Journal of Stochastic Analysis

Volume 2 | Number 2 Article 4

June 2021

The Edwards Model for Fractional Brownian Loops and Starbursts

Wolfgang Bock
Technische Universität Kaiserslautern, Technomathematics Group, 67663 Kaiserslautern, Germany, bock@mathematik.uni-kl.de

Torben Fattler
Technische Universität Kaiserslautern, Functional Analysis and Stochastic Analysis Group, 67663 Kaiserslautern, Germany, fattler@mathematik.uni-kl.de

Ludwig Streit
CIMA, University of Madeira, Campus da Penteada,, 9020-105 Funchal, Portugal, streit@uma.pt

Recommended Citation

Bock, Wolfgang; Fattler, Torben; and Streit, Ludwig (2021) "The Edwards Model for Fractional Brownian Loops and Starbursts," *Journal of Stochastic Analysis*: Vol. 2 : No. 2 , Article 4.
DOI: 10.31390/josa.2.2.04
Available at: https://digitalcommons.lsu.edu/josa/vol2/iss2/4

Journal of Stochastic Analysis
Vol. 2, No. 2 (2021) Article 4 (8 pages)
DOI: 10.31390/josa.2.2.04

THE EDWARDS MODEL FOR FRACTIONAL BROWNIAN LOOPS AND STARBURSTS

WOLFGANG BOCK*, TORBEN FATTLER, AND LUDWIG STREIT

ABSTRACT. We extend Varadhan's construction of the Edwards model for polymers to fractional Brownian loops and fractional Brownian starbursts. We show that, as in the fBm case, the Edwards density under a renormalizaion is an integrable function for the case $Hd \leq 1$.

1. Introduction

Fractional Brownian motion (fBm) has attracted considerable attention in recent years. This class of processes in general lacks the martingale and Markov properties so that many standard techniques from classical stochastic analysis are not available for them. For a detailed overview we refer to the monographs [7, 22, 24] and the references therein. There one now finds specific techniques and results developed in recent years such that these processes nowadays are more and more present in applications. Among them are models in finance, see e.g. [2, 3, 4] and physics [15, 21]. In particular they can also be used as a model for the conformations of chain polymers [8, 12, 16, 17], generalizing the classical Brownian models (see e.g. [25], and references therein). In this note we use results from [20] to show the existence of an Edwards model [14, 16] for fractional Brownian loops and starbursts. These geometrical objects can serve as models for ring polymers and so called dendrimers, see e.g. [26, 23]. The existence of the Edwards density as an integrable function gives rise to the analytical study of these objects as stochastic processes. We follow here closely the lines of [16] for the fractional Brownian motion case and prove the additional nessessary properties for loops and starbursts.

2. Fractional Rings

Conventionally, fBm $B^H(t), \quad t \geq 0$ is defined on half-lines as a centered Gaussian process with

$$E\left(\left(B^H(t) - B^H(s)\right)^2\right) = |t-s|^{2H} \tag{2.1}$$

FBm loops should be defined with parameter t on a circle of length T, with translationally invariant increments around that circle. Following J. Istas [20],

Received 2020-9-14; Accepted 2021-5-6; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60G15; Secondary 60G22 .
Key words and phrases. Edwards model, fractional Brownian motion, loops, Starbursts.
* Corresponding author.

this can be constructed replacing the distance $D = |t-s|$ by the geodesic one:

$$\mathbb{E}\left(\left(b^H(s)-b^H(t)\right)^2\right) = \left(\min\left(|s-t|, T-|s-t|\right)\right)^{2H} =: d^{2H}(s-t), \tag{2.2}$$

but with limitations: the covariance kernel so constructed is positive definite, and hence there is a corresponding Gaussian process if and only if the Hurst index H is small; for $H > 1/2$ this will not be the case, for the Brownian case in particular, with $H = 1/2$, see [9].

