The Effect of a Singular Perturbation on Nonconvex Variational Problems

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Abstract

We study the effect of a singular perturbation on certain nonconvex variational problems. The goal is to characterize the limit of minimizers as some perturbation parameter $\varepsilon \to 0$. The technique utilizes the notion of "Γ-convergence" of variational problems developed by DE GIORGI. The essential idea is to identify the first nontrivial term in an asymptotic expansion for the energy of the perturbed problem. In so doing, one characterizes the limit of minimizers as the solution of a new variational problem. For the cases considered here, these new problems have a simple geometric nature involving minimal surfaces and geodesics.

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Introduction

We are concerned with the effect of a singular perturbation on a nonconvex variational problem. The goal is to characterize the asymptotic behavior of minimizers in the limit as some perturbation parameter $\varepsilon \to 0$; this goal is achieved by showing that the minimizers converge to a limit which solves a new variational problem. For the cases considered here, these new problems have a simple geometric nature involving minimal surfaces and geodesics.

Our approach uses a tool developed by DE GIORGI called "Γ-convergence" of variational problems ([1], [7]). The fundamental idea is to identify the first nontrivial term in an asymptotic expansion for the *energy* of the perturbed problem. In doing so one characterizes the desired limit of minimizers as a solution of a new variational problem, the "Γ-limit" of the perturbed sequence of functionals.

Before perturbation, the variational problems we study are mathematically trivial. Beginning with a functional $F: L^1(\Omega) \to \boldsymbol{R}$ ($\Omega \subset \boldsymbol{R}^n$, open, bounded) given by

$$F(u) = \int_\Omega W(u)\, dx$$

with $W \geqq 0$ and $W(t) = 0$ at more than one t, consider the problem:

$$(P) \qquad \inf F(u)$$

for u possibly subject to a constraint such as $\int_\Omega u\, dx = c$, and for a variety of nonconvex W. Problem (P) has a chronic failure of uniqueness for such W: a piecewise constant absolute minimizer is determined by any partitioning of the domain into regions so as to accommodate the constraint. If, for example, minimization of F models a physical problem, then this nonuniqueness might be due to the neglect of some small effect. Restoring the effect through the addition of a singular perturbation might then resolve this failure of uniqueness. Choosing $\varepsilon^2 |\nabla u|^2$ as perhaps the simplest possible perturbation, we are led to the functional

$$F_\varepsilon(u) = \int_\Omega W(u) + \varepsilon^2 |\nabla u|^2\, dx$$

and the problem

$$(P_\varepsilon) \qquad \inf F_\varepsilon(u).$$

$$u \text{ constrained as in } (P)$$

Our goal is to characterize $u_0 = \lim_{\varepsilon \to 0} u_\varepsilon$ *(in* $L^1(\Omega)$*), where* u_ε *is a solution of* (P_ε).

Since minimizers of (P) have a purely geometric characterization, one might expect the same of the criterion which selects a limit u_0. We shall show that this is indeed the case by establishing that u_0 solves a new variational problem

$$(P_0) \qquad \inf_{u \in BV(\Omega)} F_0(u),$$

where

$$(\inf F_\varepsilon) = \varepsilon\, (\inf F_0) + o(\varepsilon).$$

Solutions of (P_0) typically involve a partition of Ω into regions separated by minimal surfaces or surfaces of constant curvature.

Often this partition problem (P_0) is easy to solve directly. In that case, the technique also yields information on the structure of constrained minimizers and the existence of local minimizers of (P_ε).

The analysis of the problem by this method clearly differs from the more classical approach of matched asymptotic expansions: the focus here is on the asymptotic behavior of the energy of (P_ε) rather than on an expansion for u_ε itself. Furthermore, in the classical approach one knows (or presumes) the location of a boundary layer, whereas one of our tasks is to *determine* its location. The two viewpoints, however, are not unrelated. The identification of (P_0) requires the construction of a sequence of functions $\{\varrho_\varepsilon\}$ which efficiently traverse this boundary layer in bridging the zeros of W, in close analogy with the notion of an "inner expansion". For a rigorous analysis of (P_ε) with a Dirichlet condition using matched asymptotic expansions, see the work of BURGER & FRAENKEL ([2]). Many others have studied similar problems by this approach. (See *e.g.* CAGINALP [4] and HOWES [17].)

When it applies, the advantages of the Γ-convergence technique are numerous: the problem (P_0) determines the location of the interior boundary layer, the analysis is considerably easier, and, as will be discussed later (see remark (1.14)), the results are immediately adaptable to continuous perturbations of (P_ε).

An earlier application of this technique to (P_ε) was carried out by MODICA & MORTOLA ([22], [23]), who obtained the Γ-limit for the unconstrained problem with various choices of scalar-dependent W. Our results generalize this work and the approach borrows many ideas from these authors.

In Section 1, we consider $W: \boldsymbol{R} \to \boldsymbol{R}$ having exactly two zeroes, a and b, and we attach the constraint

$$\int_\Omega u\,dx = c, \qquad \text{where } a\,|\Omega| < c < b\,|\Omega|$$

($|\cdot| = n$-dimensional Lebesgue measure). A typical minimizer of the unperturbed problem (P) might then take the form of Figure 1. GURTIN ([13], [14]) raised the question of describing limits of minimizers of (P_ε) with these conditions as a model for obtaining the stable density distributions u for a fluid confined to a container Ω, within the context of the Van der Waals-Cahn-Hilliard theory of phase transitions. Recent contributions to this problem include the work of NOVICK-COHEN

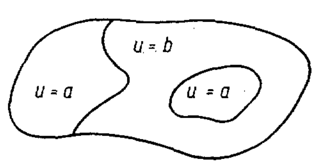

Fig. 1. A Typical Minimizer of (P)

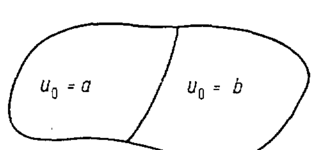

Fig. 2. Solution of (P_0): $u_0 = \lim_{\varepsilon\to 0} u_\varepsilon$.

& SEGEL ([24]) and CARR, GURTIN & SLEMROD ([5]). The latter group of authors prove that in one dimension, stable minimizers of (P_ε) are monotone, and their limit is a step function with only one discontinuity.

Our Theorem 1 generalizes this result to $\Omega \subset \boldsymbol{R}^n$. It says that any limit point of $\{u_\varepsilon\}$ must minimize

$$\inf_{\substack{u \in BV(\Omega) \\ W(u(x))=0 \text{a.e.} \\ \int_\Omega u dx = c}} \operatorname{Per}_\Omega \{u = a\},$$

where $\operatorname{Per}_\Omega A$ = perimeter of A in Ω, and $BV(\Omega)$ = space of functions of bounded variation, defined *e.g.* in ([11]). Thus, as $\varepsilon \to 0$ the minimizers of (P_ε) select a function u_0 that minimizes the area of the interface separating the states $u = a$ and $u = b$ (see Figure 2).

Essentially the same result has been proved recently also by MODICA ([20]).

Section 1 also includes, in Theorem 2, a generalization of Theorem 1 to a spatially dependent W. The associated limiting problem (P_0) which u_0 solves is then a weighted partitioning problem.

In Section 2 we consider generalizations to vector-dependent W. For W: $\boldsymbol{R}^2 \to \boldsymbol{R}$, zero on two disjoint simple closed curves, and positive elsewhere, Theorem 3 uses the techniques of Γ-convergence to show that a limit of minimizers of (P_ε) must satisfy the minimal interface criterion which arose in the scalar case. Theorem 3 also characterizes the cost—per unit area along the interface—of the transition made by the minimizers $u_\varepsilon : \Omega \to \boldsymbol{R}^2$; we show that it tends asymptotically to the distance between the two zero curves of W, measured with respect to a degenerate Riemannian metric in the plane derived from W.

1. Scalar Dependent Energy

A. Functions of Bounded Variation

We describe first some of the basic definitions and properties of functions of bounded variation; we will need these to arrive at the partitioning problem (P_0). For a more complete description, see ([11]).

Throughout the paper Ω will be an open, bounded subset of $\boldsymbol{R}^n$ with Lipschitz-continuous boundary. For $u \in L^1(\Omega)$, define:

$$\int_\Omega |\nabla u| := \sup_{\substack{g \in C_0^1(\Omega, \boldsymbol{R}^n) \\ |g| \leq 1}} \int u(x)\,(\nabla \cdot g(x))\,dx. \tag{1.1}$$

The space of functions of bounded variation, $BV(\Omega)$, consists in those $u \in L^1(\Omega)$ for which $\int_\Omega |\nabla u| < \infty$; $BV(\Omega)$ is a Banach space under the norm:

$$\|u\|_{BV(\Omega)} = \int_\Omega |u|\,dx + \int_\Omega |\nabla u|.$$

Notice that $|\nabla u|$ is not an L^1 function, but rather the total variation of the vector-valued measure ∇u. (See [9], p. 349.) If $u \in BV(\Omega)$, the integral of any positive, continuous function h with respect to the measure $|\nabla u|$ can be expressed as

$$\int_\Omega h(x)\,|\nabla u| = \sup_{\substack{g \in C_0^1(\Omega, R^n) \\ |g(x)| \leqq h(x)}} \int_\Omega u(x)\,(\nabla \cdot g(x))\,dx. \tag{1.2}$$

An important example is the case when $u = \chi_A$, the characteristic function of a subset A of R^n. Then

$$\int_\Omega |\nabla u| = \sup_{\substack{g \in C_0^1(\Omega, R^n) \\ |g| \leqq 1}} \int_A (\nabla \cdot g(x))\,dx.$$

If this supremum is finite, A is called a set of finite perimeter in Ω. If ∂A is smooth, then by the Divergence Theorem

$$\int_\Omega |\nabla u| = H^{n-1}(\partial A \cap \Omega),$$

where H^{n-1} is $(n-1)$-dimensional Hausdorff measure (surface area measure). It is therefore natural to define the perimeter of *any* subset of Ω by:

$$\operatorname{Per}_\Omega A = \text{perimeter of } A \text{ in } \Omega = \int_\Omega |\nabla \chi_A|.$$

The following two properties, easily proved, will be useful later.

Proposition 1. (Lower Semicontinuity) ([11]) *If* $u_\varepsilon \to u$ *in* $L^1(\Omega)$, *then*

$$\liminf_\varepsilon \int_\Omega |\nabla u_\varepsilon| \geqq \int_\Omega |\nabla u|.$$

Proposition 2. (Compactness of BV in L^1) ([11]) *Bounded sets in the BV norm are compact in the L^1 norm.*

We now present two technical lemmas; the first is an approximation theorem for sets of finite perimeter by sets with smooth boundary.

Lemma 1. *Let Ω be an open, bounded subset of R^n with Lipschitz-continuous boundary. Let $A \subset \Omega$ be a set of finite perimeter in Ω with $0 < |A| < |\Omega|$. Then there exists a sequence of open sets $\{A_k\}$ satisfying the following five conditions:*

(*i*) $\partial A_k \cap \Omega \in C^2$,
(*ii*) $|(A_k \cap \Omega) \Delta A| \to 0$ *as* $k \to \infty$,
(*iii*) $\operatorname{Per}_\Omega A_k \to \operatorname{Per}_\Omega A$ *as* $k \to \infty$,
(*iv*) $H^{n-1}(\partial A_k \cap \partial\Omega) = 0$,
(*v*) $|A_k \cap \Omega| = |A|$ *for all k sufficiently large.*

Here $|\cdot|$ refers to n-dimensional Lebesgue measure.

Proof. First extend χ_A to a function $\tilde{u} \in BV(R^n)$ such that

$$\tilde{u}(x) = \chi_A(x) \quad \text{for } x \in \Omega, \tag{1.3}$$

$$\int_{\partial\Omega} |\nabla \tilde{u}| = 0. \tag{1.4}$$

(See [11], 2.8, 2.16.)

Summarizing the argument of GIUSTI ([11], 1.24, 1.26), we see that a standard mollification of $\tilde{u}$ provides a sequence of C^∞ functions $\{f_\varepsilon\}$ satisfying

$$f_\varepsilon \to \tilde{u} \quad \text{in } L^1,$$

$$\lim_{\varepsilon \to 0} \int_\Omega |\nabla f_\varepsilon| = \int_\Omega |\nabla \tilde{u}|.$$

Then define sets $C_{\varepsilon,t} = \{f_\varepsilon(x) > t\}$. By use of the co-area formula ([11]) and SARD'S Theorem it can be shown that there exist a value of $t \in (0, 1)$ and a sequence $\varepsilon_k \to 0$ such that

$$\partial C_{\varepsilon_k,t} \in C^\infty,$$

$$\chi_{C_{\varepsilon_k,t}} \to \chi_A \quad \text{in } L^1(\Omega),$$

$$\operatorname{Per}_\Omega C_{\varepsilon_k,t} \to \operatorname{Per}_\Omega A,$$

and

$$H^{n-1}(\partial C_{\varepsilon_k,t} \cap \partial\Omega) = 0.$$

Such a sequence, which we denote simply by $\{C_k\}$, will not, in general, satisfy the condition (v). It therefore remains to be shown that the sequence of sets $\{C_k\}$ satisfying (i)–(iv) can be altered so as to satisfy (v) as well; that is, one must remove some measure from either $C_k \cap \Omega$ or $\Omega \setminus C_k$ (whichever is too big) and give it to the other without disrupting the smoothness of the boundary or distorting too drastically the perimeter of the boundary in Ω.

To this end, we let $E_k := C_k \cap \Omega$, and assume without loss of generality that $|E_k| - |A| > 0$.

Define

$$\lambda_k = |E_k| - |A|$$

which, by (ii), goes to 0 as $k \to \infty$.

Also define

$$L_k = \left(\frac{2\,|\Omega|}{|A|}\,\lambda_k\right)^{1/n}, \tag{1.5}$$

and impose on Ω a grid G_k of hypercubes $\{Q_i^k\}_{i=1}^{N_k}$ of side length L_k with $Q_i^k \subset \Omega$ for all i. Since $\partial\Omega$ is Lipschitz-continuous, there exists a sequence of grids $\{G_k\}$ such that:

$$\lim_{k\to\infty} |\Omega \setminus G_k| = 0. \tag{1.6}$$

It follows from (1.5) that the measure of any cube in the lattice exceeds the amount of volume which we need to transfer from E_k to $\Omega \setminus E_k$. In fact, $L_k^n > 2\lambda_k$.

Selecting the cube $\tilde{Q}^k$ which maximizes

$$\{|Q_i^k \cap E_k| : Q_i^k \in G_k\},$$

we split the argument into two cases, depending on whether or not $\tilde{Q}^k \subset E_k$.

Case 1. $\tilde{Q}^k \subset E_k$.

Since $\lambda_k < \frac{1}{2}|\tilde{Q}^k_i|$, one can remove a smooth subset of $\tilde{Q}^k$, say S_k, having volume λ_k and perimeter which goes to zero as $k \to \infty$. Placing this set S_k in the complement of E_k yields the desired set $A_k := C_k \setminus S_k$.

Case 2. $\tilde{Q}_k \setminus E_k \neq \emptyset$.

Then $|\tilde{Q}^k \cap E_k| < |\tilde{Q}^k|$. If N_k represents the number of cubes in G_k, it follows that

$$N_k \sim \frac{|\Omega|}{L_k^n} \tag{1.7}$$

and

$$N_k \leqq \frac{|\Omega|}{L_k^n}. \tag{1.8}$$

Furthermore,

$$N_k |\tilde{Q}^k \cap E_k| \geqq \sum_{Q_i^k \in G_k} |Q_i^k \cap E_k| \geqq |E_k| - |\Omega \setminus G_k|.$$

Inequality (1.8) implies

$$|\tilde{Q}^k \cap E_k| \geqq \frac{|E_k|}{|\Omega|} L_k^n - \frac{|\Omega \setminus G_k|}{N_k}.$$

Since

$$\frac{|E_k|}{|\Omega|} L_k^n > \frac{|A|}{|\Omega|} L_k^n = 2\lambda_k,$$

by (1.5), it follows that

$$|\tilde{Q}^k \cap E_k| > \lambda_k + \left(\lambda_k - \frac{|\Omega \setminus G_k|}{N_k}\right).$$

Now $\frac{1}{N_k} \sim \lambda_k$ by (1.5) and (1.7), while $|\Omega \setminus G_k| \to 0$ by (1.6), so that

$$|\tilde{Q}^k \cap E_k| > \lambda_k \quad \text{for sufficiently large } k.$$

This last inequality asserts that $\tilde{Q}^k$ contains enough of E_k to achieve (v). We now collapse the cube continuously towards its center through a family R_k of sets which have smooth boundary and which satisfy a uniform bound:

$$\sup_{T \in R_k} \mathrm{Per}_\Omega T < M_k \quad \text{for some } M_k = O(L_k^{n-1}).$$

At some point in this process one must obtain a set $T_k^\alpha \in R_k$ with

$$|T_k^\alpha \cap E_k| = \lambda_k.$$

If we remove this set from E_k, the boundary of the resulting set $E_k \setminus T_k^\alpha$ will fail to be smooth only on an $(n-1)$-dimensional set in $\partial E_k \cap \partial T_k^\alpha$. Near this set, one smooths the boundary of $E_k \setminus T_k^\alpha$ in such a way as to leave $|T_k^\alpha \cap E_k| = \lambda_k$. Actually, it is conceivable that smoothness could be lacking on a larger set if ∂E_k has high oscillation while approaching ∂T_k^α tangentially, but this can be averted through a slight modification of R_k; *e.g.* through a small rotation.

