_\infty$.

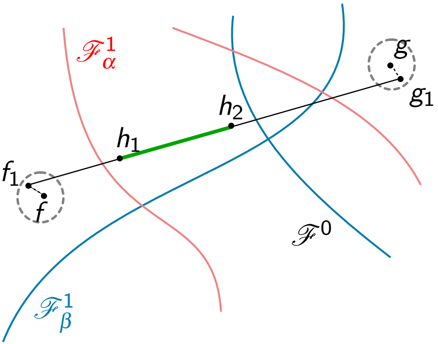
Stability property (sketch of the proof)



There exist $f_1, g_1 \in \mathscr{F}^0$ arbitrarily near to f, g, resp., for which the path $h(\lambda) = (1-\lambda)f_1 + \lambda g_1$, $\lambda \in [0,1]$, is such that

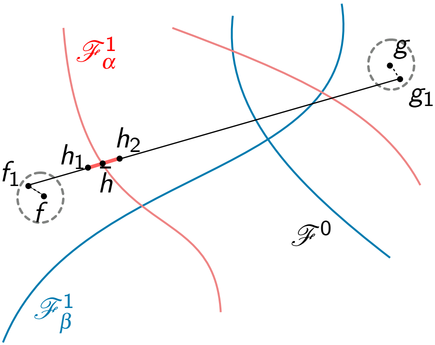
- $h(\lambda)$ belongs to $\mathscr{F}^0 \cup \mathscr{F}^1$ for every $\lambda \in [0,1]$;
- $h(\lambda)$ is transversal to $\mathscr{F}^1$.

Stability property (sketch of the proof)



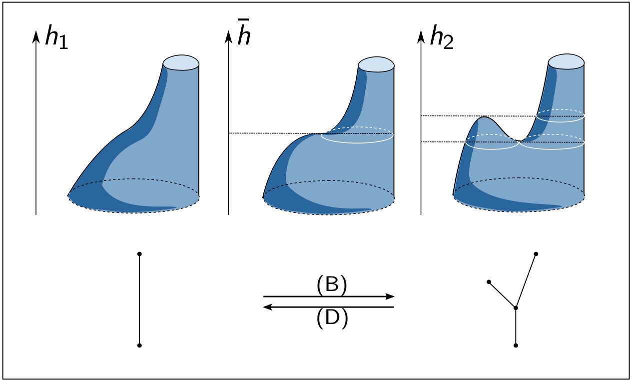
A linear path between two functions h_1, h_2 in the same connected component of $\mathscr{F}^0$ corresponds to deformations of type (R) with cost less than $\|h_1 - h_2\|_\infty$.

Stability property (sketch of the proof)

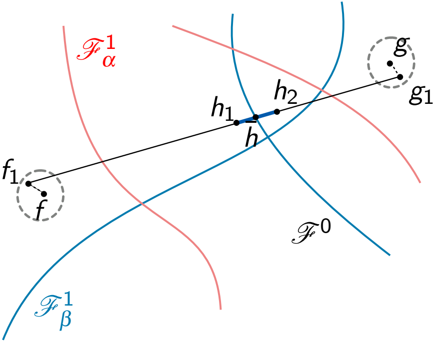


A linear path between two functions h_1, h_2 across $\mathscr{F}^1_\alpha$ corresponds to deformations of type (B) or (D) with cost less than $\|h_1 - h_2\|_\infty$.

Stability property (sketch of the proof)

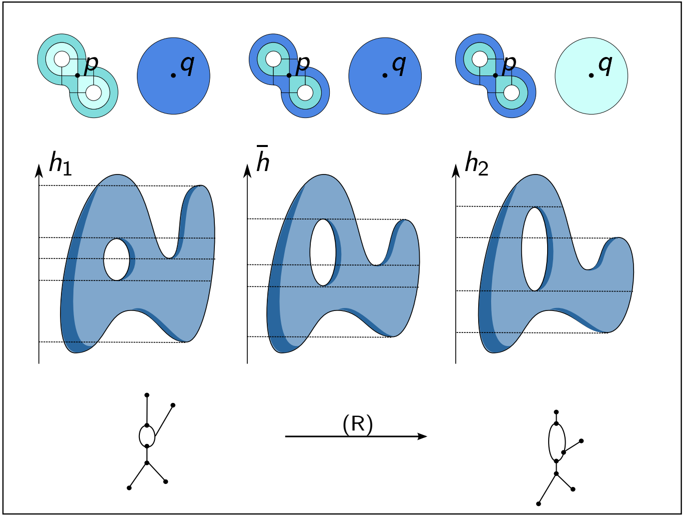


Stability property (sketch of the proof)