For $H \leq 1/2$ one defines d-dimensional fBm loops via a d-tuple $\mathbf{b}^H = \left(b_1^H, ...b_d^H\right)$ of independent copies of b^H.We note that d^{2H} is concave and positive on $(0,T)$ which implies (see p.89 of [5]) that $\mathbf{b}_H$ is locally non-deterministic and hence, for $0 < t_1 < \ldots < t_n$ there is a $k > 0$ such that for any vector $\mathbf{u} := (u_2, \ldots u_n) \in \mathbb{R}^{n-1}$

$$\mathbb{E}\left(\left(\sum_{i=2}^{n} u_i\left(\mathbf{b}^H(t_i)-\mathbf{b}^H(t_{i-1})\right)\right)^2\right) \geq k\sum_{i=2}^{n} u_i^2\mathbb{E}\left(\left(\mathbf{b}^H(t_i)-\mathbf{b}^H(t_{i-1})\right)^2\right), \tag{2.3}$$

by equation (2.1) of [6] and in equation (3.4) of [18].

2.1. The Self-Intersection Local Time. We define, first informally, the self-intersection local time of fBm loops as the integral

$$L = \iint_{0<s<t<T} dsdt\ \delta\left(\mathbf{b}^H(s)-\mathbf{b}^H(t)\right), \tag{2.4}$$

an expression which calls for a regularization of the Dirac δ-function, such as by the heat kernel

$$\delta_\varepsilon(x) \equiv \frac{1}{(2\pi\varepsilon)^{d/2}}e^{-\frac{|x|^2}{2\varepsilon}}, \quad x \in \mathbb{R}^d, \varepsilon > 0, \tag{2.5}$$

For $Hd < 1$, similar to the usual fBm case,

$$L = \lim_{\varepsilon\searrow 0} L_\varepsilon := \lim_{\varepsilon\searrow 0}\iint_{0<s<t<T} dsdt\ \delta_\varepsilon\left(\mathbf{b}^H(s)-\mathbf{b}^H(t)\right) \tag{2.6}$$

exists, see e.g. Theorem 1 of [19]. In particular one finds in our case from (2.4) and (2.5) that

$$\mathbb{E}(L) = \lim_{\varepsilon\searrow 0}\frac{1}{(2\pi)^{d/2}}\int_0^T dt\int_0^t ds\frac{1}{\left(d^H(t-s)+\varepsilon\right)^{d/2}}. \tag{2.7}$$

Further note that, if we split the outer integration in the middle, we can use $|s-t|^{2H}$ and $(T-|s-t|)^{2H}$ for the different areas respectively. By substitution rule we obtain from the above expression

$$\frac{1}{(2\pi)^{d/2}}\int_0^{T/2} d\tau\frac{T-\tau}{\tau^{dH}} + \frac{1}{(2\pi)^{d/2}}\int_{T/2}^{T} d\tau\frac{T-\tau}{(T-\tau)^{dH}} = \frac{T}{(2\pi)^{d/2}}\int_0^{T/2} d\tau\frac{1}{\tau^{dH}}, \tag{2.8}$$

which is finite for $Hd < 1$. The same holds true for $\mathbb{E}(L^2)$.

Theorem 2.1. *For $H \leq 1/2$ and $Hd < 1$ there exists the L^2- limit*

$$\lim_{\varepsilon\searrow 0} L_\varepsilon > 0. \tag{2.9}$$

Hence, for any $g > 0$ there exists the Edwards model, with

$$\frac{\exp(-gL)}{\mathbb{E}(\exp(-gL))} \in L^1(\nu_H) \tag{2.10}$$

as a the probability density w.r.t. the fBm measure ν_H.

2.2. The Case Hd=1. As in the fBm case for $Hd = 1$ one has to center the local time. Hence we define

$$L_{\varepsilon,c} \equiv L_\varepsilon - \mathbb{E}(L_\varepsilon). \tag{2.11}$$

Theorem 2.2. *Assume that $Hd = 1$, $d \geq 2$. Then the limit*

$$L_c \equiv \lim_{\varepsilon \searrow 0} L_{\varepsilon,c} \in L^2(\nu_H) \tag{2.12}$$

exists in $L^2(\nu_H)$ and there is a positive constant M such that for all $0 \leq g \leq M$

$$\exp(-gL_c) \tag{2.13}$$

is an integrable function.

Hence, also in this case, we have an Edwards measure, with

$$\frac{\exp(-gL_c)}{\mathbb{E}(\exp(-gL_c))} \in L^1(d\nu_H) \tag{2.14}$$

as probability density w.r.t. the fBm measure $d\nu_H$.