Now we define $A_k := C_k \setminus T_k^\alpha$. Recall that $E_k = C_k \cap \Omega$ and note that:

$$\limsup_k \operatorname{Per}_\Omega A_k \leqq \lim_k (\operatorname{Per}_\Omega C_k + \operatorname{Per}_\Omega T_k^\alpha)$$

$$\leqq \lim_k (\operatorname{Per}_\Omega C_k + M_k),$$

so that

$$\limsup_k \operatorname{Per}_\Omega A_k \leqq \operatorname{Per}_\Omega A,$$

since $\{C_k\}$ satisfies condition (iii) and $M_k = O(L_k^{n-1}) \to 0$. On the other hand, $\chi_{A_k} \to \chi_A$ in $L^1(\Omega)$ so that, by Proposition 1,

$$\liminf_k \operatorname{Per}_\Omega A_k \geqq \operatorname{Per}_\Omega A;$$

hence we conclude that

$$\lim_{k\to\infty} \operatorname{Per}_\Omega A_k = \operatorname{Per}_\Omega A.$$

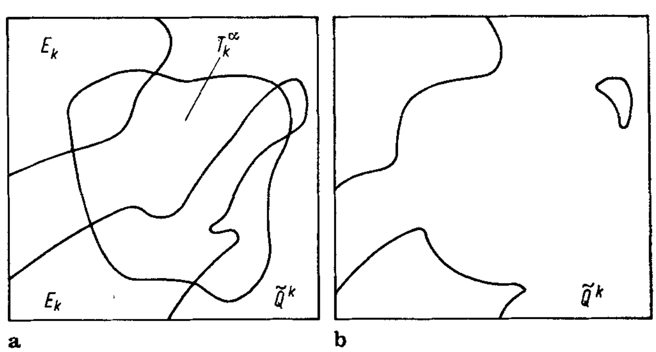

Fig. 3a. $T_k^\alpha \subset \tilde{Q}^k$ satisfying $|T_k^\alpha \cap E_k| = \lambda_k$; b. $E_k \setminus T_k^\alpha$ in $\tilde{Q}^k$ with boundary smoothed

Combining Cases 1 and 2, we obtain a sequence $\{A_k\}$ satisfying conditions (i)–(v).

Note. We could actually find sets with boundary C^k, $k > 2$, by this process, but C^2 will suffice for our purposes.

The next lemma does not concern functions of bounded variation, but rather asserts the existence of a smooth function measuring the distance from a smooth hypersurface to a nearby point not on the surface.

Lemma 2. *Let Ω be an open bounded subset of $\mathbf{R}^n$ with Lipschitz-continuous boundary. Let A be an open subset of $\mathbf{R}^n$ with C^2, compact, non-empty boundary such that* $H^{n-1}(\partial A \cap \partial\Omega) = 0$.

Define the distance function to ∂A, $d: \Omega \to \boldsymbol{R}$, *by*

$$d(x) = \begin{cases} \operatorname{dist}(x, A) & x \in \Omega \setminus A \\ -\operatorname{dist}(x, A) & x \in A \cap \Omega. \end{cases}$$

Then for some $s > 0$, d *is a* C^2 *function in* $\{|d(x)| < s\}$ *with*

$$|\nabla d| = 1. \tag{1.9}$$

Furthermore,

$$\lim_{s \to 0} H^{n-1}(\{d(x) = s\}) = H^{n-1}(\partial A). \tag{1.10}$$

Proof. When restricted to $\{0 < d(x) < s\}$ or $\{-s < d(x) < 0\}$, d will be C^k provided $\partial A \in C^k$ ([10], App. A, and [19]). The triangle inequality yields $|d(x) - d(y)| \leqq |x - y|$; (1.9) then follows from noting that, for x and y on the same normal to ∂A, $|d(x) - d(y)| = |x - y|$. Finally, (1.10) is classical; see *e.g.* Modica ([20]) for a proof.

Note. We will later apply Lemma 2 to $\{A_k\}$ constructed in Lemma 1. In the proof of (1.10) by Modica, it suffices to have a C^2 distance function, which is why the same degree of smoothness is desired for ∂A_k. We also remark that while $d(x)$ is only locally smooth, it is globally Lipschitz-continuous. (Lemma 11 proves this fact in a more general setting.)

B. The Result for $W: \boldsymbol{R} \to \boldsymbol{R}$

We consider first a non convex energy density $W: \boldsymbol{R} \to \boldsymbol{R}$ having the following properties:

(a) $W \in C^2$. (b) $W \geqq 0$. (c) W has exactly two roots, which we label a and b, with $a < b$. (d) $W'(a) = W'(b) = 0$, $W''(a) > 0$, $W''(b) > 0$. (See Fig. 4).

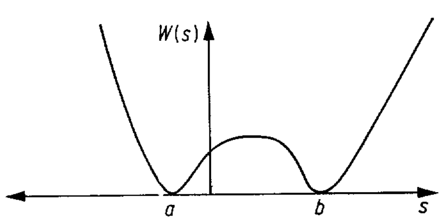

Fig. 4. Graph of W

Restating the unperturbed problem (P) for this W, we begin with the variational problem:

$$(P) \qquad \inf_{\substack{u \in L^1(\Omega) \\ \int_\Omega u\,dx = c}} \int_\Omega W(u)\,dx$$

where c is any number satisfying

$$a\,|\Omega| < c < b\,|\Omega|.$$

The minimizers of (P) are precisely the set of L^1 functions taking only the values a or b in such a way as to satisfy the integral constraint. Equivalently, minimizers correspond to partitions of Ω into measurable sets A and B such that $a\,|A| + b\,|B| = c$.

Through the introduction of the singular perturbation $\varepsilon^2\,|\nabla u|^2$, one obtains the associated perturbed problem (P_ε):

$$\inf_{\substack{u\in H^1(\Omega)\\ \int_\Omega u\,dx = c}} \int_\Omega W(u) + \varepsilon^2\,|\nabla u|^2\,dx.$$

Let u_ε denote a minimizer of (P_ε). Existence of such a minimizer can be shown using the direct method of the calculus of variations. (In general, minimizers will not be unique.) The goal is to characterize $u_0 = \lim_{\varepsilon_j\to 0} u_{\varepsilon_j}$ for any L^1-convergent subsequence of $\{u_\varepsilon\}$. A compactness argument asserting the existence of a convergent subsequence will be given later using Proposition 2.

Theorem 1 gives a purely geometric criterion to select the possible limit points u_0 from the large set of minimizers to (P): a "preferred" solution to (P) is one that minimizes interfacial area between the states $u = a$ and $u = b$.

Theorem 1. *Suppose $u_{\varepsilon_j} \to u_0$ in $L^1(\Omega)$ for some sequence of numbers $\varepsilon_j \to 0$, where u_{ε_j} is a solution of (P_{ε_j}).*

Then u_0 is a solution of (P_0):

$$(P_0) \qquad \inf_{\substack{u\in BV(\Omega)\\ W(u(x))=0 \text{ a.e.}\\ \int_\Omega u\,dx = c}} \operatorname{Per}_\Omega\{u = a\}.$$

The proof relies on correctly identifying the first non-trivial term in an asymptotic expansion for the energy of (P_ε). It is easy to construct a function in $H^1(\Omega)$ having energy $O(\varepsilon)$. Such a function will take on only the values a and b except in a transition layer of width ε between the two states. Thus, anticipating the order of the first term, we rescale the problem and consider the functionals $F_\varepsilon : L^1(\Omega) \to \boldsymbol{R}$ given by

$$F_\varepsilon(u) = \begin{cases} \int_\Omega \frac{1}{\varepsilon} W(u) + \varepsilon\,|\nabla u|^2\,dx & u \in H^1(\Omega),\ \int_\Omega u\,dx = c \\ +\infty & \text{otherwise}. \end{cases}$$

At the same time, define $F_0 : L^1(\Omega) \to \boldsymbol{R}$ by

$$F_0(u) = \begin{cases} \left(2\int_a^b \sqrt{W(s)}\,ds\right)\operatorname{Per}_\Omega\{u = a\} & u \in BV(\Omega),\ W(u(x)) = 0 \text{ a.e.},\ \int_\Omega u\,dx = c \\ +\infty & \text{otherwise}. \end{cases}$$

The penalties of $+\infty$ in the two previous definitions allow us to define F_ε and F_0 on $L^1(\Omega)$, a space whose topology has desirable compactness properties with respect to H^1 and BV.

The theorem follows easily from the two properties listed below, which comprise a working definition of the Γ-convergence of a sequence of functionals $\{F_\varepsilon\}$ to a Γ-limit, F_0, with respect to the L^1 topology ([7]):

(i) For each $v \in L^1(\Omega)$, and for each sequence $\{v_\varepsilon\}$ in $L^1(\Omega)$,

$$v_\varepsilon \to v \quad \text{in } L^1(\Omega) \quad \text{implies } \liminf_\varepsilon F_\varepsilon(v_\varepsilon) \geqq F_0(v). \tag{1.11}$$

(ii) For each $v \in L^1(\Omega)$, there exists a sequence $\{\varrho_{\varepsilon_j}\}$ in $L^1(\Omega)$ satisfying

$$\varrho_{\varepsilon_j} \to v \quad \text{in } L^1(\Omega) \tag{1.12}$$

and

$$\lim_{j\to\infty} F_{\varepsilon_j}(\varrho_{\varepsilon_j}) = F_0(v). \tag{1.13}$$

Notation. If $\{F_\varepsilon\}$, F_0 satisfy (1.11)–(1.13), we write

$$\Gamma(L^1(\Omega)^-) \lim_{\substack{\varepsilon\to 0\\ \varrho\to v}} F_\varepsilon(\varrho) = F_0(v).$$

Remark 1.14. The real advantage of proving Γ-convergence, rather than simply the convergence of minimizers, is that the results adapt immediately to continuous perturbations of F_ε. This is clear from (1.11)–(1.13). Thus one can characterize the asymptotic behavior of minimizers of a whole family of problems obtained from F_ε by the addition of a functional continuous with respect to $L^1(\Omega)$ (e.g. $\int \frac{1}{\varepsilon} W(u) + ug(x) + \varepsilon\,|\nabla u|^2\,dx$ for $g \in L^\infty(\Omega)$).

Proof of Theorem 1. For the moment we delay the proof of inequality (1.11) and the construction of a sequence yielding (1.12) and (1.13) and show how Theorem 1 follows from these claims.

Let $w_0 \in BV(\Omega)$ be a minimizer of F_0. Existence of such a function follows from the direct method using the compactness and lower semicontinuity of $BV(\Omega)$ with respect to $L^1(\Omega)$ (*i.e.* Propositions 1 and 2). In fact, minimizers will have an interface which is analytic and of constant mean curvature for dimension $n < 8$. For a more complete description of minimizers of F_0 see the work of GONZALEZ, MASSARI & TAMANINI ([12]).

Let $\{w_{\varepsilon_j}\}$ be the sequence satisfying (1.12), (1.13) for w_0. Assuming that the minimizers $\{u_{\varepsilon_j}\}$ converge in $L^1(\Omega)$ to a limit u_0, it follows from (1.11) that

$$\liminf F_{\varepsilon_j}(u_{\varepsilon_j}) \geqq F_0(u_0).$$

Using that $F_{\varepsilon_j}(u_{\varepsilon_j}) \leqq F_{\varepsilon_j}(w_{\varepsilon_j})$, one has

$$F_0(u_0) \leqq \liminf F_{\varepsilon_j}(u_{\varepsilon_j}) \leqq \lim_{j\to\infty} F_{\varepsilon_j}(w_{\varepsilon_j}) = F(w_0).$$

Thus u_0 must be a minimizer of F_0 and Theorem 1 follows.

We now return to the task of proving Γ-convergence: (1.11)–(1.13). Before proving (1.11), we should make some preliminary observations about the kinds of L^1-convergent sequences $\{v_\varepsilon\}$ and limits v that needs be considered.

If $W(v(x)) \neq 0$ on a set of positive measure, then $F_0(v) = +\infty$. But

$$\liminf F_\varepsilon(v_\varepsilon) \geqq \liminf \frac{1}{\varepsilon} \int_\Omega W(v_\varepsilon(x))\, dx = +\infty$$

as well, so that (1.11) is immediate. Equally simple is the case in which

$$\int_\Omega v\, dx \neq c,$$

for here

$$\int_\Omega v_\varepsilon\, dx \neq c$$

for all small ε, again yielding

$$\liminf F_\varepsilon(v_\varepsilon) = +\infty.$$

Therefore, consider only those $v \in L^1(\Omega)$ satisfying

$$W(v(x)) = 0 \quad \text{a.e.,} \quad \int_\Omega u\, dx = c.$$

Proof of Inequality (1.11). First we assume that the sequence $\{v_\varepsilon\}$ satisfies

$$a \leqq v_\varepsilon \leqq b. \tag{1.15}$$

Applying the Cauchy-Schwarz inequality to $F_\varepsilon(v_\varepsilon)$, we obtain

$$F_\varepsilon(v_\varepsilon) \geqq 2 \int_\Omega \sqrt{W(v(x))}\, |\nabla v_\varepsilon(x)|\, dx.$$

Let $\phi: \boldsymbol{R} \to \boldsymbol{R}$ be defined by

$$\phi(t) = 2 \int_a^t \sqrt{W(s)}\, ds, \tag{1.16}$$

so that

$$F_\varepsilon(v_\varepsilon) \geqq \int_\Omega |\nabla \phi(v_\varepsilon(x))|\, dx.$$

Then, from (1.15) and the L^1 convergence of v_ε to v, it follows that

$$\phi(v_\varepsilon) \to \phi(v) \text{ in } L^1(\Omega).$$

By the lower semicontinuity shown in Propositon 1, we conclude that

$$\liminf F_\varepsilon(v_\varepsilon) \geqq \liminf \int_\Omega |\nabla \phi(v_\varepsilon)|\, dx \geqq \int_\Omega |\nabla \phi(v)|.$$

Now

$$\phi(v(x)) = \begin{cases} 0 & \{v = a\} \\ 2\int_a^b \sqrt{W(s)}\,ds & \{v = b\}, \end{cases}$$

since $W(v(x)) = 0$ a.e., and therefore

$$\int_\Omega |\nabla\phi(v)| = \left(2\int_a^b \sqrt{W(s)}\,ds\right) \operatorname{Per}_\Omega\{v = a\} = F_0(v),$$

which establishes (1.11).

To justify assumption (1.15), we compare $\{v_\varepsilon\}$ to the truncated sequence $\{v_\varepsilon^*\}$ defined by:

$$v_\varepsilon^* = \begin{cases} a & \{v_\varepsilon(x) < a\} \\ v_\varepsilon(x) & \{a \leqq v_\varepsilon(x) \leqq b\} \\ b & \{v_\varepsilon(x) > b\}. \end{cases}$$

First note that $v_\varepsilon \to v$ in $L^1(\Omega)$ implies that $v_\varepsilon^* \to v$ in $L^1(\Omega)$. Also,

$$\begin{aligned} F_\varepsilon(v_\varepsilon) &\geqq \int_\Omega \frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon\,|\nabla v_\varepsilon|^2\,dx \\ &= F_\varepsilon(v_\varepsilon^*) + \int_{\{v_\varepsilon < a\}\cup\{v_\varepsilon > b\}} \frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon\,|\nabla v_\varepsilon|^2\,dx \\ &\geqq F_\varepsilon(v_\varepsilon^*). \end{aligned}$$

Since the proof of (1.11) made no use of the constraint

$$\int_\Omega v_\varepsilon\,dx = c,$$

this last inequality shows that it suffices to consider only sequences bounded as in (1.15).

The proof of (1.12), (1.13) involves the construction of a sequence of functions such as to traverse efficiently a boundary layer while bridging the values a and b. Before presenting the proof, we discuss some properties of the solution $z(s)$ of the following ordinary differential equation, which will be used in the construction:

$$\frac{dz}{ds} = \sqrt{W(z)}, \tag{1.17}$$

$$z(0) = \tfrac{1}{2}(a + b).$$

Local existence is clear since $\sqrt{W(z)}$ will be Lipschitz-continuous in a neighborhood of $\frac{1}{2}(a + b)$. However, by writing

$$\int_{\frac{1}{2}(a+b)}^{z(s)} \frac{d\eta}{\sqrt{W(\eta)}} = s \tag{1.18}$$

and noting that $W(\eta) > 0$ for $a < \eta < b$, one sees that local solutions may be extended to all of $\boldsymbol{R}$. Furthermore,

$$a < z(s) < b \quad \text{for all } s, \tag{1.19}$$

and

$$\lim_{s\to\infty} z(s) = b, \qquad \lim_{s\to-\infty} z(s) = a. \tag{1.20}$$

In fact, since $W''(a) > 0$ and $W''(b) > 0$, it follows from Taylor's Theorem that

$$\frac{1}{\sqrt{W(\eta)}} \leqq \frac{c_1}{|\eta - a|} \quad \text{for } |\eta - a| \text{ small},$$

$$\frac{1}{\sqrt{W(\eta)}} \leqq \frac{c_2}{|\eta - b|} \quad \text{for } |\eta - b| \text{ small},$$

where c_1 and c_2 are positive constants depending on W.

This implies the decay estimates:

$$|b - z(s)| \leqq c_3 e^{-c_4 s} \quad \text{as } s \to \infty, \qquad |a - z(s)| \leqq c_3 e^{c_4 s} \quad \text{as } s \to -\infty, \tag{1.21}$$

where c_3 and c_4 are again positive constants depending on W.