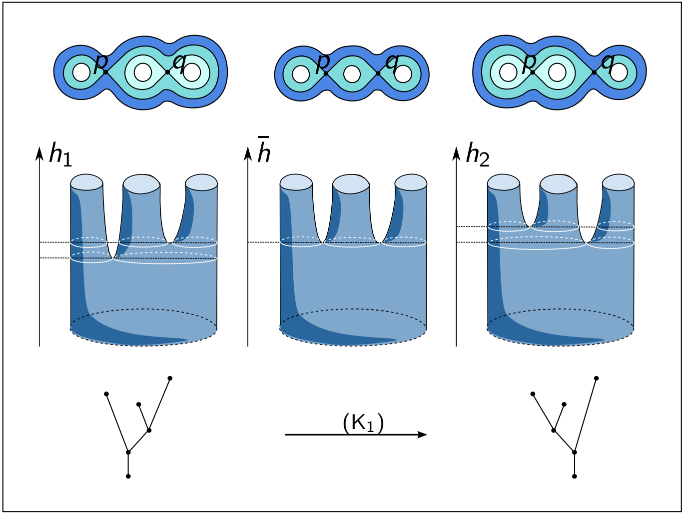


A linear path between two functions h_1, h_2 across $\mathscr{F}^1_\beta$ correspond to a deformation of type (R) or (K$_i$), $i = 1, 2, 3$ with cost less than $\|h_1 - h_2\|_\infty$.

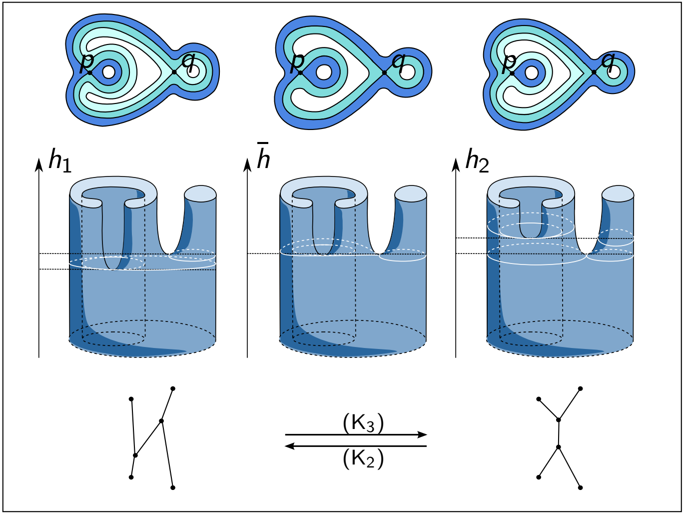
Stability property (sketch of the proof)



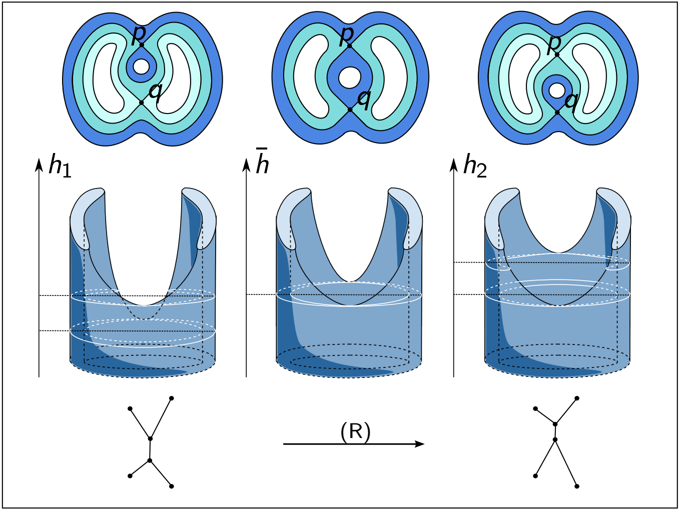
Stability property (sketch of the proof)