Proof. For the case $Hd = 1$ singularities arise for $\tau = d(t-s) \gtrsim 0$, so that the expectation of the local time diverges. The Varadhan construction requires two estimates [27, 16], namely:

$$\mathbb{E}(L_\varepsilon) = O(|\ln \varepsilon|) \tag{2.15}$$

and, after centering, i.e.

$$L_{\varepsilon,c} \equiv L_\varepsilon - \mathbb{E}(L_\varepsilon) \tag{2.16}$$

we need to show for some $K > 0$, that

$$\mathbb{E}\left((L_{\varepsilon,c} - L_c)^2\right) \leq K\varepsilon^{1/2}. \tag{2.17}$$

In this proof we elaborate these estimates for the case of loops. The first bound can be verified directly, see (2.8). To adapt the proof in [16] of the second estimate it is useful to introduce

$$\Gamma_\varepsilon = \iint\limits_{d(t-s)\geq\Delta} \delta_\varepsilon(\mathbf{b}^H(t) - \mathbf{b}^H(s)) \tag{2.18}$$

for a small positive Δ. The "gap-renormalized" Γ_ε is non-negative and finite in the limit

$$\Gamma \equiv \lim_{\varepsilon \searrow 0} \Gamma_\varepsilon \in L^2(\nu_H). \tag{2.19}$$

As a result $\exp(-g\Gamma)$ is finite for any $g > 0$.

Remark 2.3. Note that Γ_ε is not only dependent on ε but also on Δ, the same holds for Γ. To avoid too many indices it is however not included this in the notation.

Due to the fact that $\exp(-g\Gamma)$ is finite for any $g > 0$, we only need to verify the validity of the Varadhan construction for

$$\Lambda_\varepsilon \equiv \iint\limits_{d(t-s)<\Delta} \delta_\varepsilon(\mathbf{b}^H(t) - \mathbf{b}^H(s)).$$

As a first step we center Λ_ε:

$$\Lambda_{\varepsilon,c} \equiv \Lambda_\varepsilon - \mathbb{E}(\Lambda_\varepsilon). \tag{2.20}$$

For positive ε one computes

$$\begin{aligned} \mathbb{E}(\Lambda_\varepsilon^2) &= \int_{\mathcal{T}_\Delta} dsdtds'dt'\, \mathbb{E}(\delta_\varepsilon(\mathbf{b}^H(t) - \mathbf{b}^H(s))\delta_\varepsilon(\mathbf{b}^H(t') - \mathbf{b}^H(s'))) && (2.21)\\ &= \frac{1}{(2\pi)^d}\int_{\mathcal{T}_\Delta} dsdtds'dt' \left((\lambda+\varepsilon)(\rho+\varepsilon) - \mu^2\right)^{-d/2} && (2.22) \end{aligned}$$

- as in equation (13) of [19] - where now

$$\mathcal{T}_\Delta \equiv \{(s,t,s',t') \in [0,T]^4 :\ |t-s| < \Delta, |t'-s'| < \Delta\}, \tag{2.23}$$

and for $\Delta \le T/2$ with

$$\begin{aligned} \lambda &\equiv \mathbb{E}\left(\left(b^H(s) - b^H(t)\right)^2\right) = |t-s|^{2H} && (2.24)\\ \rho &\equiv \mathbb{E}\left(\left(b^H(s') - b^H(t')\right)^2\right) = |t'-s'|^{2H} \\ \mu &\equiv \mathbb{E}\left(\left(b^H(s) - b^H(t)\right)\left(b^H(s') - b^H(t')\right)\right) \\ &= \frac{1}{2}\left(d^{2H}(s-t') + d^{2H}(s'-t) - d^{2H}(t-t') - d^{2H}(s-s')\right). && (2.25) \end{aligned}$$