Construction of $\{\varrho_{\varepsilon_j}\}$ satisfying (1.12), (1.13). Let $v \in L^1(\Omega)$. We may immediately assume that

$$v \in BV(\Omega), \qquad W(v(x)) = 0 \quad \text{a.e.}, \qquad \int_\Omega v\,dx = c$$

(otherwise $F_0(v) = \infty$ and the choice $\varrho_\varepsilon = v$ for each ε achieves (1.12), (1.13)). Therefore we may write

$$v(x) = \begin{cases} a & x \in A \\ b & x \in B, \end{cases}$$

where A and B are sets of finite perimeter in Ω and

$$a\,|A| + b\,|B| = c.$$

Let $\Gamma := \partial A \cap \partial B$ and assume $\Gamma \in C^2$. At the conclusion of the proof we will show that this represents no loss of generality.

Recalling Lemma 2, consider the function $d: \Omega \to \boldsymbol{R}$, given by

$$d(x) = \begin{cases} \operatorname{dist}(x, \Gamma) & x \in B \\ -\operatorname{dist}(x, \Gamma) & x \in A, \end{cases}$$

which represents the signed distance to Γ.

Now define a sequence of functions $g_\varepsilon: \boldsymbol{R} \to \boldsymbol{R}$ which effect the transition between the zeroes of W:

$$g_\varepsilon(s) = \begin{cases} b & s > 2\sqrt{\varepsilon} \\ \left(\dfrac{b - z\left(\frac{1}{\sqrt{\varepsilon}}\right)}{\sqrt{\varepsilon}}\right)(s - 2\sqrt{\varepsilon}) + b & \sqrt{\varepsilon} \leqq s \leqq 2\sqrt{\varepsilon} \\ z\left(\dfrac{s}{\varepsilon}\right) & |s| \leqq \sqrt{\varepsilon} \\ \left(\dfrac{z\left(-\frac{1}{\sqrt{\varepsilon}}\right) - a}{\sqrt{\varepsilon}}\right)(s + 2\sqrt{\varepsilon}) + a & -2\sqrt{\varepsilon} \leqq s \leqq -\sqrt{\varepsilon} \\ a & s < -2\sqrt{\varepsilon}. \end{cases} \tag{1.22}$$

Replacing s by $d(x)$, we obtain a sequence $\{\tilde{\varrho}_\varepsilon\}$ given by

$$\tilde{\varrho}_\varepsilon(x) = g_\varepsilon(d(x)) \tag{1.23}$$

Notice that for ε small, $d(x)$ is Lipschitz-continuous in $\{|d(x)| < 2\sqrt{\varepsilon}\}$, so that $\tilde{\varrho}_\varepsilon \in H^1(\Omega)$.

As will be shown, this sequence would serve to verify (1.12), (1.13) if

$$\int_\Omega \tilde{\varrho}_\varepsilon \, dx = c.$$

This, however, is not generally the case, and the sequence must be altered by an additive constant so as to meet the integral constraint.

We split the argument into three steps, the first of which is to prove that the additive constant is $O(\varepsilon)$.

Step 1. Claim

$$\tilde{\varrho}_\varepsilon \to v \quad \text{in } L^1(\Omega) \tag{1.24}$$

with

$$\int_\Omega \tilde{\varrho}_\varepsilon \, dx = c + \delta_\varepsilon, \quad \text{where } \delta_\varepsilon = O(\varepsilon). \tag{1.25}$$

From (1.23)

$$\int_\Omega \tilde{\varrho}_\varepsilon \, dx = \int_\Omega v \, dx + \int_\Omega (\tilde{\varrho}_\varepsilon - v) \, dx = c + \int_{\{|d(x)| < 2\sqrt{\varepsilon}\}} (\tilde{\varrho}_\varepsilon - v) \, dx,$$

so the claim is that

$$\int_{\{|d(x)| < 2\sqrt{\varepsilon}\}} (\tilde{\varrho}_\varepsilon - v) \, dx = O(\varepsilon).$$

First consider

$$\int_{\{0<d(x)<2\sqrt{\varepsilon}\}} (\tilde{\varrho}_\varepsilon - v)\,dx = \int_{\{0<d(x)<\sqrt{\varepsilon}\}} \left(z\left(\frac{d(x)}{\varepsilon}\right) - b\right) dx \tag{1.26}$$

$$+ \int_{\{\sqrt{\varepsilon}<d(x)<2\sqrt{\varepsilon}\}} \left(\frac{b - z\left(\frac{1}{\sqrt{\varepsilon}}\right)}{\sqrt{\varepsilon}}\right) (d(x) - 2\sqrt{\varepsilon})\,dx.$$

In light of (1.21), the last integral is $O(e^{-c_4/\sqrt{\varepsilon}})$. From the co-area formula ([9])

$$\int_\Omega f(h(x))\,|\nabla h|\,dx = \int_R f(s)\,H^{n-1}\{x : h(x) = s\}\,ds, \tag{1.27}$$

which holds for any Lebesgue measurable f and Lipschitz-continuous h, we find for the first integral on the right hand side of (1.26),

$$\int_{\{0<d(x)<\sqrt{\varepsilon}\}} \left(z\left(\frac{d(x)}{\varepsilon}\right) - b\right) dx$$

$$= \int_{\{0<d(x)<\sqrt{\varepsilon}\}} \left(z\left(\frac{d(x)}{\varepsilon}\right) - b\right) |\nabla d|\,dx \quad (\text{since } |\nabla d| = 1 \text{ by } (1.9))$$

$$= \int_0^{\sqrt{\varepsilon}} \left(z\left(\frac{s}{\varepsilon}\right) - b\right) H^{n-1}\{d(x) = s\}\,ds$$

$$\leqq \left(\max_{0\leqq s\leqq\sqrt{\varepsilon}} H^{n-1}\{d(x) = s\}\right) \int_0^{\sqrt{\varepsilon}} \left|z\left(\frac{s}{\varepsilon}\right) - b\right| ds$$

$$\leqq \left(\max_{0\leqq s\leqq\sqrt{\varepsilon}} H^{n-1}\{d(x) = s\}\right) \varepsilon \int_0^{1/\sqrt{\varepsilon}} |z(\eta) - b|\,d\eta.$$

Then (1.10) and (1.21) imply that

$$\int_{\{0<d(x)<\sqrt{\varepsilon}\}} \left(z\left(\frac{d(x)}{\varepsilon}\right) - b\right) dx \leqq \text{const. } \varepsilon.$$

Hence

$$\int_{\{0<d(x)<2\sqrt{\varepsilon}\}} (\tilde{\varrho}_\varepsilon - v)\,dx = O(\varepsilon).$$

A similar argument works for

$$\int_{\{-2\sqrt{\varepsilon}<d(x)<0\}} (\tilde{\varrho}_\varepsilon - v)\,dx$$

and (1.24), (1.25) follow.

Step 2. Here we show that, as $\varepsilon \to 0$, the energy of $\{\tilde{\varrho}_\varepsilon\}$ approaches $F_0(v)$,

$$\text{Claim:} \quad \lim_{\varepsilon\to 0} \int \frac{1}{\varepsilon} W(\tilde{\varrho}_\varepsilon) + \varepsilon\,|\nabla\tilde{\varrho}_\varepsilon|^2\,dx \leqq F_0(v), \tag{1.28}$$

To confirm (1.28), first note that

$$\int_{\{|d(x)|>2\sqrt{\varepsilon}\}} \frac{1}{\varepsilon} W(\tilde{\varrho}_\varepsilon) + \varepsilon |\nabla \tilde{\varrho}_\varepsilon|^2 \, dx = 0,$$

so that, by (1.9),

$$\int_\Omega \cdot = \int_{\{|d(x)|<2\sqrt{\varepsilon}\}} \frac{1}{\varepsilon} W(\tilde{\varrho}_\varepsilon) + \varepsilon |\nabla \tilde{\varrho}_\varepsilon|^2 \, dx$$

$$= \int_{\{|d(x)|<2\sqrt{\varepsilon}\}} \left(\frac{1}{\varepsilon} W(\tilde{\varrho}_\varepsilon) + \varepsilon |\nabla \tilde{\varrho}_\varepsilon|^2\right) |\nabla d| \, dx.$$

Applying (1.22) and the co-area formula (1.27), one finds that

$$\int_\Omega \frac{1}{\varepsilon} W(\tilde{\varrho}_\varepsilon) + \varepsilon |\nabla \tilde{\varrho}_\varepsilon|^2 \, dx \tag{1.29}$$

$$= \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left[\frac{1}{\varepsilon} W\left(z\left(\frac{s}{\varepsilon}\right)\right) + \varepsilon\left(\frac{d}{ds} z\left(\frac{s}{\varepsilon}\right)\right)^2\right] H^{n-1}\{d(x) = s\} \, ds$$

$$+ \int_{\sqrt{\varepsilon}}^{2\sqrt{\varepsilon}} \left(\frac{1}{\varepsilon} W(g_\varepsilon(s)) + \varepsilon g_\varepsilon'(s)^2\right) H^{n-1}\{d(x) = s\} \, ds$$

$$+ \int_{-2\sqrt{\varepsilon}}^{-\sqrt{\varepsilon}} \left(\frac{1}{\varepsilon} W(g_\varepsilon(s)) + \varepsilon g_\varepsilon'(s)^2\right) H^{n-1}\{d(x) = s\} \, ds.$$

Next, by use of a Taylor expansion about b to approximate $W(g_\varepsilon(s))$,

$$\int_{\sqrt{\varepsilon}}^{2\sqrt{\varepsilon}} \left(\frac{1}{\varepsilon} W(g_\varepsilon(s)) + \varepsilon g_\varepsilon'(s)^2\right) H^{n-1}\{d(x) = s\} \, ds \leqq$$

$$\left(\max_{\sqrt{\varepsilon} \leqq s \leqq 2\sqrt{\varepsilon}} H^{n-1}\{d(x) = s\}\right)$$

$$\int_{\sqrt{\varepsilon}}^{2\sqrt{\varepsilon}} \left[\frac{1}{\varepsilon} \frac{W''(\xi)}{2} \left(\frac{b - z\left(\frac{1}{\sqrt{\varepsilon}}\right)}{\sqrt{\varepsilon}}\right)^2 (s - 2\sqrt{\varepsilon})^2 + \varepsilon \left(\frac{b - z\left(\frac{1}{\sqrt{\varepsilon}}\right)}{\sqrt{\varepsilon}}\right)^2\right] ds$$

for some $\xi = \xi(s)$ near b, and it follows from (1.10) and the decay estimate (1.21) that this integral approaches zero with ε. A similar approach leads to the same conclusion concerning the last integral in (1.29).

Turning to the first integral in (1.29), we observe that (1.17) implies

$$\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left[\frac{1}{\varepsilon} W\left(z\left(\frac{s}{\varepsilon}\right)\right) + \varepsilon \frac{d}{ds}\left(z\left(\frac{s}{\varepsilon}\right)\right)^2\right] H^{n-1}\{d(x) = s\}\, ds$$

$$= \frac{2}{\varepsilon} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} W\left(z\left(\frac{s}{\varepsilon}\right)\right) H^{n-1}\{d(x) = s\}\, ds$$

$$= \leqq \left(\frac{2}{\varepsilon} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} W\left(z\left(\frac{s}{\varepsilon}\right)\right) ds\right) \left(\sup_{|s| \leqq \sqrt{\varepsilon}} H^{n-1}\{d(x) = s\}\right).$$

Then, since z is monotone, letting $t = z\left(\frac{s}{\varepsilon}\right)$, we find $\frac{\varepsilon}{\sqrt{W(t)}}\, dt = ds$ and arrive at

$$2\left(\int_{z\left(\frac{1}{-\sqrt{\varepsilon}}\right)}^{z\left(\frac{1}{\sqrt{\varepsilon}}\right)} \sqrt{W(t)}\, dt\right) \sup_{|s| \leqq \sqrt{\varepsilon}} H^{n-1}\{d(x) = s\} \leqq 2\left(\int_a^b \sqrt{W(t)}\, dt\right) \sup_{|s| \leqq \sqrt{\varepsilon}} \{\{d(x) = s\}.$$

From (1.10) in Lemma 2, one can pass to the limit as $\varepsilon \to 0$ to conclude (1.28).

Step 3. It remains to show that the addition of a constant to each $\tilde{\varrho}_\varepsilon$ so as to satisfy the integral constraint will not disturb inequality (1.28). Define

$$\eta_\varepsilon = \frac{-\delta_\varepsilon}{|\Omega|}.$$

It was shown in Step 1 that $\eta_\varepsilon = O(\varepsilon)$. We now define a candidate for a sequence satisfying (1.12), (1.13) through

$$\varrho_\varepsilon = \tilde{\varrho}_\varepsilon + \eta_\varepsilon.$$

Clearly

$$\int_\Omega \varrho_\varepsilon(x)\, dx = c,$$

but it remains to verify that

$$\lim_{\varepsilon \to 0} F_\varepsilon(\varrho_\varepsilon) \leqq \lim_{\varepsilon \to 0} \int_\Omega \frac{1}{\varepsilon} W(\tilde{\varrho}_\varepsilon) + \varepsilon\, |\nabla \tilde{\varrho}_\varepsilon|^2\, dx. \tag{1.30}$$

One finds

$$F_\varepsilon(\varrho_\varepsilon) = \frac{1}{\varepsilon} W(a + \eta_\varepsilon)\, |\{d(x) < -2\sqrt{\varepsilon}\}|$$

$$+ \int_{\{|d(x)| < 2\sqrt{\varepsilon}\}} \frac{1}{\varepsilon} W(\tilde{\varrho}_\varepsilon + \eta_\varepsilon) + \varepsilon\, |\nabla \tilde{\varrho}_\varepsilon|^2\, dx \tag{1.31}$$

$$+ \frac{1}{\varepsilon} W(b + \eta_\varepsilon)\, |\{d(x) > 2\sqrt{\varepsilon}\}|.$$

The first term in (1.31) can be estimated by Taylor's Theorem:

$$\frac{1}{\varepsilon} W(a+\eta_\varepsilon)\,|\{d(x) < -2\sqrt{\varepsilon}\}| \leqq \frac{|\Omega|}{2\varepsilon} W''(\xi_\varepsilon)\,\eta_\varepsilon^2$$

for some $\xi_\varepsilon \in (a - |\eta_\varepsilon|, a + |\eta_\varepsilon|)$. Hence this term approaches zero with ε since $\eta_\varepsilon = O(\varepsilon)$. The last term in (1.31) is treated similarly.

To establish (1.30) thus reduces to showing that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{|d|<2\sqrt{\varepsilon}\}} (W(\tilde{\varrho}_\varepsilon + \eta_\varepsilon) - W(\tilde{\varrho}_\varepsilon))\,dx = 0.$$

From the Mean Value Theorem we find

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{|d|<2\sqrt{\varepsilon}\}} (W(\tilde{\varrho}_\varepsilon + \eta_\varepsilon) - W(\tilde{\varrho}_\varepsilon))\,dx \leqq \max_{a-\sigma\leqq s\leqq b+\sigma} |W'(s)|\,\frac{\eta_\varepsilon}{\varepsilon}\,|\{|d(x)| < 2\sqrt{\varepsilon}\}|$$

for some $\sigma > 0$ small. Since this approaches zero with ε, (1.30) follows. Equations (1.24), (1.28), and (1.30) together with (1.11) imply (1.12), (1.13).

Our final task is to show that to assume A smooth does not lessen generality. We therefore relax this assumption and consider $v \in BV(\Omega)$ where

$$\int_\Omega v\,dx = c, \qquad v(x) = \begin{cases} a & x \in A \\ b & x \in \Omega \setminus A, \end{cases}$$

and A is a set of finite perimeter in Ω.

Now let $\{A_k\}$ be the sequence of approximating sets described in Lemma 1, and define $\{v_k\}$ by

$$v_k(x) = \begin{cases} a & x \in A_k \cap \Omega \\ b & x \in \Omega \setminus A_k. \end{cases}$$

Property (iii) of the lemma implies that

$$\lim_{k\to\infty} F_0(v_k) = F_0(v),$$

and from property (ii), $v_k \to v$ in $L^1(\Omega)$.

A sequence satisfying (1.12), (1.13) with v replaced by v_k exists since ∂A_k is smooth. A diagonalization argument then yields a sequence $\{\varrho_{\varepsilon_{j_k}}\}$ in $H^1(\Omega)$ satisfying (1.12), (1.13) for a general $v \in BV(\Omega)$.

This completes the proof of Theorem 1.

We turn now to the question of compactness for the minimizers of (P_ε). Some additional hypothesis on W seems to be required; it is sufficient to assume that W has polynomial growth:

Proposition 3. *Let $\{u_\varepsilon\}$ be a sequence of minimizers of (P_ε). Suppose that there exist positive numbers c_1, c_2, s_0 and a number $p \geqq 2$ such that*

$$c_1 |s|^p \leqq W(s) \leqq c_2 |s|^p \quad \textit{for} \quad |s| \geqq s_0. \tag{1.32}$$

Then there exists a subsequence $\{u_{\varepsilon_j}\}$ which converges to a limit u_0 in $L^1(\Omega)$.

Proof. Recall the definition of ϕ from (1.16). Notice that ϕ is a monotone increasing function, and that from (1.32) we have

$$\phi'(s) = \sqrt{W(s)} \geqq \sqrt{c_1}\, |s|^{p/2} \quad \text{for } |s| \geqq s_0.$$

We conclude that ϕ^{-1} exists and is uniformly continuous on compact sets in $\boldsymbol{R}$.