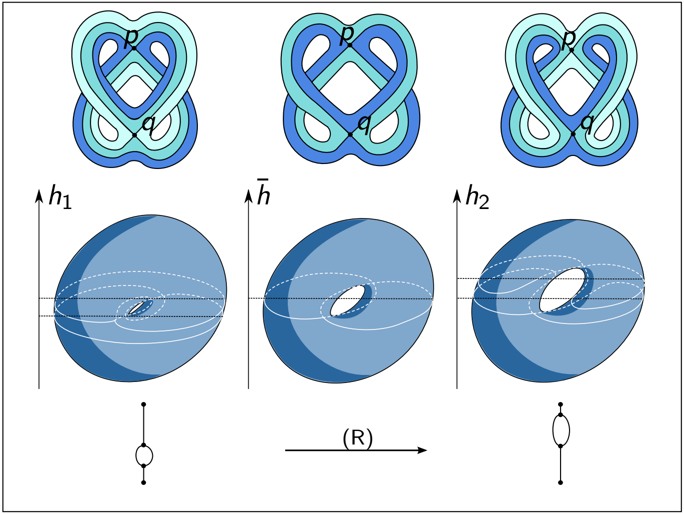
Stability property (sketch of the proof)



Stability property (sketch of the proof)



Stability property (sketch of the proof)



Optimality of the edit distance

Theorem

$$d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = \inf_{\xi \in Diff(\mathscr{M})} \|f - g \circ \xi\|_\infty.$$

Optimality of the edit distance

Theorem

$d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = \inf_{\xi \in Diff(\mathscr{M})} \|f - g \circ \xi\|_\infty.$

Theorem (Cagliari, Di Fabio, L., Forum Mathematicum)

$\delta([f],[g]) := \inf_{\xi \in Diff(\mathscr{M})} \|f - g \circ \xi\|_\infty$ is a metric on classes of simple Morse functions of surfaces up to composition with diffeomorphisms.

Corollary

d is a metric on isomorphism classes of labeled Reeb graphs.

Relationship with the bottleneck distance

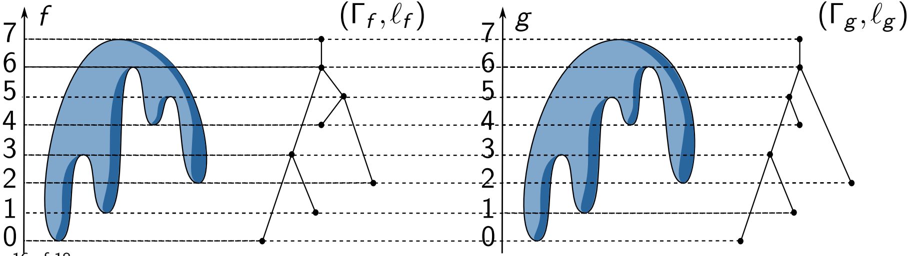
Corollary

Let D_f, D_g *denote the persistence diagrams of* f, g*, and* d_B *the bottleneck distance. It holds that* $d_B(D_f, D_g) \leq d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ *and the inequality may be strict.*

Relationship with the bottleneck distance

Corollary

Let D_f, D_g denote the persistence diagrams of f, g, and d_B the bottleneck distance. It holds that $d_B(D_f, D_g) \leq d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ and the inequality may be strict.



Relationship with the functional distortion distance

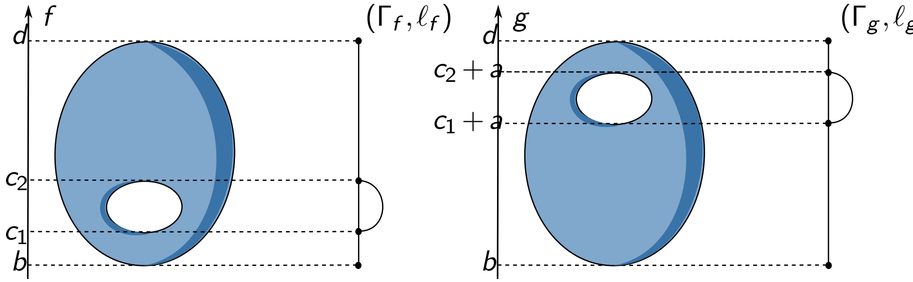
Corollary

Let R_f, R_g denote the Reeb spaces of f, g, and d_{FD} the functional distortion distance. It holds that $d_{FD}(R_f, R_g) \leq d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ and the inequality may be strict.

Relationship with the functional distortion distance

Corollary

Let R_f, R_g denote the Reeb spaces of f, g, and d_{FD} the functional distortion distance. It holds that $d_{FD}(R_f, R_g) \leq d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ and the inequality may be strict.



To do

- Generalization to the piecewise-linear case
- Generalization to the comparison of non-diffeomorphic surfaces
- Algorithm

To do

- Generalization to the piecewise-linear case
- Generalization to the comparison of non-diffeomorphic surfaces
- Algorithm

Thank you for your attention!

Preprint: http://arxiv.org/abs/1411.1544