Following the argument in [16] we have here again the estimate

$$\mathbb{E}\left((\Lambda_{\varepsilon,c} - \Lambda_c)^2\right) \le \frac{d}{2(2\pi)^d}\int_{\mathcal{T}_\Delta} d\tau\,\rho\int_0^\varepsilon dx\left(\frac{1}{(\delta+x\rho)^{d/2+1}} - \frac{1}{((\lambda+x)\rho)^{d/2+1}}\right), \tag{2.26}$$

so, following [16], it is sufficient to show that also in the case of loops the rhs is of order $\varepsilon^{1/2}$. We then decompose $\mathcal{T}_\Delta$ into two subsets, adapting the notation of [16]

$$\mathcal{T}_{\Delta 1,2} \equiv \{(s,t,s',t') \in \mathcal{T}_\Delta :\ [s,t] \cap [s',t'] \neq \varnothing\} \tag{2.27}$$

and

$$\mathcal{T}_{\Delta 3} \equiv \{(s,t,s',t') \in \mathcal{T}_\Delta :\ [s,t] \cap [s',t'] = \varnothing\} \tag{2.28}$$

In the first subset, for any $\Delta \le T/4$, "geodesic" distances d between any pair of points are less than $T/2$, hence are equal to the ordinary ones: $d(t'-s) = |t'-s|$ etc.. Hence the estimates given in [16] for the domains $\mathcal{T}_{1,2}$ apply to the present case, and the contribution from this subdomain of the integral (2.26) is of order $\varepsilon^{1/2}$. For $\mathcal{T}_{\Delta 3}$ we assume without loss of generality $0 < s < t < s' < t'$. If $t'-s \le T/2$, all distances between points (s,t,s',t') are again less than $T/2$, and as above, the contribution from this subdomain of the integral (2.26) for sufficiently small Δ is of order $\varepsilon^{1/2}$ too. Likewise, with an exchange of variables, for the case $d(s'-t) \leq T/2$.

In the remaining case, for sufficiently small Δ, the geodesic distance b between the intervals $[s,t]$ and $[s',t']$ is large in comparison to Δ This corresponds to the second sub-region

$$t-s<\Delta \ll b, \quad t'-s'<\Delta \ll b \tag{2.29}$$

of $\mathcal{T}_3$ considered in the proof of Proposition 1 in [16]. Recalling that the periodic fBm also is locally nondeterministic, we conclude as in the corresponding proof of Lemma3.1(3) of [18] that for a sufficiently small $k>0$,

$$\lambda\rho - \mu^2 \geq k\lambda\rho \tag{2.30}$$

also here. Hence the arguments in Lemma 6 and Lemma 7 of [16] carry over to the case at hand, and in conclusion the $O(\varepsilon^{1/2})$ bound holds for $\mathbb{E}\left((L_{\varepsilon,c}-L_c)^2\right)$. □

3. Starbursts

A generalization of centered Gaussian random paths, such as e.g.

$$\mathbf{X} = \{\mathbf{x}_k(t_k) : \quad \mathbf{x}_k(0)=0; \quad k=1,\dots,n; \quad 0 \leq t_k \leq T_k\}, \tag{3.1}$$

branching out from a common starting point, is often called a "starburst" or "dendrimer" in applications, see e.g. [13] and chapter 1 in[25]. For a Brownian motion, one simply obtains n Brownian motions starting in zero.

For the definition of an fBm starburst $\mathbf{X} = \vec{\beta}^H$, following [20] one will want to maintain the characteristic fractional correlations between different branches, i.e.

$$\mathbb{E}\left(\left(\vec{\beta}^H_k(s) - \vec{\beta}^H_l(t)\right)^2\right) = d^{2H}_{kl}(s,t), \tag{3.2}$$

where now the geodesic distance

$$d_{kl}(s,t) = \begin{cases} |s-t| & \text{if} \quad k=l \\ s+t & \text{if} \quad k\neq l \end{cases}. \tag{3.3}$$

This means that on the same branch, i.e. in the case $k=l$ the distance is given by the absolute value, hence as for standard processes defined on the real line. For the case of two different branches, one connects the distance via the common starting point, which is set to zero. Hence one obtains $s+t$ for $k\neq l$. We hence obtain for $H \neq \frac{1}{2}$ a long-range interaction also between different branches.

We denote the corresponding Gaussian measure by $\mu(H,n)$.

As shown by Istas [20] such an extension of fBm is again viable whenever the Hurst index H is no larger than 1/2. (For $H=1/2$ this produces simply an n-tuple of independent Brownian motions.)