Letting $\{v_\varepsilon\}$ denote the sequence $\{\phi(u_\varepsilon)\}$, we seek a uniform $BV(\Omega)$ bound on this sequence so as to exploit the compactness of BV in L^1. By comparing the energy of $\{u_\varepsilon\}$ to the energy of the constructed sequence $\{\varrho_\varepsilon\}$ used in Theorem 1, we infer that

$$\int_\Omega |\nabla\phi(u_\varepsilon)| \leqq F_\varepsilon(u_\varepsilon) \leqq F_\varepsilon(\varrho_\varepsilon) < C \tag{1.33}$$

for some positive C.

Also, from (1.32):

$$\int_\Omega |\phi(u_\varepsilon)| = \int_\Omega \int_a^{u_\varepsilon(x)} \sqrt{W(s)}\, ds\, dx \leqq c_3 + c_4 \int_\Omega u_\varepsilon^{\frac{p}{2}+1}\, dx$$

for some positive constants c_3, c_4. But (1.32) implies that

$$\int_\Omega u_\varepsilon^p\, dx \leqq |\Omega|\, s_0^p + \frac{1}{c_1} \int_\Omega W(u_\varepsilon)\, dx \leqq |\Omega|\, s_0^p + C. \tag{1.34}$$

Since $p \geqq 2$, it follows that $p \geqq \frac{1}{2}p + 1$, and so $\|\phi(u_\varepsilon)\|_{BV(\Omega)}$ is uniformly bounded in ε. Thus, by Proposition 2, we may pass to an L^1-convergent subsequence

$$v_{\varepsilon_j} = \phi(u_{\varepsilon_j}) \to v_0 \quad \text{in } L^1(\Omega).$$

Using the uniform continuity of ϕ^{-1} it is then easy to show that $\{u_{\varepsilon_j}\} = \{\phi^{-1}(v_{\varepsilon_j})\}$ converges in measure. Since the u_{ε_j} are uniformly bounded in L^p, their convergence in $L^1(\Omega)$ follows.

Remark (1.35). One can replace the growth assumption on W in Proposition 2 with the assumption that the minimizers be uniformly bounded in L^∞; a similar argument then yields compactness. In dimension $n = 1$ such an assumption is easily justified from the monotonicity of minimizers (see [5]). For $n \geqq 2$, this bound was proved by GURTIN & MATANO ([15]).

Remark (1.36). MODICA ([20]) proves a result very similar to Theorem 1. His argument is more general in that it makes no regularity hypothesis on W beyond continuity. However, instead of establishing the Γ-convergence of F_ε to F_0 as is done here, he makes use of results by GONZALEZ, MASSARI & TAMANINI ([12]) about the nature of minimizers of (P_0) to achieve the conclusion of Theorem 1 without the full Γ-convergence. The full Γ-convergence is needed in proving existence of *local* minimizers (see [18]). MODICA'S construction of the transition layer satisfying (1.12), (1.13) is also somewhat different, suggesting that there is considerable flexibility in the argument just presented. MODICA has also recently proved a generalization of Theorem 1 which includes a term for contact energy along $\partial\Omega$ ([21]).

C. Generalization to an Integrand with Spatial Dependence

In this section we adapt the techniques of the previous section to the case where the nonconvex integrand contains some spatial dependence. Choosing a simple model which preserves the essential two phase nature of the problem, we consider

$$\inf_{\substack{u\in L^1(\Omega)\\ \int_\Omega u\,dx=c}} \int_\Omega (u(x)-g_1(x))^2\,(u(x)-g_2(x))^2\,dx, \tag{1.37}$$

where $g_1, g_2 : \Omega \to \boldsymbol{R}$ satisfy $g_1(x) < g_2(x)$ and are both bounded in the C^1 topology, while c is any number satisfying

$$\int_\Omega g_1\,dx < c < \int_\Omega g_2\,dx.$$

As before, any solution of (1.37) corresponds to a partition of Ω into two sets A and B, where now $u(x) = g_1(x)$ in A and $u(x) = g_2(x)$ in B, so as to satisfy the constraint.

Introducing the singular perturbation $\varepsilon^2\,|\nabla u|^2$, we let u_ε denote a solution of the perturbed problem:

$$\inf_{\substack{u\in H^1(\Omega)\\ \int_\Omega u\,dx=c}} \int_\Omega (u(x)-g_1(x))^2\,(u(x)-g_2(x))^2 + \varepsilon^2\,|\nabla u|^2\,dx. \tag{1.38}$$

Here again we expect a geometric characterization of $u_0 = \lim_{\varepsilon\to 0} u_\varepsilon$ involving interfacial area. The dependence of the integrand upon x, however, changes the limiting problem to one which might be called a weighted perimeter problem. Define

$$h(x) = 2\int_{g_1(x)}^{g_2(x)} (s-g_1(x))\,(g_2(x)-s)\,ds.$$

We now turn to

Theorem 2. *Suppose that* $u_{\varepsilon_j} \to u_0$ *in* $L^1(\Omega)$ *for some sequence of numbers* $\varepsilon_j \to 0$. *Then* u_0 *is a solution of*

$$\inf_{\substack{u\in BV(\Omega)\\ u(x)\in\{g_1(x),g_2(x)\}\text{a.e.}\\ \int_\Omega u\,dx=c}} \int_\Omega h(x)\,|\nabla\chi_{\{u=g_2\}}|. \tag{1.39}$$

Remark. If $\partial\{u=g_2\}$ is smooth for u in (1.39), we can apply the Divergence Theorem to definition (1.2) and so obtain

$$\int_\Omega h(x)\,|\nabla\chi_{\{u=g_2\}}| = \int_{\partial\{u=g_2\}\cap\Omega} h(x)\,dH^{n-1}(x).$$

The proof of Theorem 2 follows the same outline as that of Theorem 1. Therefore, rather than detailing the whole proof, we present only those parts of the argument that involve notable alterations.

Proof of Theorem 2. We define the functionals $G_\varepsilon, G : L^1(\Omega) \to \boldsymbol{R}$ by

$$G_\varepsilon(v) = \begin{cases} \int_\Omega \frac{1}{\varepsilon}(v(x) - g_1(x))^2 (v(x) - g_2(x))^2 + \varepsilon |\nabla v|^2 \, dx, & v \in H^1(\Omega), \quad \int_\Omega v \, dx = c, \\ +\infty & \text{otherwise}, \end{cases}$$

$$G_0(v) = \begin{cases} \int_\Omega h(x) |\nabla \chi_{\{v=g_2\}}| & v \in BV(\Omega), \quad \int_\Omega v \, dx = c, \quad v(x) \in \{g_1(x), g_2(x)\} \text{ a.e.} \\ +\infty & \text{otherwise}. \end{cases}$$

Since $h(x)$ is a uniformly bounded, positive function with uniformly bounded gradient, it is clear from (1.2) that $G_0(v)$ is finite for $v \in BV(\Omega)$, provided

$$\int_\Omega v \, dx = c \quad \text{and } v(x) \in \{g_1(x), g_2(x)\} \text{ a.e.}$$

while $G_0(v) = \infty$ for any v such that $\{v = g_2\}$ is not a set of finite perimeter in Ω.

As before, it suffices to establish the Γ-convergence of G_ε to G_0, *i.e.* the analogues of (1.11), and (1.12), (1.13). To obtain the analogue of (1.11) we consider $\{v_\varepsilon\}$, $v \in L^1(\Omega)$ such that $v_\varepsilon \to v$ in $L^1(\Omega)$ with

$$v \in BV(\Omega), \quad \int_\Omega v \, dx = c, \quad v(x) \in \{g_1(x), g_2(x)\} \text{ a.e.}$$

Again, in case any of these conditions on v fails to hold, the inequality is trivial. We may also assume that $g_1 \leqq v_\varepsilon \leqq g_2(x)$ since the truncated sequence

$$\tilde{v}_\varepsilon(x) = \begin{cases} g_1(x) & \{v_\varepsilon(x) < g_1(x)\} \\ v_\varepsilon(x) & \{g_1(x) \leqq v_\varepsilon(x) \leqq g_2(x)\} \\ g_2(x) & \{v_\varepsilon(x) > g_2(x)\} \end{cases}$$

satisfies $F_\varepsilon(v_\varepsilon) \geqq F_\varepsilon(\tilde{v}_\varepsilon)$ and $\tilde{v}_\varepsilon \to v$ in $L^1(\Omega)$. Now define $f : \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ by

$$f(x, s) = 2(s - g_1(x))(g_2(x) - s)$$

and $\psi_\varepsilon : \Omega \to \boldsymbol{R}$ by

$$\psi_\varepsilon(x) = \int_{g_1(x)}^{v_\varepsilon(x)} f(x, s) \, ds.$$

An application of the Cauchy-Schwarz inequality leads to

$$G_\varepsilon(v_\varepsilon) \geqq \int_\Omega f(x, v_\varepsilon) |\nabla v_\varepsilon| \, dx = \sup_{\substack{\sigma \in C_0^1(\Omega, R^n) \\ |\sigma| \leqq 1}} \int_\Omega f(x, v_\varepsilon) \langle \nabla v_\varepsilon, \sigma \rangle \, dx.$$

For fixed $\sigma \in C_0^1(\Omega, \boldsymbol{R}^n)$, $|\sigma| \leqq 1$, it follows that

$$G_\varepsilon(v_\varepsilon) \geqq \int_\Omega \langle \nabla \psi_\varepsilon(x), \sigma(x) \rangle - \left\langle \int_{g_1(x)}^{v_\varepsilon(x)} \nabla_x f(x, s)\, ds, \sigma(x) \right\rangle dx,$$

and an integration by parts yields

$$G_\varepsilon(v_\varepsilon) \geqq - \int_\Omega \int_{g_1(x)}^{v_\varepsilon(x)} (f(x, s)\, (\nabla \cdot \sigma(x))) + \langle \nabla_x f(x, s), \sigma(x) \rangle\, ds\, dx.$$

Using the L^∞ bounds on g_1, v_ε, f, $\nabla_x f$, σ and $\nabla \cdot \sigma$, we pass to the limit as $\varepsilon \to 0$ Thus,

$$\begin{aligned}
\liminf G_\varepsilon(v_\varepsilon) &\geqq - \int_\Omega \int_{g_1(x)}^{v(x)} f(x, s)\, (\nabla \cdot \sigma(x)) + \langle \nabla_x f(x, s), \sigma(x) \rangle\, ds\, dx \\
&= - \int_\Omega \chi_{\{v = g_2(x)\}} \int_{g_1(x)}^{g_2(x)} \nabla_x \cdot (f(x, s)\, \sigma(x))\, ds\, dx \\
&= - \int_\Omega \chi_{\{v = g_2(x)\}} \nabla \cdot (h(x)\, \sigma(x))\, dx.
\end{aligned}$$

Finally, taking the supremum over all admissible σ, we obtain an expression equivalent to (1.2). Therefore,

$$\liminf G_\varepsilon(v_\varepsilon) \geqq G_0(v).$$

To construct a sequence $\{\varrho_{\varepsilon_j}\}$ satisfying the analogues of (1.12), (1.13), *i.e.*

$$\varrho_{\varepsilon_j} \to v \quad \text{in } L^1(\Omega), \tag{1.40}$$

$$\lim_{j \to \infty} G_{\varepsilon_j}(\varrho_{\varepsilon_j}) = G_0(v), \tag{1.41}$$

we again first suppose that $v \in BV(\Omega)$ takes the form

$$v(x) = \begin{cases} g_1(x) & x \in A \\ g_2(x) & x \in B \end{cases}$$

with

$$\int_\Omega v\, dx = c$$

and $\partial A \cap \partial B$ smooth.

In constructing the transition layer sequence, the differential equation (1.17) of Theorem 1 is replaced by

$$\begin{aligned}
\frac{\partial z}{\partial s}(x, s) &= (z - g_1(x))\,(g_2(x) - z), \\
z(x, 0) &= \tfrac{1}{2}\,(g_1(x) + g_2(x)) =: \bar{g}(x).
\end{aligned} \tag{1.42}$$

Since g_1, g_2 are C^1 functions we obtain a solution $z: \Omega \times \mathbf{R} \to \mathbf{R}$ with $z \in C^1(\Omega \times \mathbf{R})$ ([7]). Arguing as before we find that

$$g_1(x) < z(x, s) < g_2(x) \quad \text{for all } s, \tag{1.43}$$

$$\lim_{s \to \infty} z(x, s) = g_2(x), \qquad \lim_{s \to -\infty} z(x, s) = g_1(x), \tag{1.44}$$

the limits on the right being approached at an exponential rate.

We also need an $L^\infty(\Omega \times \mathbf{R})$ bound on $z_x(x, s)$ (here z_x denotes the spatial gradient of z). To this end we note that

$$\int_{\bar{g}(x)}^{z(x,s)} \frac{d\eta}{(\eta - g_1(x))\,(g_2(x) - \eta)} = s. \tag{1.45}$$

Differentiating both sides with respect to x and solving the resulting equation for z_x, we obtain

$$z_x(x, s) = (z(x, s) - g_1(x))\,(g_2(x) - z(x, s)) \times \left[\frac{\nabla \bar{g}}{(\bar{g} - g_1)(\bar{g} - g_2)} + \nabla g_2 \int_{\bar{g}(x)}^{z(x,s)} \frac{d\eta}{(\eta - g_1)(\eta - g_2)^2} - \nabla g_1 \int_{\bar{g}(x)}^{z(x,s)} \frac{d\eta}{(\eta - g_1)^2 (g_2 - \eta)}\right]. \tag{1.46}$$

Since $g_1(x) < z(x, s) < g_2(x)$ by (1.43) and g_1 and g_2 are bounded in the $C^1(\Omega)$ topology, it follows that z_x is bounded for any finite s. Passing to the limit as $s \to \pm\infty$ in (1.46) and using L'Hospital's Rule, we conclude from (1.44) that

$$\lim_{s \to \infty} z_x(x, s) = \nabla g_2(x) \quad \text{and} \quad \lim_{s \to -\infty} z_x(x, s) = -\nabla g_1.$$

We thus infer that

$$\sup_{\Omega \times \mathbf{R}} |z_x(x, s)| < \infty. \tag{1.47}$$

Reintroducing the distance function d given by

$$d(x) = \begin{cases} -\operatorname{dist}(x, \partial A \cap \partial B) & x \in A \\ \operatorname{dist}(x, \partial A \cap \partial B) & x \in B, \end{cases}$$

one can define a boundary layer sequence $\{\tilde{\varrho}_\varepsilon\}$ through

$$\tilde{\varrho}_\varepsilon(x) = \begin{cases} g_2(x) & \{d > 2\sqrt{\varepsilon}\} \\ z(x, d(x)/\varepsilon) & \{|d| < \sqrt{\varepsilon}\} \\ g_1(x) & \{d < -2\sqrt{\varepsilon}\}, \end{cases}$$

where $\tilde{\varrho}_\varepsilon$ is linear in $d(x)$ on $\{\sqrt{\varepsilon} < |d| < 2\sqrt{\varepsilon}\}$ so as to be continuous for all x.

From here on the proof of (1.40), (1.41) follows in the same manner as did (1.12), (1.13), except that one must use (1.47) to estimate $|\nabla \tilde{\varrho}_\varepsilon|^2$ in proving the analogue of inequality (1.28).

2. Vector Dependent Energy

In this section, we consider a generalization of Theorem 1 to a variational problem in which the nonconvex integrand is vector-dependent. In order to preserve the "two-phase" nature of minimizers, we consider a nonnegative integrand $W: \boldsymbol{R}^2 \to \boldsymbol{R}$ which is zero on two disjoint closed curves Γ_1 and Γ_2, where Γ_1 lies in the interior of Γ_2.

As in Section 1, one goal is a characterization of the limit of minimizers of the perturbed problem. Theorem 3 shows that such a limit will again minimize interfacial surface area in Ω. As before, Ω is an open, bounded subset of $\boldsymbol{R}^n$ with Lipschitz-continuous boundary. However, this characterization is incomplete since the limit problem does not determine *where* on $\Gamma_1 \cup \Gamma_2$ the limit takes its values. We also characterize the cost per unit area along the interface of the transition made by the perturbed minimizers from Γ_1 to Γ_2. The latter is measured (asymptotically) by the length of a geodesic that minimizes distance with respect to a degenerate Riemannian metric on the plane. This is accomplished in Part A by identifying the Γ-limit and by proving Γ-convergence in this setting. In Part B we establish certain properties of the degenerate metric which were needed in Part A, including the existence of geodesics that minimize distance.

A. Generalization to $W: \boldsymbol{R}^2 \to \boldsymbol{R}$

Consider first a model in which W is only radially dependent. Let $u: \Omega \to \boldsymbol{R}^2$ and $W(u) = (|u| - a)^2 (|u| - b)^2$ with $0 < a < b$. Then the unperturbed problem is

$$\inf_{\substack{u \in L^1(\Omega, \boldsymbol{R}^2) \\ \int_\Omega |u| = c}} \int_\Omega (|u| - a)^2 (|u| - b)^2 \, dx, \tag{2.1}$$

where $a |\Omega| < c < b |\Omega|$.