For $k\neq l$ we set, first informally,

$$L_{kl} = \int_0^{T_k} ds \int_0^{T_l} dt \delta\left(\beta^H_k(s) - \beta^H_l(t)\right) \tag{3.4}$$

and for $k=l$ we define the centered local times $L_{k,c}$ as in [16].

Then consider

$$L(g) \equiv \sum_k g_k L_{k,c} + \sum_{l<k} g_{kl} L_{kl} \tag{3.5}$$

for positive g_k and g_{kl}, $\quad 1 \leq k, l \leq n$.

For shorthand we write $g > 0$ for $g_k > 0$ and $g_{kl} > 0$, for all $1 \leq k, l \leq n$.

$L_{k,c}$ by itself is well-defined for $Hd < 1$ and controllable à la Varadhan for $Hd = 1$ and small positive g; and the L_{kl} are bounded. Hence, for $Hd < 1$ we have as before

Theorem 3.1. *For $H \leq 1/2$ and $H < 1/d$ there exists the L^2 limit*

$$\lim_{\varepsilon \searrow 0} L_\varepsilon(g) > 0. \tag{3.6}$$

So for any $g > 0$ there exists the Edwards model, with

$$\frac{\exp\left(-L(g)\right)}{\mathbb{E}\left(-L(g)\right)} \in L^1\left(\mu(H,n)\right) \tag{3.7}$$

as a probability density w.r.t. the Gaussian measure measure $d\mu(H,n)$.

Theorem 3.2. *Assume that $Hd = 1$, $d \geq 2$. Then the limit*

$$L_c(g) \equiv \lim_{\varepsilon \searrow 0} L_{\varepsilon,c}(g) \in L^2\left(\mu(H,n)\right) \tag{3.8}$$

exists and there exists a positive constant M such that for all $0 \leq g_k \leq M$

$$\exp(-L_c(g)) \tag{3.9}$$

is an integrable function.

Proof. For sufficiently small $g > 0$ we have that $\exp\left(-g_k L_{k,c}\right) \epsilon L^1(\mu(H,n))$. Hence we can choose $g > 0$ such that $\exp\left(-g_k L_{k,c}\right) \epsilon L^n$ and have

$$\exp\left(-\sum_{k=1}^{n} g_k L_{k,c}\right) \in L^1(\mu(H,n)). \tag{3.10}$$

The L_{kl} are non-negative and bounded, hence, for arbitrary $g_{kl} \geq 0$

$$\exp(-L_c(g)) = \exp\left(-\sum_{k=1}^{n} g_k L_{k,c} - \sum_{k>1}^{n} g_{kl} L_{kl}\right) \in L^1(\mu(H,n)). \tag{3.11}$$

□

Remark 3.3. Based on the construction of fBm on metric trees as parameter space by [20], vast generalizations of this last result seem possible.

4. Conclusion and Outlook

In this note we have generalized the methods from [16] from fBm to fractional loops and starbursts. These loops and starbursts can serve as models for the coformations of ring polymers and dendrimers in solvents. The existence of the Edwards density is the starting point for further analytical study of these models.