Clearly any function with range on the circles of radii a and b that satisfies the constraint will minimize (2.1). Now introduce the perturbation

$$\varepsilon^2 |\nabla u|^2 (= \varepsilon^2 |\nabla u_1|^2 + \varepsilon^2 |\nabla u_2|^2)$$

and let u_ε denote a solution of

$$\inf_{\substack{u \in H^1(\Omega) \\ \int_\Omega |u| = c}} \int_\Omega (|u| - a)^2 (|u| - b)^2 + \varepsilon^2 |\nabla u|^2 \, dx. \tag{2.2}$$

Proposition 4. *Suppose there exists a scalar function* $R_0 \in L^1(\Omega)$ *such that* $|u_\varepsilon| \to R_0$ *in* $L^1(\Omega)$. *Then* R_0 *solves*

$$\inf_{\substack{R \in BV(\Omega) \\ R(x) \in \{a,b\} \text{ a.e.} \\ \int_\Omega R\,dx = c}} \operatorname{Per}_\Omega \{R = a\}. \tag{2.3}$$

Proof. If one writes

$$u(x) = R(x)\,(\cos\theta(x), \sin\theta(x)) \quad \text{with } R \geqq 0,$$

(2.2) becomes

$$\inf_{\substack{u\in H^1(\Omega)\\ \int_\Omega R dx = c}} \int_\Omega (R(x) - a)^2 ((R(x) - b)^2 + \varepsilon^2 |\nabla R|^2 + \varepsilon^2 R^2 |\nabla\theta|^2\, dx. \tag{2.4}$$

Then from

$$u_\varepsilon(x) = R_\varepsilon(x)\,(\cos\theta_\varepsilon(x), \sin\theta_\varepsilon(x)),$$

it is evident that a minimizer must satisfy $\nabla\theta_\varepsilon = 0$. The value of the constant θ_ε is arbitrary without any further boundary conditions or constraint. Since the infimum in (2.4) must be achieved by a function of the form

$$u(x) = R(x)\,(\cos\bar\theta, \sin\bar\theta),$$

with $\bar\theta \in \boldsymbol{R}$, $R \in H^1(\Omega)$, the problem reduces to the scalar case of Section 1. The proposition follows from Theorem 1.

Thus, the moduli of the minimizers of (2.2) converge in $L^1(\Omega)$ to a solution of the partition problem, and the phase is such as to effect the transition between the two zero states of W along a radial path in the plane.

Remark. The existence of a subsequential limit R_0 follows from Proposition 3. Note that since the value of the constant θ_ε is arbitrary, one cannot expect any determination of the constant phase θ_0 of the limit of minimizers $u_0 = R_0(\cos\theta_0, \sin\theta_0)$.

Generalization. Our model problem reduced to the scalar case because W was only radially dependent and the phase θ_ε of u_ε was constant. To generalize the problem we distort the radial dependence and consider $W = T^2$, where $T: \boldsymbol{R}^2 \to \boldsymbol{R}$ has the following properties:

$$T \in C^2, \quad T = 0 \quad \textit{only} \text{ on } \Gamma_1 \cup \Gamma_2,$$

where Γ_1, Γ_2 are two disjoint simple closed curves on the plane that admit C^3 regular parametrizations $\alpha: [0, 1] \to \Gamma_1$, $\beta: [0, 1] \to \Gamma_2$, respectively. Furthermore, we assume $\Gamma_1 \subset$ interior of Γ_2 and

$$T > 0 \text{ in } \mathcal{D}, \tag{2.5}$$

where $\mathcal{D}$ denotes the subset of $\boldsymbol{R}^2$ lying exterior to Γ_1 but interior to Γ_2. Finally, we suppose

$$|\nabla T(y)| \geqq m_0 \quad \text{for } y \in \partial\mathcal{D}\ (= \Gamma_1 \cup \Gamma_2) \quad \text{for some } m_0 > 0. \tag{2.6}$$

The unperturbed problem is now

$$\inf_{u\in L^1(\Omega,\boldsymbol{R}^2)} \int_\Omega T^2(u)\, dx. \tag{2.7}$$

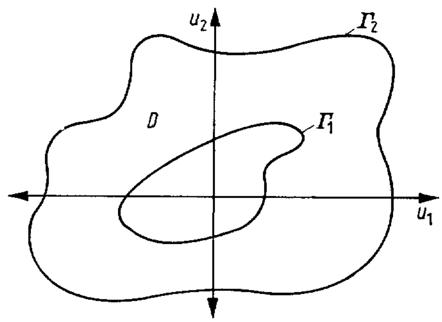

Fig. 5. $T = 0$ on $\Gamma_1 \cup \Gamma_2$

Its solutions u are in one-to-one correspondence with the partitions of Ω into sets A and B such that $u(x) \in \Gamma_1$ on A and $u(x) \in \Gamma_2$ on B. Choosing the usual perturbation, we obtain the perturbed problem:

$$\inf_{u \in H^1(\Omega, \boldsymbol{R}^2)} \int_\Omega T^2(u) + \varepsilon^2 |\nabla u|^2 \, dx. \tag{2.8}$$

As in Section 1, any L^1-convergent subsequence of minimizers must converge to a solution of the Γ-limit problem, so that the task is to evaluate this Γ-limit. However, in contrast to the scalar case, we are as yet unable to prove the compactness of minimizers (see Remark (2.30)).

We begin by defining the rescaled sequence of functionals $H_\varepsilon : L^1(\Omega) \to \boldsymbol{R}$ through

$$H_\varepsilon(u) = \begin{cases} \int_\Omega \frac{1}{\varepsilon} T^2(u) + \varepsilon |\nabla u|^2 \, dx & u \in H^1(\Omega, \boldsymbol{R}^2) \\ +\infty & \text{otherwise.} \end{cases}$$

The proposed limit functional $H_0 : L^1(\Omega) \to \boldsymbol{R}$ is given by

$$H_0(u) = \begin{cases} 2L(\underline{\gamma}) \operatorname{Per}_\Omega \{u \in \Gamma_1\} & T(u(x)) = 0 \text{ a.e., } \chi_{\{u \in \Gamma_1\}} \in BV(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$L(\gamma) := \int_{t_1}^{t_2} T(\gamma(t)) \, |\dot{\gamma}(t)| \, dt.$$

$L(\gamma)$ is defined for $\gamma : [t_1, t_2] \to \overline{\mathscr{D}}$, Lipschitz-continuous, and $\underline{\gamma}$ is a minimizer of

$$\inf_{\substack{\gamma(t_1) \in \Gamma_1 \\ \gamma(t_2) \in \Gamma_2}} L(\gamma). \tag{2.9}$$

The existence of $\underline{\gamma}$ is proved below, in Lemma 9.

Theorem 3.

$$\Gamma(L^1(\Omega)^-) \lim_{\varepsilon \to 0} H_\varepsilon(\varrho) = H_0(v).$$

Thus

(*i*) *for each* $v \in L^1(\Omega)$, *and for each sequence* $\{v_\varepsilon\}$ *in* $L^1(\Omega)$,

$$v_\varepsilon \to v \quad \text{in } L^1(\Omega) \text{ implies } \liminf H_\varepsilon(v_\varepsilon) \geqq H_0(v); \tag{2.10}$$

(*ii*) *for each* $v \in L^1(\Omega)$ *there exists a sequence* $\{\varrho_{\varepsilon_j}\}$ *in* $L^1(\Omega)$ *satisfying*

$$\varrho_{\varepsilon_j} \to v \text{ in } L^1(\Omega), \tag{2.11}$$

$$\lim_{j\to\infty} H_{\varepsilon_j}(\varrho_{\varepsilon_j}) = H_0(v). \tag{2.12}$$

As a prerequisite for the proof, we remark on the kinds of L^1-convergent sequences $\{v_\varepsilon\}$ and limits v that need to be considered in demonstrating the inequality (2.10).

One may immediately assume

$$T(v(x)) = 0 \quad \text{a.e. in } \Omega,$$

since otherwise $v_\varepsilon \to v$ implies that $\lim_{\varepsilon\to 0} H_\varepsilon(v_\varepsilon) = \infty$. Hence we assume v takes the form

$$v(x) = \begin{cases} a(x) & x \in A \\ b(x) & x \in B, \end{cases}$$

where $A \cup B = \Omega$ and $a, b \in L^1(\Omega)$ with $a: A \to \Gamma_1$, $b: B \to \Gamma_2$.

Concerning the sequence $\{v_\varepsilon\}$, one may suppose $v_\varepsilon \in H^1(\Omega)$ since otherwise $H_\varepsilon(v_\varepsilon) = \infty$. In fact, one may suppose $v_\varepsilon \in C^\infty$ since $C^\infty(\Omega)$ is dense in $H^1(\Omega)$.

In proving (2.10) we make use of properties established in Part B of this section concerning the degenerate Riemannian metric d_T defined by

$$d_T(y_1, y_2) = \inf_{\substack{\gamma \text{ Lipschitz-continuous} \\ \gamma(t_1)=y_1 \\ \gamma(t_2)=y_2}} \int_{t_1}^{t_2} T(\gamma(t))\, |\dot{\gamma}(t)|\, dt \quad \text{for } y_1, y_2 \in \bar{\mathscr{D}} \tag{2.13}$$

and the associated "distance to Γ_1", given by

$$h(y) = \inf_{y_0 \in \Gamma_1} d_T(y_0, y).$$

In particular, we note that h is Lipschitz-continuous in $\mathscr{D}$ (and therefore differentiable a.e.), and that

$$|\nabla h(y)| = T(y) \quad \text{for a.e. } y \in \mathscr{D}. \tag{2.14}$$

These facts are confirmed below in Lemma 11.

Now define

$$g_\varepsilon(x) = \begin{cases} h(v_\varepsilon(x)) & \text{for } \{v_\varepsilon \in \mathscr{D}\} \\ 0 & \text{elsewhere}. \end{cases}$$

Then the restriction of g_ε to $\{v_\varepsilon \in \mathscr{D}\}$ is a Lipschitz-continuous function satisfying

$$\nabla g_\varepsilon(x) = \nabla_y h(v_\varepsilon(x)) \cdot \nabla_x v_\varepsilon(x) \text{ a.e.}, \tag{2.15}$$

so that, in view of (2.14),

$$|\nabla g_\varepsilon| \leqq T(v_\varepsilon(x))\,|\nabla v_\varepsilon(x)| \quad \text{for a.e. } x \in \{v_\varepsilon \in \mathscr{D}\}. \tag{2.16}$$

Since $v_\varepsilon \to v$ in $L^1(\Omega)$, it follows that $g_\varepsilon \to h(v(x))$ in $L^1(\Omega)$, where

$$h(v(x)) = 0 \quad \text{in } A, \qquad h(v(x)) \geqq \underline{L}(\gamma) \quad \text{in } B. \tag{2.17}$$

As a final preliminary to the proof, observe that, for fixed $t \in (0, \underline{L}(\gamma))$, (2.17) implies

$$g_\varepsilon(x) - h(v(x)) > t \quad \text{in } \{g_\varepsilon > t\} \setminus B,$$
$$h(v(x)) - g_\varepsilon(x) \geqq \underline{L}(\gamma) - t \quad \text{in } B \setminus \{g_\varepsilon > t\}.$$

Thus

$$\int_\Omega |g_\varepsilon(x) - h(v(x))|\,dx \geqq \int_{\{g_\varepsilon > t\} \Delta B} |g_\varepsilon(x) - h(v(x))|\,dx$$
$$\geqq \min\{t, \underline{L}(\gamma) - t\}\,|\{g_\varepsilon(x) > t\} \Delta B| = \min\{t, \underline{L}(\gamma) - t\} \int_\Omega |\chi_{\{g_\varepsilon > t\}} - \chi_B|\,dx.$$

Consequently, $\chi_{\{g_\varepsilon > t\}} \to \chi_B$ in $L^1(\Omega)$ as $\varepsilon \to 0$ for all $t \in (0, \underline{L}(\gamma))$.

Proof of (2.10). To establish (2.10), we note first, using (2.16), that

$$H_\varepsilon(v_\varepsilon) \geqq \int_{\{0 < g_\varepsilon < \underline{L}(\gamma)\}} \frac{1}{\varepsilon} T^2(v_\varepsilon(x)) + \varepsilon\,|\nabla v_\varepsilon|^2\,dx$$
$$\geqq 2 \int_{\{0 < g_\varepsilon < \underline{L}(\gamma)\}} T(v_\varepsilon(x))\,|\nabla v_\varepsilon|\,dx$$
$$\geqq 2 \int_{\{0 < g_\varepsilon < \underline{L}(\gamma)\}} |\nabla g_\varepsilon(x)|\,dx.$$

Next, we apply the co-area formula for BV functions ([11], pg. 20). Since the support of $|\nabla \chi_{\{g_\varepsilon > t\}}| \subseteq \{g_\varepsilon = t\}$, we arrive at

$$H_\varepsilon(v_\varepsilon) \geqq 2 \int_0^{\underline{L}(\gamma)} \int_{\{0 < g_\varepsilon < \underline{L}(\gamma)\}} |\nabla \chi_{\{g_\varepsilon > t\}}|\,dt$$
$$= 2 \int_0^{\underline{L}(\gamma)} \left(\int_\Omega |\nabla \chi_{\{g_\varepsilon > t\}}| \right) dt.$$

Fatou's Lemma and Proposition 1 (lower semi-continuity) now yield:

$$\liminf H_\varepsilon(v_\varepsilon) \geqq 2 \int_0^{\underline{L}(\gamma)} \liminf \int_\Omega |\nabla \chi_{\{g_\varepsilon > t\}}|\,dt$$
$$\geqq 2 \int_0^{\underline{L}(\gamma)} \int_\Omega |\nabla \chi_B|\,dt$$
$$= 2\underline{L}(\gamma)\,\mathrm{Per}_\Omega\, B = 2\underline{L}(\gamma)\,\mathrm{Per}_\Omega\, A = H_0(v).$$

This establishes (2.10).

It remains to construct a sequence of functions satisfying (2.11) and (2.12). Toward this end, we insert here a remark about the solution of the following differential equation, which will be used in the boundary layer:

$$\frac{dz}{ds} = \frac{T(\gamma_{\varepsilon_j}(z))}{|\dot{\gamma}_{\varepsilon_j}(z)|}, \quad z(0) = \tfrac{1}{2}. \tag{2.18}$$

For any $\delta > 0$, $\gamma_\delta : [0, 1] \to \boldsymbol{R}^2$ is a C^1, regular curve that minimizes the distance between Γ_1 and Γ_2 in the metric $d_{T_\delta}(y_1, y_2)$ given by

$$d_{T_\delta}(y_1, y_2) = \inf_{\substack{\gamma(t_1)=y_1 \\ \gamma(t_2)=y_2}} \int_{t_1}^{t_2} (T(\gamma(t)) + \delta)\,|\dot{\gamma}(t)|\,dt.$$

The existence of such a geodesic is demonstrated in Lemma 4. For the purpose of the construction, δ is chosen equal to ε_j, although it would suffice to admit any $\delta = \delta(\varepsilon_j)$ that approaches zero with ε_j.

Since the value of d_{T_ε} is independent of parametrization, we require γ_{ε_j} to have constant speed, take

$$|\dot{\gamma}_{\varepsilon_j}(t)| = s_{\varepsilon_j} = \text{Euclidean arclength},$$

and write $\gamma_{\varepsilon_j}(0) = a_{\varepsilon_j} \in \Gamma_1$, and $\gamma_{\varepsilon_j}(1) = b_{\varepsilon_j} \in \Gamma_2$. For $t \in (0, 1)$, $\gamma_{\varepsilon_j} \in \mathcal{D}$ and γ_{ε_j} tends uniformly to $\underline{\gamma}$ (see Lemma 9). Finally, Lemma 6 asserts that $s_\varepsilon < c_1$, where c_1 is a positive constant independent of ε.

Denote by z_{ε_j} the solution of (2.18). From (2.18) follows

$$\int_{\frac{1}{2}}^{z_{\varepsilon_j}} \frac{|\dot{\gamma}_{\varepsilon_j}(\eta)|}{T(\gamma_{\varepsilon_j}(\eta))}\,d\eta = s. \tag{2.19}$$

For $\eta \in (0, 1)$, the integrand is positive and has singularities at the endpoints of this interval. Furthermore, Taylor's Theorem yields

$$T(\gamma_{\varepsilon_j}(\eta)) = |\langle \nabla T(\gamma_{\varepsilon_j}(\tilde{\varrho})), \dot{\gamma}_{\varepsilon_j}(\tilde{\varrho})\rangle|\,|1 - \eta| \quad \text{for some } \tilde{\varrho} \in (\eta, 1).$$

It then follows from the estimate (2.86) of Lemma 10, proved below, that there exist positive numbers $\bar{s}$ and $\bar{m}$, independent of ε, such that:

$$T(\gamma_{\varepsilon_j}(\eta)) \geqq \bar{m}\,|1 - \eta| \quad \text{provided } 1 - \bar{s} \leqq \eta \leqq 1.$$

A similar estimate holds for η near 0. We now conclude from (2.19) that

$$\lim_{s \to \infty} z_{\varepsilon_j}(s) = 1, \tag{2.20}$$

$$\lim_{s \to -\infty} z_{\varepsilon_j}(s) = 0. \tag{2.21}$$

In fact, our estimate implies

$$|1 - z_{\varepsilon_j}(s)| \leqq \tilde{c}e^{-\frac{\bar{m}}{c_1}s} \quad \text{as } s \to \infty, \tag{2.22}$$

where $\tilde{c}$ is another constant independent of ε. An analogous statement applies to the rate of convergence of the limit (2.21).

Proof of (2.11), (2.12). The proof now proceeds in two steps. In the first, one supposes that v takes on only two values; in the second, the construction is adapted to cope with a more general v. In either case, we may assume $T(v(x)) = 0$ a.e. and $\chi_{\{v \in \Gamma_1\}} \in BV(\Omega)$ since otherwise $H_0(v) = \infty$ and the construction is trivial.

Step 1. Assume v takes the form

$$v(x)=\begin{cases}a_0\in\Gamma_1 & x\in A\\ b_0\in\Gamma_2 & x\in B,\end{cases}$$

where $A\cup B=\Omega$, $\chi_A\in BV(\Omega)$ and $\underline{\gamma}(0)=a_0$, $\underline{\gamma}(1)=b_0$. As in the scalar case, one may also assume without loss of generality that $\Gamma:=\partial A\cap\partial B$ is smooth (see Lemma 1).

Using Lemma 2, we introduce the distance function $d:\Omega\to\boldsymbol{R}$ by means of

$$d(x)=\begin{cases}\operatorname{dist}(x,\Gamma) & x\in B\\ -\operatorname{dist}(x,\Gamma) & x\in A,\end{cases}$$

and define the construction $\{\varrho_\varepsilon\}$ by the formula

$$\varrho_{\varepsilon_j}(x)=\begin{cases}a_{\varepsilon_j} & \text{for } \{d(x)<-2\sqrt{\varepsilon_j}\}\\ \gamma_{\varepsilon_j}\left(z_{\varepsilon_j}\left(\dfrac{d(x)}{\varepsilon_j}\right)\right) & \text{for } \{|d(x)|<\sqrt{\varepsilon_j}\}\\ b_{\varepsilon_j} & \text{for } \{d(x)>2\sqrt{\varepsilon_j}\},\end{cases}\tag{2.23}$$

with ϱ_{ε_j} linear in $d(x)$ for $\{\sqrt{\varepsilon_j}\leqq|d(x)|\leqq 2\sqrt{\varepsilon_j}\}$, so that ϱ_{ε_j} is continuous and in $H^1(\Omega)$. Note that the uniform convergence of γ_{ε_j} to γ (Lemma 9) implies that $a_{\varepsilon_j}\to a_0$ and $b_{\varepsilon_j}\to b_0$, so that (2.11) is immediate.

To prove (2.12), write

$$\begin{aligned}H_{\varepsilon_j}(\varrho_{\varepsilon_j})=\int\limits_{\{|d|<\sqrt{\varepsilon_j}\}}&\frac{1}{\varepsilon_j}T^2\left(\gamma_{\varepsilon_j}\left(z_{\varepsilon_j}\left(\frac{d(x)}{\varepsilon_j}\right)\right)\right)\\ &+\varepsilon_j\left|\dot{\gamma}_{\varepsilon_j}\left(z_{\varepsilon_j}\left(\frac{d(x)}{\varepsilon_j}\right)\right)\right|^2\left|\dot{z}_{\varepsilon_j}\left(\frac{d(x)}{\varepsilon_j}\right)\right|^2\frac{1}{\varepsilon_j^2}|\nabla d|^2\,dx\\ &+\int\limits_{\{\sqrt{\varepsilon_j}<|d|<2\sqrt{\varepsilon_j}\}}\text{linear piece}.\end{aligned}\tag{2.24}$$

First consider the integral of the linear piece over $\{\sqrt{\varepsilon_j}<d<2\sqrt{\varepsilon_j}\}$:

$$\begin{aligned}&\int\limits_{\{\sqrt{\varepsilon_j}<d<2\sqrt{\varepsilon_j}\}}\frac{1}{\varepsilon_j}T^2(\varrho_{\varepsilon_j})+\varepsilon_j|\nabla\varrho_{\varepsilon_j}|^2\,dx\\ &\quad=\int\limits_{\{\sqrt{\varepsilon_j}<d<2\sqrt{\varepsilon_j}\}}\frac{1}{\varepsilon_j}T^2\left(\left[\frac{b_{\varepsilon_j}-\gamma_{\varepsilon_j}\left(z_{\varepsilon_j}\left(\frac{1}{\sqrt{\varepsilon_j}}\right)\right)}{\sqrt{\varepsilon_j}}\right](d(x)-2\sqrt{\varepsilon_j})+b_{\varepsilon_j}\right)\\ &\qquad+\varepsilon_j\left[\frac{b_{\varepsilon_j}-\gamma_{\varepsilon_j}\left(z_{\varepsilon_j}\left(\frac{1}{\sqrt{\varepsilon_j}}\right)\right)}{\sqrt{\varepsilon_j}}\right]^2|\nabla d|^2\,dx.\end{aligned}$$

Since $|\nabla d| = 1$ for ε_j small by (1.9), the estimate (2.22) implies that the integral above approaches 0 as $\varepsilon_j \to 0$. The same is true of

$$\int_{-2\sqrt{\varepsilon_j}}^{-\sqrt{\varepsilon_j}} \text{linear piece}.$$

Then we use (2.18) and the co-area formula (1.27) to arrive at

$$\limsup_{j\to\infty} H_{\varepsilon_j}(\varrho_{\varepsilon_j}) = \limsup \frac{2}{\varepsilon_j} \int_{\{|d|<\sqrt{\varepsilon_j}\}} T^2\left(\dot{\gamma}_{\varepsilon_j}\left(z_{\varepsilon_j}\left(\frac{d(x)}{\varepsilon_j}\right)\right)\right) dx$$

$$= \limsup \frac{2}{\varepsilon_j} \int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} T^2\left(\gamma_{\varepsilon_j}\left(z_{\varepsilon_j}\left(\frac{s}{\varepsilon_j}\right)\right)\right) H^{n-1}\{d(x) = s\}\, ds.$$

Making the change of variables $\eta = z_{\varepsilon_j}\left(\dfrac{s}{\varepsilon_j}\right)$, we obtain with the aid of (1.10),

$$\limsup_{j\to\infty} H_{\varepsilon_j}(\varrho_{\varepsilon_j}) = \limsup 2 \int_{z_{\varepsilon_j}(-1/\sqrt{\varepsilon_j})}^{z_{\varepsilon_j}(1/\sqrt{\varepsilon_j})} T(\gamma_{\varepsilon_j}(\eta))\, |\dot{\gamma}_{\varepsilon_j}(\eta)|\, H^{n-1}\{d(x) = \varepsilon_j z_{\varepsilon_j}^{-1}(\eta)\}\, d\eta$$

$$\leq \limsup 2 \left(\sup_{|s|\leq\sqrt{\varepsilon_j}} H^{n-1}\{d(x) = s\}\right) \int_0^1 T(\gamma_{\varepsilon_j}(\eta))\, |\dot{\gamma}_{\varepsilon_j}(\eta)|\, d\eta$$

$$= 2H^{n-1}(\Gamma) \limsup_{j\to\infty} L(\gamma_{\varepsilon_j}) = 2\, \mathrm{Per}_\Omega\{v \in \Gamma_1\}\, L(\underline{\gamma}),$$

since $\lim_{j\to\infty} L(\gamma_{\varepsilon_j}) = L(\underline{\gamma})$ from (2.85). Combining this inequality with the reverse inequality from (2.10) yields (2.12).

Step 2. To establish (2.11), (2.12) for a general v consider $v \in L^1(\Omega)$ satisfying

$$v(x) = \begin{cases} a(x) & \text{for } x \in A \\ b(x) & \text{for } x \in B, \end{cases}$$

where $A \cup B = \Omega$, $\chi_A \in BV(\Omega)$, $a : A \to \Gamma_1$, $b : B \to \Gamma_2$. Without loss of generality, again assume $\Gamma := \partial A \cup \partial B$ is smooth. Further, we extend $a(x) \equiv a_0 = \underline{\gamma}(0)$ for $x \notin A$ and $b(x) \equiv b_0 = \underline{\gamma}(1)$ for $x \notin B$ whenever it is necessary to consider these functions on points beyond their original domains of definition.

Recall the assumption that Γ_1 and Γ_2 admit C^3 parametrizations $\alpha : [0, 1] \to \Gamma_1$, $\beta : [0, 1] \to \Gamma_2$, which are 1–1, surjective maps restricted to [0, 1).

In the construction of a sequence $\{\varrho_{\varepsilon_j}\}$ satisfying (2.11), (2.12) in this more general setting, our strategy is as follows: first smooth $a(x)$ and $b(x)$ away from Γ using mollification, the mollification radius being dependent on the distance to Γ; then bridge from a_{ε_j} to b_{ε_j} across Γ by using the construction (2.23).

In order to keep $H_{\varepsilon_j}(\varrho_{\varepsilon_j})$ finite, the mollification of $a(x)$ and $b(x)$ must be effected in such a way as to leave the values of the functions on $\Gamma_1 \cup \Gamma_2$. To this end, we introduce the L^1 map $q_\varepsilon : \Omega \to [0, 1)$ defined by

$$q_\varepsilon(x) = \begin{cases} \alpha^{-1}(a(x)) & \text{for } \{d(x) < -\varepsilon^{1/4}\} \\ \beta^{-1}(b(x)) & \text{for } \{d(x) > \varepsilon^{1/4}\} \\ 0 & \text{elsewhere in } \boldsymbol{R}^n. \end{cases} \tag{2.25}$$

Now let $\eta \in C_0^\infty(\boldsymbol{R}^n)$ satisfy

$$0 \leqq \eta \leqq 1, \quad \eta(x) = \eta(|x|), \quad \text{support } \eta \subseteqq \{|x| < 1\}, \quad \int_{\boldsymbol{R}^n} \eta(x)\, dx = 1,$$

and let

$$\eta_\varepsilon := \varepsilon^{-\frac{1}{3}n} \eta\left(\frac{x}{\varepsilon^{1/3}}\right),$$

so that

$$\int_{\boldsymbol{R}^n} \eta_\varepsilon(x) = 1.$$

Then define

$$t_\varepsilon := \eta_\varepsilon * q_\varepsilon = \int_{\boldsymbol{R}^n} \eta_\varepsilon(x - y)\, q_\varepsilon(y)\, dy.$$

Claim. $t_\varepsilon : \Omega \to [0, 1)$ is smooth and has the following properties:

$$\alpha(t_\varepsilon) \xrightarrow{L^1(A)} a, \qquad \beta(t_\varepsilon) \xrightarrow{L^1(B)} b, \tag{2.26}$$

$$|d(x)| < 4\varepsilon^{1/2} \Rightarrow t_\varepsilon(x) = 0 \quad \text{for } \varepsilon \text{ small}, \tag{2.27}$$

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega |\nabla t_\varepsilon|^2\, dx = 0. \tag{2.28}$$

To prove (2.26), note first that

$$\int_A |\alpha(t_\varepsilon(x)) - a(x)|\, dx \leqq |\alpha'|_{L^\infty} \int_A |t_\varepsilon(x) - \alpha^{-1}(a(x))|\, dx.$$

Hence it will suffice to show that the quantity on the right tends to zero with ε. From the triangle inequality follows

$$|t_\varepsilon - \alpha^{-1}(a)|_{L^1(A)} \leqq |\eta_\varepsilon * (q_\varepsilon - \alpha^{-1}(a))|_{L^1(A)} + |\eta_\varepsilon * \alpha^{-1}(a) - \alpha^{-1}(a)|_{L^1(A)}.$$

Since the last term clearly approaches zero, we only need to show that the same is true of the first term on the right hand side of this inequality. Now

$$\begin{aligned} &\int_A \left| \int_{\boldsymbol{R}^n} \eta_\varepsilon(x - y)\,(q_\varepsilon(y) - \alpha^{-1}(a(y)))\, dy \right| dx \\ &\qquad \leqq \int_A \int_{\{-\varepsilon^{1/4} < d(y) < 0\}} \eta_\varepsilon(x - y)\, |q_\varepsilon(y) - \alpha^{-1}(a(y))|\, dy\, dx \\ &\qquad \leqq \int_{\{-\varepsilon^{1/4} < d(y) < 0\}} |\alpha^{-1}(a(y))| \int_A \eta_\varepsilon(x - y)\, dx\, dy = O(\varepsilon^{1/4}). \end{aligned}$$

The second part of (2.26) follows similarly. To confirm (2.27), suppose $|d(x)| \leqq 4\varepsilon^{\frac{1}{2}}$. Then

$$t_\varepsilon(x) = \int_{R^n} q_\varepsilon(x-y)\,\varepsilon^{-\frac{1}{3}n}\eta\left(\frac{y}{\varepsilon^{1/3}}\right)dy = \int_{\{|y|<\varepsilon^{1/3}\}} q_\varepsilon(x-y)\,\varepsilon^{-\frac{1}{3}n}\eta\left(\frac{y}{\varepsilon^{1/3}}\right)dy. \tag{2.29}$$

Now

$$|d(x)| \leqq 4\varepsilon^{\frac{1}{2}} \quad \text{implies} \quad |d(x-y)| \leqq 4\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{3}}.$$

But for ε sufficiently small, $4\varepsilon^{1/2} + \varepsilon^{1/3} < \varepsilon^{1/4}$, so (2.25) and (2.29) imply (2.27).

Finally, to prove (2.28), note that since

$$\text{support } \eta_\varepsilon \subset \{|x| < \varepsilon^{1/3}\},$$

one has

$$|\nabla t(x)| \leqq c\varepsilon^{-1/3}$$

where c depends on η, but not on ε. Then

$$\varepsilon\,|\nabla t_\varepsilon(x)|^2 \leqq c^2\varepsilon^{1/3}$$

and (2.28) follows.

We now define a sequence $\{\varrho_{\varepsilon_j}\} \in H^1(\Omega)$ which will serve to verify (2.11), (2.12).

$$\text{Let } \tilde{\varrho}_{\varepsilon_j}(x) := \begin{cases} \alpha(t_{\varepsilon_j}(x)) & \text{for } \{d(x) < -4\sqrt{\varepsilon_j}\} \\ \alpha\left(\dfrac{\varkappa_j}{\sqrt{\varepsilon_j}}(d(x) + 4\sqrt{\varepsilon_j})\right) & \text{for } \{-4\sqrt{\varepsilon_j} \leqq d(x) \leqq -3\sqrt{\varepsilon_j}\} \\ \varrho_{\varepsilon_j}(x) & \text{for } \{|d(x)| < 3\sqrt{\varepsilon_j}\} \\ \beta\left(\dfrac{\tilde{\varkappa}_j}{\sqrt{\varepsilon_j}}(4\sqrt{\varepsilon_j} - d(x))\right) & \text{for } \{3\sqrt{\varepsilon_j} \leqq d(x) \leqq 4\sqrt{\varepsilon_j}\} \\ \beta(t_{\varepsilon_j}(x)) & \text{for } \{d(x) > 4\sqrt{\varepsilon_j}\}, \end{cases}$$

where $\varrho_{\varepsilon_j}(x)$ is given by (2.23) and $\varkappa_j$, $\tilde{\varkappa}_j$ are defined by

$$\alpha(\varkappa_j) = a_{\varepsilon_j}, \qquad \beta(\tilde{\varkappa}_j) = b_{\varepsilon_j}.$$

The continuity of $\tilde{\varrho}_{\varepsilon_j}$ along $|d(x)| = 4\sqrt{\varepsilon_j}$ is guaranteed by (2.27).

The verification of (2.11) follows from (2.26) since

$$\int_\Omega |\tilde{\varrho}_{\varepsilon_j}(x) - v(x)|\,dx = \int_A |\alpha(t_{\varepsilon_j}(x)) - a(x)|\,dx + \int_B |\beta(t_{\varepsilon_j}(x)) - b(x)|\,dx + O(\sqrt{\varepsilon_j}).$$

To obtain (2.12) we infer by calculation that

$$H_{\varepsilon_j}(\tilde{\varrho}_{\varepsilon_j}) = \varepsilon_j \int\limits_{\{d<-4\sqrt{\varepsilon_j}\}} |\alpha'(t_{\varepsilon_j}(x))|^2 \, |\nabla t_{\varepsilon_j}(x)|^2 \, dx$$

$$+ \varepsilon_j \int\limits_{\{d>4\sqrt{\varepsilon_j}\}} |\beta'(t_{\varepsilon_j}(x))|^2 \, |\nabla t_{\varepsilon_j}(x)|^2 \, dx$$

$$+ \int\limits_{\{-4\sqrt{\varepsilon_j}\leq d\leq -3\sqrt{\varepsilon_j}\}} \left|\alpha'\left(\frac{\varkappa_j}{\sqrt{\varepsilon_j}}(d(x) + 4\sqrt{\varepsilon_j})\right)\right|^2 \varkappa_j^2 \, |\nabla d|^2 \, dx$$

$$+ \int\limits_{\{3\sqrt{\varepsilon_j}\leq d\leq 4\sqrt{\varepsilon_j}\}} \left|\beta'\left(\frac{\tilde{\varkappa}_j}{\sqrt{\varepsilon_j}}(4\sqrt{\varepsilon_j} - d(x))\right)\right|^2 \tilde{\varkappa}_j^2 \, |\nabla d|^2 \, dx$$

$$+ \int\limits_{\{|d|<3\sqrt{\varepsilon_j}\}} \frac{1}{\varepsilon_j} T^2(\varrho_{\varepsilon_j}(x)) + \varepsilon_j \, |\nabla \varrho_{\varepsilon_j}|^2 \, dx.$$

The first two of these integrals approach zero with ε because of (2.28). The next two terms involve bounded integrands taken over sets of measure $O(\sqrt{\varepsilon})$ and hence also approach zero. Therefore,

$$\lim_{j\to\infty} H_{\varepsilon_j}(\tilde{\varrho}_{\varepsilon_j}) = \lim_{j\to\infty} \int\limits_{\{|d|<3\sqrt{\varepsilon_j}\}} \frac{1}{\varepsilon_j} T^2(\varrho_{\varepsilon_j}) + \varepsilon_j \, |\nabla \varrho_{\varepsilon_j}|^2 \, dx = 2L(\underline{\gamma}) \operatorname{Per}_\Omega \{v \in \Gamma_1\},$$

as was shown in Step 1. This completes the proof of Theorem 3.