References

1. Albeverio, S., Oliveira, M. J. and Streit, L., *Intersection Local Times of Independent Brownian Motions as Generalized White Noise Functionals*. Acta Appl. Math. **69**, (2001).
2. Androshchuk, T. and Mishura, Y., *Mixed Brownian fractional Brownian model: absence of arbitrage and related topics*. Stochastics. **78** (2006).
3. Bäuerle, N. and Desmettre, S., *Portfolio Optimization in Fractional and Rough Heston Models*, Papers 1809.10716, arXiv.org, revised May 2019.
4. Bender, C., Sottinen, T., and Valkeila, E., *Fractional processes as models in stochastic finance*. In Advanced mathematical methods for finance (pp. 75-103). Springer, Berlin, Heidelberg.(2011)
5. Berman, S., *Local Nondeterminism and Local Times of Gaussian Processes*. Indiana Univ. Math. J. **23**, 69–94 (1974).
6. Berman, S., *Self-intersections and local nondeterminism of Gaussian processes*. Ann. Probab. **19**, 160–191 (1991).
7. Biagini, F. , Hu, Y., Øksendal, B., and Zhang, T., *Stochastic Calculus for Fractional Brownian Motion and Applications*. Probability and Its Applications. Springer, 2007.
8. Bock, W., Bornales, J., Cabahug, C., and Streit, L. : *Scaling Properties of Weakly Self-Avoiding Fractional Brownian Motion in One Dimension.* J. Stat. Phys. **161**, 1155–1162 (2015).
9. Bock, W., Bornales, J. and Streit, L., *Dynamical Properties of Gaussian Chains and Rings with Long Range Interactions*. Papers 1907.09850, arxiv.org, July, 2019.
10. Bock, W., Oliveira, M. J., Silva, J. L. and Streit, L., *Polymer Measure: Varadhan's Renormalization Revisited*. Rev. Math. Phys. **27**, (2015).
11. Bornales, J., Oliveira, M. J. and Streit, L., *Chaos Decomposition and Gap Renormalization of Brownian Self-Intersection Local Times*. Rep. Math. Physics **77**, 141-152 (2016).
12. Cherayil, B. J. and Biswas, P., *Path integral description of polymers using fractional Brownian walks.* J. Chem. Phys. **99**, 9230 (1993).
13. de Gennes, P.G., and Hervet, H. Statistics of starburst polymers. Journal de Physique Lettres, 44(9), 351-360. (1983).
14. Edwards, S. F., The statistical mechanics of polymers with excluded volume. *Proc. Phys. Soc.*, **85**:613–624, (1965).
15. Ernst, D., Hellmann, M., Köhler, J. and Weiss, M., *Fractional Brownian motion in crowded fluids.* Soft Matter, **8** (18), (2012).
16. Grothaus, M., Oliveira, M.J., Silva, J.-L., Streit, L., *Self-avoiding fractional Brownian motion - The Edwards model*. J. Stat. Phys. **145**, 1513–1523 (2011).
17. Hammouda, B., *Probing Nanoscale Structures - The SANS Toolbox*, p.293ff. http://www.ncnr.nist.gov/staff/hammouda/
18. Hu, Y., *Self-intersection local time of fractional Brownian motions - via chaos expansion. J. Math. Kyoto Univ.*, **41**, 233–250 (2001).
19. Hu, Y. and Nualart,D., Renormalized self-intersection local time for fractional Brownian motion. Ann. Probab. **33**, 948-983 (2005).
20. Istas, J., *Hyperbolic Fractional Brownian Motion*. Elect. Comm. in Probab. **10**, 254-262 (2005).
21. Jae-Hyung, J. and Metzler, R., *Fractional Brownian motion and motion governed by the fractional Langevin equation in confined geometries*. Physical Review E **81**, (2) (2010).
22. Mishura, Y. S., *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg, Berlin, Heidelberg, 2008.
23. Mousavia, S. M., Gompper, G., and Winkler, R. G. *Active Brownian ring polymers*- J. Chem. Phys. **150**, (2019).
24. Nourdin, I., *Selected aspects of fractional Brownian motion*. Vol. 4. Springer, 2012.
25. Rubinstein, M., Colby, R.H., *Polymer Physics*. Oxford U. Press 2003.
26. Tomalia, D. A., et al. . *A New Class of Polymers: Starburst-Dendritic Macromolecules. Polymer J.* ***17**, 117-132 (1985).*

27. Varadhan, S. R. S., Appendix to "Euclidean quantum field theory" by K. Symanzik. In R. Jost, editor, *Local Quantum Theory*, New York, 1969. Academic Press.

Wolfgang Bock: Technische Universität Kaiserslautern, Technomathematics Group, 67663 Kaiserslautern, Germany
E-mail address: bock@mathematik.uni-kl.de

Torben Fattler: Technische Universität Kaiserslautern, Functional Analysis and Stochastic Analysis Group, 67663 Kaiserslautern, Germany
E-mail address: fattler@mathematik.uni-kl.de

Ludwig Streit: CIMA, University of Madeira, Campus da Penteada,, 9020-105 Funchal, Portugal, & BiBoS, Universität Bielefeld, Germany
E-mail address: streit@uma.pt