Remark (2.30). The failure of $\{\theta_\varepsilon\}$ to converge in our model problem (2.4) emerges here as well. This is reflected in the fact that the Γ-limit H_0 does not characterize where on $\Gamma_1 \cup \Gamma_2$ a minimizer takes its values. At present, this indeterminacy is a hindrance in proving compactness of minimizers of H_ε, as well as in proving the existence of local minimizers (see [18]). Presumably, a clearer description of the limits of minimizers could be obtained by finding one more term in the expansion of the minimum energy with respect to ε. Nonetheless, Theorem 3 does give a partial characterization of the limit points of minimizers of H_ε.

Remark (2.31). We have presented Theorem 3 without an integral constraint, such as

$$\int\limits_\Omega |u| \, dx = c,$$

in order to simplify the proof and focus on the identification of the Γ-limit H_0. Such a constraint could be included in a similar manner as in Theorem 1.

B. Properties of the Degenerate Metric d_T

This section establishes the existence of distance-minimizing geodesics and other related properties of the degenerate Riemannian metric d_T (see 2.13) given by $dy^2 = T^2 \, dx^2$, which were used in Part A. The approach adopted is as follows:

first we prove the existence of geodesics γ_δ that minimize distance in the metric d_{T_δ} given by $dy^2 = (T + \delta)^2\, dx^2$; then we obtain a uniform bound on the Euclidean arc-length of γ_δ; finally, we pass to the limit as $\delta \to 0$ and obtain a geodesic in the metric d_T.

Begin by letting $\mathscr{D}'$ denote an open, bounded subset of $\boldsymbol{R}^2$ with $\overline{\mathscr{D}} \subset \mathscr{D}'$. Then, for $y \in \boldsymbol{R}^2$ and $\delta > 0$, we define the map $T_\delta: \boldsymbol{R}^2 \to \boldsymbol{R}$ by

$$T_\delta(y) = \begin{cases} T(y) + \delta & y \in \overline{\mathscr{D}} \\ \frac{1}{2}\,\delta & y \in \boldsymbol{R}^2 \setminus \mathscr{D}' \end{cases} \tag{2.32}$$

with $T_\delta \in C^2(\boldsymbol{R}^2)$ and satisfying

$$\tfrac{1}{2}\,\delta \leqq T_\delta(y) \leqq \delta \quad \text{for } y \in \mathscr{D}' \setminus \overline{\mathscr{D}}.$$

Next define the functional L_δ by

$$L_\delta(\gamma) = \int_{t_1}^{t_2} T_\delta(\gamma(t))\, |\dot{\gamma}(t)|\, dt \tag{2.33}$$

for $\gamma: [t_1, t_2] \to \boldsymbol{R}^2$ Lipschitz-continuous. Note that the value of $L_\delta(\gamma)$ does not depend on the parametrization of γ.

We can now introduce a (nondegenerate) Riemannian metric on the plane d_{T_δ}, which is, in fact, conformally equivalent to the standard one:

$$d_{T_\delta}(y_1, y_2) = \inf_{\substack{\gamma(t_1) = y_1 \\ \gamma(t_2) = y_2}} L_\delta(\gamma). \tag{2.34}$$

The first task is to establish the existence of geodesics that minimize distance with respect to d_{T_δ}. This follows from the Hopf-Rinow Theorem once it is shown that the plane endowed with this metric is geodesically complete. A geodesic is an extremal for L_δ, and thus must satisfy the Euler-Lagrange equation

$$|\dot{\gamma}|\, \nabla T_\delta(\gamma) = \frac{d}{dt}\left(T_\delta(\gamma)\frac{\dot{\gamma}}{|\dot{\gamma}|}\right). \tag{2.35}$$

Lemma 3. (Geodesic completeness) *Every locally defined solution of* (2.35) *can be extended to a solution defined for all t in* $\boldsymbol{R}$.

Proof. It is useful to write (2.35) as a first-order 4×4 system with a view toward appeal to standard existence and extension theorems. This process is simplified by seeking a solution for which $|\dot{\gamma}(t)| \equiv c > 0$. Such a solution to (2.35) must satisfy

$$c^2\, \nabla T_\delta(\gamma) = (\nabla T_\delta(\gamma) \cdot \dot{\gamma})\, \dot{\gamma} + T_\delta(\gamma)\, \ddot{\gamma}, \tag{2.36}$$

which we wish to solve for $\gamma(0) = p$, $\dot{\gamma}(0) = v$, $p, v \in \boldsymbol{R}^2$ and c chosen equal to $|v|$. Writing $\eta = (\eta_1, \eta_2, \eta_3, \eta_4) = (\gamma_1, \gamma_2, \dot{\gamma}_1, \dot{\gamma}_2)$, (2.36) has an equivalent representation as the first order 4×4 system

$$\eta' = f(\eta) = \left(\eta_3, \eta_4, \frac{c^2\,\partial_1 T_\delta(\eta_1,\eta_2) - \nabla T_\delta(\eta_1,\eta_2)\cdot(\eta_3,\eta_4)\,\eta_3}{T_\delta(\eta_1,\eta_2)},\right.$$
$$\left.\frac{c^2\,\partial_2 T_\delta(\eta_1,\eta_2) - \nabla T_\delta(\eta_1,\eta_2)\cdot(\eta_3,\eta_4)\,\eta_4}{T_\delta(\eta_1,\eta_2)}\right)$$

with $(\eta_1(0),\eta_2(0)) = p$, $(\eta_3(0),\eta_4(0)) = v$ as initial conditions. Here $\partial_1 = \frac{\partial}{\partial\eta_1}$, $\partial_2 = \frac{\partial}{\partial\eta_2}$.

Now $T_\delta \in C^2$ and from (2.32) it follows that $f(\eta) = (\eta_3,\eta_4,0,0)$ for $(\eta_1,\eta_2) \in \mathbf{R}^2 \setminus \mathscr{D}'$. Thus we conclude that f is a globally Lipschitz-continuous function. Hence the local solution of $\eta' = f(\eta)$ has a globally defined extension ([6]).

It remains to show that any solution of (2.36) does indeed have constant speed. Let γ be the local solution of (2.36) with $c = |v|$ and consider the inner product of $\dot\gamma$ with both sides of (2.36):

$$|v|^2\,\nabla T_\delta(\gamma)\cdot\dot\gamma = (\nabla T_\delta(\gamma)\cdot\dot\gamma)\,|\dot\gamma|^2 + T_\delta(\gamma)\,\dot\gamma\cdot\ddot\gamma.$$

Thus,

$$\frac{(|v|^2 - |\dot\gamma|^2)}{T_\delta(\gamma)}\,\nabla T_\delta(\gamma)\cdot\dot\gamma = \frac{d}{dt}\,|\dot\gamma|^2. \tag{2.37}$$

Viewing (2.37) as an equation determining $|\dot\gamma|^2$ with initial condition $|\dot\gamma|^2(0) = |v|^2$, one sees that local uniqueness of the solution to this differential equation implies $|\dot\gamma|^2 \equiv |v|^2$. This completes the proof of Lemma 3.

Using Lemma 3, we can assert:

Lemma 4. (Hopf-Rinow) *Given any points* $y_1, y_2 \in \mathbf{R}^2$, *there exists a geodesic* $\gamma: [t_1, t_2] \to \mathbf{R}^2$ *with* $\gamma(t_1) = y_1, \gamma(t_2) = y_2$, *such that*

$$d_{T_\delta}(y_1, y_2) = L_\delta(\gamma).$$

This is a classical result proved, for example, in HERMANN ([16]) and DOCARMO ([8]).

Next consider the variable endpoint problem

$$\inf_{\substack{y_1\in\Gamma_1\\ y_2\in\Gamma_2}} d_\delta(y_1, y_2). \tag{2.38}$$

One seeks a geodesic that achieves this infimum. Define $H_\delta: \Gamma_1\times\Gamma_2 \to \mathbf{R}$ by

$$H_\delta(y_1, y_2) = \inf_{\substack{\gamma(t_1)=y_1\\ \gamma(t_2)=y_2}} L_\delta(\gamma).$$

This map is continuous for all (y_1, y_2) in the compact set $\Gamma_1\times\Gamma_2$ and hence achieves its minimum at some pair (a_δ, b_δ) with $a_\delta \in \Gamma_1$ and $b_\delta \in \Gamma_2$. Lemma 4 then guarantees the existence of a geodesic $\gamma_\delta: [0, t_\delta] \to \mathbf{R}^2$, such that $\gamma_\delta(0) = a_\delta$, $\gamma_\delta(t_\delta) = b_\delta$ and $L_\delta(\gamma_\delta)$ achieves the infimum in (2.38). It must also be true that

$\gamma_\delta(t) \in \mathscr{D}$ for $0 < t < t_\delta$, for otherwise it would not be a minimizer of (2.38). The following lemma asserts that γ_δ satisfies a transversality condition at its endpoints.

Lemma 5. *The geodesic γ_δ minimizing* (2.38), *at both of its endpoints, satisfies the condition*

$$\dot{\gamma}_\delta \parallel \nabla T(\gamma_\delta). \tag{2.39}$$

Proof. Define a family of competing curves by a map

$$k(s, t): [0, 1] \times [0, t_\delta] \to \bar{\mathscr{D}}$$

satisfying

$$k(\tilde{s}_\delta, t) = \gamma_\delta(t), \tag{2.40}$$

$$k(s, 0) = \alpha(s), \tag{2.41}$$

$$k(s, t_\delta) = \gamma_\delta(t_\delta), \tag{2.42}$$

where, as before, $\alpha: [0, 1] \to \Gamma_1$ is a parametrization of Γ_1 and $\alpha(\tilde{s}_\delta) = \gamma_\delta(0) = a_\delta$.

Since γ_δ minimizes L_δ,

$$\frac{d}{ds} L_\delta(k(s, t))\bigg|_{s=\tilde{s}_\delta} = 0.$$

Use of (2.40) shows that

$$\int_0^{t_\delta} |\dot{\gamma}_\delta| \nabla T(\gamma_\delta) \cdot \frac{\partial k}{\partial s}(\tilde{s}_\delta, t) + (T(\gamma_\delta) + \delta) \frac{\dot{\gamma}_\delta}{|\dot{\gamma}_\delta|} \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s} k(\tilde{s}_\delta, t)\right) dt = 0.$$

Integrating by parts yields

$$\int_0^{t_\delta} \left\langle |\dot{\gamma}_\delta| \nabla T(\gamma_\delta) - \frac{d}{dt}\left((T(\dot{\gamma}_\delta) + \delta) \frac{\dot{\gamma}_\delta}{|\dot{\gamma}_\delta|}\right), \frac{\partial k}{\partial s}(\tilde{s}_\delta, t)\right\rangle dt \tag{2.43}$$

$$+ \frac{(T(\gamma_\delta(t)) + \delta)}{|\dot{\gamma}_\delta(t)|} \left\langle \dot{\gamma}_\delta(t), \frac{\partial k}{\partial s}(\tilde{s}_\delta, t)\right\rangle \Bigg]_0^{t_\delta} = 0.$$

Since γ_δ solves (2.35), the integral in (2.43) vanishes. From (2.41) and (2.42) follows

$$\frac{\partial k}{\partial s}(\tilde{s}_\delta, 0) = \alpha'(\tilde{s}_\delta)$$

and

$$\frac{\partial k}{\partial s}(\tilde{s}_\delta, t_\delta) = \frac{\partial}{\partial s}(\gamma_\delta(t_\delta)) = 0.$$

Then (2.43) implies

$$\frac{\delta}{|\dot{\gamma}_\delta(0)|}\langle\dot{\gamma}_\delta(0), \alpha'(\tilde{s}_\delta)\rangle = 0.$$

Since

$$\langle\alpha'(\tilde{s}_\delta), \nabla T(\alpha(\tilde{s}_\delta))\rangle = 0,$$

we have

$$\dot{\gamma}_\delta \parallel \nabla T(\gamma_\delta) \text{ at } t = 0.$$

Choosing a family of curves with a variable endpoint along Γ_2 yields the corresponding condition at $t = t_\delta$, and (2.39) is established.

In order to assert the existence of a distance-minimizing geodesic between Γ_1 and Γ_2 for the metric d_T defined by (2.13), one must pass to the limit as $\delta \to 0$ along $\{\gamma_\delta\}$. The following lemma establishes the compactness necessary to obtain a subsequential limit curve.

Lemma 6. *Let s_δ be the Euclidean arc-length of γ_δ. Then there exists $c_1 > 0$, independent of δ, such that $s_\delta < c_1$ for all δ.*

The proof of Lemma 6, which we present later, relies on Lemmas 7 and 8 below. Lemma 7 gives uniform bounds on the arc-length of γ_δ when the curve is near Γ_1 or Γ_2. We then show in Lemma 8 that once γ_δ departs from the boundaries, it never again comes too close. This conclusion supplies a bound on the arc-length of the middle piece of γ_δ.

Assume $\gamma_\delta : [0, s_\delta] \to \bar{\mathscr{D}}$ is parametrized by Euclidean arc-length. To analyze γ_δ near Γ_1, we introduce local coordinates (u, v) in a tubular neighborhood of Γ_1:

$$y = (y_1, y_2) = M(u, v) := \alpha(u) + vn(u). \tag{2.44}$$

Here $\alpha : [0, L_1] \to \Gamma_1$ is taken be to parametrized by Euclidean arc-length and $n(u)$ is the unit normal to Γ_1 at $\alpha(u)$, pointing into $\mathscr{D}$. Since $\partial\mathscr{D}$ is smooth and compact, a uniform interior disk condition holds along $\partial\mathscr{D}$: for each $y \in \partial\mathscr{D}$, there is a disk D_y of radius r_y such that

$$\bar{D}_y \cap (R^2 \setminus \mathscr{D}) = y$$

and such that $\inf_{y\in\partial\mathscr{D}} r_y = \mu$ for some $\mu > 0$. In particular, for all $y \in \partial\mathscr{D}$,

$$|k| \leqq \frac{1}{\mu}, \tag{2.45}$$

where $k = k(u)$ represents the curvature of $\partial\mathscr{D}$ at y. The coordinate map M defined by (2.44) is a C^2-diffeomorphism for $0 < v < \mu$. (See [10], Appendix A, as well as [19]).

In this neighborhood of Γ_1, let $z_\delta(s) = (u_\delta(s), v_\delta(s))$ and define $\tilde{T}$ by

$$M(z_\delta(s)) = \gamma_\delta(s) \tag{2.46}$$

and

$$\tilde{T}(u, v) = T(M(u, v)). \tag{2.47}$$

Consider the function $\lambda_\delta : \boldsymbol{R} \to \boldsymbol{R}$ given by

$$\lambda_\delta(s) = \frac{\dot{u}_\delta(s)}{\dot{v}_\delta(s)}, \tag{2.48}$$

in which the superior dots indicate differentiation with respect to s. The quantity λ_δ measures the tangential speed of γ_δ relative to the normal speed with respect to Γ_1. Thus λ_δ small means that γ_δ is progressing efficiently away from Γ_1 towards Γ_2. The following lemma enables us to bound the arc-length near Γ_1.

Lemma 7. *There exists a number* $c_2 \in (0, \mu)$, *independent of* δ, *such that*

$$|\lambda_\delta(s)| \leqq \tfrac{1}{2} \quad \text{for } 0 \leqq s \leqq \min\{s : v_\delta(s) = c_2\}. \tag{2.49}$$

This estimate is needed to prove

Lemma 8. *There is a positive number* c_3 *independent of* δ *such that*

$$\operatorname{dist}(\gamma_\delta(s), \Gamma_1) \geqq c_3, \tag{2.50}$$

provided $s \geqq \min\{s : \operatorname{dist}(\gamma_\delta(s), \Gamma_1) = c_2\}$, *where* "dist" *refers to Euclidean distance.*

Proof of Lemma 7. The proof of Lemma 7 is split into three steps: first, we derive a differential equation satisfied by λ_δ; then we use this to obtain a differential inequality; finally, we integrate this inequality to obtain (2.49).

Step 1. To derive a differential equation for λ_δ, note first that $\lambda_\delta(0) = 0$ since (2.44), (2.46) imply

$$\dot{\gamma}_\delta(s) = \dot{u}_\delta(s)\,\alpha'(u_\delta(s)) + v_\delta(s)\,\dot{u}_\delta(s)\,n'(u_\delta(s)) + \dot{v}_\delta(s)\,n(u_\delta(s)). \tag{2.51}$$

By (2.39), $\dot{\gamma}_\delta(0)$ is orthogonal to $\alpha'(u_\delta(0))$, while $v_\delta(0) = 0$, so that

$$0 = \langle \dot{\gamma}_\delta(0), \alpha'(u_\delta(0)) \rangle = \dot{u}_\delta(0)\,\langle \alpha'(u_\delta(0)), \alpha'(u_\delta(0)) \rangle = \dot{u}_\delta(0),$$

the primes denoting differentiation with respect to u.

It then follows from (2.48) that λ_δ solves the initial value problem

$$\dot{\lambda}_\delta = \frac{\dot{v}_\delta \ddot{u}_\delta - \dot{u}_\delta \ddot{v}_\delta}{(\dot{v}_\delta)^2} = \frac{\ddot{u}_\delta}{\dot{v}_\delta} - \lambda \frac{\ddot{v}_\delta}{\dot{v}_\delta}, \qquad \lambda_\delta(0) = 0. \tag{2.52}$$

Now, since γ_δ is an extremal for the functional L_δ, z_δ must be an extremal for the functional

$$z \to \int_{t_1}^{t_2} (\tilde{T}(z) + \delta) \left| \frac{d}{dt} M(z(t)) \right| dt.$$

Setting the first variation equal to zero, we obtain the Euler-Lagrange equation

$$|\dot{M}(z_\delta)|\,\nabla\tilde{T}(z_\delta) = \frac{d}{ds}\left((\tilde{T}(z_\delta)+\delta)\frac{\dot{M}(z_\delta)}{|\dot{M}(z_\delta)|}\right) J_M(z_\delta),$$

where $\dot{M}(z_\delta) = \dfrac{d}{ds} M(z_\delta(s))$ and J_M is the Jacobian matrix of the coordinate map M. Since $|\dot{M}(z_\delta)| = |\dot{\gamma}| = 1$, the equation reduces to

$$\nabla\tilde{T}(z_\delta) = \frac{d}{ds}((\tilde{T}(z_\delta)+\delta)\,\dot{M}(z_\delta))\,J_M(z_\delta)$$

or

$$\nabla\tilde{T}(z_\delta) = \langle\nabla\tilde{T}(z_\delta), \dot{z}_\delta\rangle\,\dot{M}(z_\delta)\,J_M(z_\delta) + (\tilde{T}(z_\delta)+\delta)\,\ddot{M}(z_\delta)\,J_M(z_\delta). \tag{2.53}$$

Use of (2.46) and (2.51) yields

$$\ddot{M}(z_\delta) = \ddot{u}_\delta\alpha' + \ddot{v}_\delta n + (v_\delta\ddot{u}_\delta + 2\dot{u}_\delta\dot{v}_\delta)\,n' + (\dot{u}_\delta)^2\,\alpha'' + v_\delta(\dot{u}_\delta)^2\,n'',$$

while (2.44) gives

$$M_u(z_\delta) = \alpha'(u_\delta) + v_\delta n'(u_\delta),\; M_v(z_\delta) = n(u_\delta).$$

Applying the Frenet equations ([9])

$$\alpha'' = kn, \quad n' = -k\alpha', \tag{2.54}$$

we can calculate

$$\dot{M}(z_\delta)\,J_M(z_\delta) = (\langle\dot{M}(z_\delta), M_u(z_\delta)\rangle, \langle\dot{M}(z_\delta), M_v(z_\delta)\rangle).$$

$$\ddot{M}(z_\delta)\,J_M(z_\delta) = (\langle\ddot{M}(z_\delta), M_u(z_\delta)\rangle, \langle\ddot{M}(z_\delta), M_v(z_\delta)\rangle).$$

In this manner we arrive at

$$\dot{M}(z_\delta)\,J_M(z_\delta) = (\dot{u}_\delta(1-kv_\delta)^2, \dot{v}_\delta), \tag{2.55}$$

$$\begin{aligned}\ddot{M}(z_\delta)\,J_M(z_\delta) = (&\ddot{u}_\delta(1-kv_\delta)^2 - (1-kv_\delta)(2k\dot{u}_\delta\dot{v}_\delta + k'v_\delta(\dot{u}_\delta)^2),\\ &\ddot{v}_\delta + k(\dot{u}_\delta)^2(1-kv_\delta)),\end{aligned} \tag{2.56}$$

where

$$k' = \frac{d}{du}(k(u))\bigg|_{u=u_\delta}.$$

Note that $\alpha \in C^3$ assures that the signed curvature k defined by (2.54) is C^1.

Substituting (2.55), (2.56) into the Euler-Lagrange equation (2.53) and solving for $\ddot{u}_\delta$ and $\ddot{v}_\delta$, one finds that

$$\ddot{u}_\delta = \frac{\tilde{T}_u - \langle\nabla\tilde{T}, \dot{z}_\delta\rangle(1-kv_\delta)^2\,\dot{u}_\delta}{(\tilde{T}(z_\delta)+\delta)(1-kv_\delta)^2} + \frac{\omega_\delta}{(1-kv_\delta)}, \tag{2.57}$$

$$\ddot{v}_\delta = \frac{\tilde{T}_v - \langle\nabla\tilde{T}, \dot{z}_\delta\rangle\,\dot{v}_\delta}{\tilde{T}(z_\delta)+\delta} - k(\dot{u}_\delta)^2(1-kv_\delta), \tag{2.58}$$

where

$$\tilde{T}_u = \frac{\partial \tilde{T}}{\partial u}(u, v)\bigg|_{(u,v)=(u_\delta,v_\delta)}, \quad \tilde{T}_v = \frac{\partial \tilde{T}}{\partial v}(u, v)\bigg|_{(u,v)=(u_\delta,v_\delta)},$$

and

$$\omega_\delta := 2k\dot{u}_\delta \dot{v}_\delta + k' v_\delta (\dot{u}_\delta)^2. \tag{2.59}$$

Note that the (u, v) coordinates are only defined for $v < \mu$, so that $1 - kv_\delta > 0$ by (2.45). Thus $\ddot{u}_\delta$ is finite. Substitution from (2.57) and (2.58) yields

$$\dot{\lambda}_\delta = \frac{\tilde{T}_u - \langle \nabla \tilde{T}, \dot{z}_\delta \rangle (1 - kv_\delta)^2 \dot{u}_\delta}{(\tilde{T}(z_\delta) + \delta)(1 - kv_\delta)^2 \dot{v}_\delta} - \lambda_\delta \left(\frac{\tilde{T}_v - \langle \nabla \tilde{T}, \dot{z}_\delta \rangle \dot{v}_\delta}{(\tilde{T}(z_\delta) + \delta) \dot{v}_\delta} \right) + \frac{\omega_\delta + \lambda_\delta k (\dot{u}_\delta)^2 (1 - kv_\delta)^2}{(1 - kv_\delta) \dot{v}_\delta}.$$

Since $\lambda_\delta = \dot{u}_\delta / \dot{v}_\delta$, one arrives at

$$\dot{\lambda}_\delta = \frac{\tilde{T}_u}{(\tilde{T}(z_\delta) + \delta)(1 - kv_\delta)^2 \dot{v}_\delta} - \frac{\lambda_\delta \tilde{T}_v}{(\tilde{T}(z_\delta) + \delta) \dot{v}_\delta} + \frac{\omega_\delta + \lambda_\delta k (\dot{u}_\delta)^2 (1 - kv_\delta)^2}{(1 - kv_\delta) \dot{v}_\delta}. \tag{2.60}$$

This completes step 1.

Step 2. We now estimate the terms on the right side of (2.60) to obtain differential inequalities that control λ_δ.

From here on we shall restrict our attention to $\{s : v_\delta(s) \leqq \mu/2\}$, so that (2.45) implies

$$\tfrac{1}{2} \leqq 1 - kv_\delta \leqq \tfrac{3}{2}. \tag{2.61}$$

According to Taylor's Theorem and (2.61), there are positive constants c_4 and $v_1 \leqq \frac{\mu}{2}$, such that

$$\left| \frac{\tilde{T}_u(u_\delta, v_\delta)}{(\tilde{T}(u_\delta, v_\delta) + \delta)(1 - kv_\delta)^2} \right| \leqq \tfrac{1}{4} \left| \frac{\tilde{T}_u(u_\delta, 0) + O(v_\delta)}{\tilde{T}(u_\delta, 0) + \delta + \tilde{T}_v(u_\delta, 0) v_\delta + O(v_\delta^2)} \right| \leqq c_4 \tag{2.62}$$

for $0 \leqq v_\delta \leqq v_1$, since $\tilde{T}(u_\delta, 0) = \tilde{T}_u(u_\delta, 0) = 0$, while $\tilde{T}_v(u_\delta, 0) = |\nabla T(a_\delta)| \geqq m_0$ by (2.6). It also follows from (2.6) that there are positive constants c_5 and c_6 satisfying

$$\begin{aligned} \frac{\tilde{T}_v(u_\delta, v_\delta)}{\tilde{T}(u_\delta, v_\delta) + \delta} &= \frac{\tilde{T}_v(u_\delta, 0) + O(v_\delta)}{\tilde{T}(u_\delta, 0) + \delta + \tilde{T}_v(u_\delta, 0) v_\delta + O(v_\delta^2)} \\ &= \frac{|\nabla T(a_\delta)| + O(v_\delta)}{|\nabla T(a_\delta)| v_\delta + \delta + O(v_\delta^2)} \\ &\geqq \frac{c_5}{c_6 v_\delta + \delta} \end{aligned} \tag{2.63}$$

for $0 \leqq v_\delta \leqq v_1$, in which $a_\delta := \gamma_\delta(0)$. We emphasize the fact that v_1, c_4, c_5, c_6 are constants depending on T and its derivatives, but independent of δ.

Using (2.51), the Frenet equations, and the fact that γ_δ is parametrized by arc-length, one obtains

$$(\dot{u}_\delta)^2 (1 - kv_\delta)^2 + (\dot{v}_\delta)^2 = 1. \tag{2.64}$$

In view of (2.59), this implies the existence of another positive constant, c_7, independent of δ, such that

$$|\omega_\delta| < c_7, \tag{2.65}$$

since k, k' are continuous functions on the compact set Γ_1, and hence are bounded.

To establish (2.49), first consider λ_δ restricted to

$$\{s : 0 \leqq \lambda_\delta(s') \leqq 1 \text{ for all } s' \in [0, s]\}.$$

This may contain only $s = 0$ if $\lambda_\delta(s) < 0$ for all small $s \neq 0$.

Using the definition of λ_δ and (2.64), one is led to

$$\frac{1}{\dot{v}_\delta^2} = (1 + (1 - kv_\delta)^2 \lambda_\delta^2). \tag{2.66}$$

Thus (2.61) leads to

$$\frac{1}{\dot{v}_\delta^2} \leqq (1 + \tfrac{9}{4} \lambda_\delta^2) < 4 \tag{2.67}$$

for $\lambda_\delta(s)$ restricted as above. Noting that $\dot{v}_\delta(0) = 1$ implies $\dot{v}_\delta(s) > 0$ for the values of s under consideration, one draws from (2.66) that

$$\frac{1}{\dot{v}_\delta} \geqq \dot{v}_\delta. \tag{2.68}$$

We now apply the preceding estimates to control $\dot{\lambda}_\delta$ in (2.60). Estimates (2.61), (2.62), (2.64), (2.65), (2.67) and (2.68) combine to yield an L^∞ bound on the first and third terms in (2.60):

$$\frac{\tilde{T}_u}{(\tilde{T}(z_\delta) + \delta)(1 - kv_\delta)^2 \dot{v}_\delta} + \frac{\omega_\delta + \lambda_\delta k (\dot{u}_\delta)^2 (1 - kv_\delta)^2}{(1 - kv_\delta)\dot{v}_\delta} \tag{2.69}$$

$$\leqq \frac{c_4}{\dot{v}_\delta} + \frac{2(c_7 + |k|_{L^\infty})}{\dot{v}_\delta} < 2c_4 + 4c_7 + 4|k|_{L^\infty} := c_8,$$

where the last equality defines the positive constant c_8.

On applying (2.63) and (2.68) to (2.60), one arrives at the desired differential inequality

$$\dot{\lambda}_\delta(s) \leqq - \frac{c_5 \dot{v}_\delta}{c_6 v_\delta + \delta} \lambda_\delta(s) + c_8 \tag{2.70}$$

for λ_δ restricted to $\{s : 0 \leqq \lambda_\delta(s') \leqq 1 \text{ for all } s' \in [0, s]\}$.

It could be the case, however, that $\lambda_\delta(s) \leqq 0$ as z_δ departs from Γ_1. Entertaining this possibility, consider λ_δ restricted to $\{s : -1 \leqq \lambda_\delta(s') \leqq 0$ for all $s' \in [0, s]\}$. Then inequalities (2.67) and (2.68) still apply, and combine with (2.63) and (2.69) to imply the differential inequality

$$\dot{\lambda}_\delta \geqq -\frac{c_5 \dot{v}_\delta}{c_6 v_\delta + \delta} \lambda_\delta - c_8. \tag{2.71}$$

Step 3. We now integrate inequalities (2.70) and (2.71) to obtain (2.49).

First suppose λ_δ remains nonnegative as z_δ departs from Γ_1. Then (2.70) applies. Multiplying (2.70) by $(c_6 v_\delta + \delta)^{c_5/c_6}$, one has

$$\frac{d}{ds}\left(\lambda_\delta (c_6 v_\delta + \delta)^{c_5/c_6}\right) \leqq c_8 (c_6 v_\delta + \delta)^{c_5/c_6}.$$

Since $\lambda_\delta(0) = 0$, we can use (2.67) and integrate this inequality to find

$$\begin{aligned}\lambda_\delta(s)\,(c_6 v_\delta(s) + \delta)^{c_5/c_6} &\leqq c_8 \int_0^s (c_6 v_\delta(s') + \delta)^{c_5/c_6}\, ds' \\ &< 2c_8 \int_0^s (c_6 v_\delta(s') + \delta)^{c_5/c_6}\, \dot{v}_\delta(s')\, ds'.\end{aligned}$$

Thus one arrives at

$$\begin{aligned}\lambda_\delta(s)\,(c_6 v_\delta(s) + \delta)^{c_5/c_6} &\leqq \frac{2c_8}{c_5 + c_6}\left[(c_6 v_\delta(s) + \delta)^{c_5/c_6 + 1} - \delta^{c_5/c_6 + 1}\right] \\ &\leqq \frac{2c_8}{c_5 + c_6}(c_6 v_\delta(s) + \delta)^{c_5/c_6 + 1}.\end{aligned}$$

Consequently,

$$\lambda_\delta(s) < \left(\frac{2c_8}{c_5 + c_6}\right)(c_6 v_\delta(s) + \delta). \tag{2.72}$$

We conclude that if $\lambda_\delta(s)$ remains positive as z_δ departs from Γ_1, then

$$\lambda_\delta(s) \leqq \tfrac{1}{2}, \tag{2.73}$$

provided

$$\delta < \frac{c_5 + c_6}{8c_8} \quad \text{and } 0 \leqq s \leqq \min\left\{s : v(s) = \frac{c_5 + c_6}{8c_6 c_8} \text{ or } v_\delta(s) = v_1\right\}.$$

If, instead, $\lambda_\delta(s)$ remains negative as z_δ departs from Γ_1, the same analysis as before yields

$$\lambda_\delta(s) \geqq -\left(\frac{2c_8}{c_5 + c_6}\right)(c_6 v_\delta(s) + \delta), \tag{2.74}$$

so that in this case,

$$\lambda_\delta(s) \geqq -\tfrac{1}{2}, \tag{2.75}$$

provided

$$\delta < \frac{c_5 + c_6}{8c_8} \quad \text{and } 0 \leqq s \leqq \min\left\{s : v_\delta(s) = \frac{c_5 + c_6}{8c_6c_8} \text{ or } v_\delta(s) = v_1\right\}.$$

Finally, λ_δ might change sign before v_δ reaches the value

$$\min\left\{\frac{c_5 + c_6}{8c_6c_8}, v_1\right\}.$$

Thus, there may be one or more positive parameter values $\{\tau_\delta\}$ such that $\lambda_\delta(\tau_\delta) = 0$, while

$$v_\delta(\tau_\delta) < \min\left\{\frac{c_5 + c_6}{8c_6c_8}, v_1\right\}.$$

In this case we repeat the preceding argument using (2.70) or (2.71) on each parameter interval between successive zeroes of λ_δ, depending on the sign of λ_δ in each interval, and we again reach eventually the conclusion that $|\lambda_\delta(s)| \leqq \tfrac{1}{2}$.

Combining (2.73) and (2.75) with the preceding remark, we infer (2.49) with

$$c_2 = \min\left\{\frac{c_5 + c_6}{8c_6c_8}, v_1\right\}.$$

Proof of Lemma 8. To show that once γ_δ departs from Γ_1, it never again comes too close in the sense of (2.50), we suppose otherwise and seek a contradiction. Thus, we suppose that for all positive $\eta < c_2$ there exists a $\delta > 0$ and a parameter value

$$\bar{s}_\delta > \min\{s : \text{dist}(\gamma_\delta(s), \Gamma_1) = c_2\}$$

such that

$$\text{dist}(\gamma_\delta(\bar{s}_\delta), \Gamma_1) = \eta. \tag{2.76}$$

If γ_δ is to minimize L_δ among all curves joining Γ_1 to Γ_2, then in particular it must yield the lowest value of L_δ calculated between its initial point a_δ and any intermediate point p on its graph, when compared to any other curve joining Γ_1 to p. We now construct a competing curve that gives a lower value to L_δ under the hypothesis (2.76).

For each $\eta < c_2$ there must exist a constant $\bar{u}_\delta$ where

$$\gamma_\delta(\bar{s}_\delta) = M(\bar{u}_\delta, \eta).$$

Define the competing curve in (u, v)-coordinates by

$$\zeta_\delta(s) := (\bar{u}_\delta, s) \quad \text{for } 0 \leqq s \leqq \eta.$$

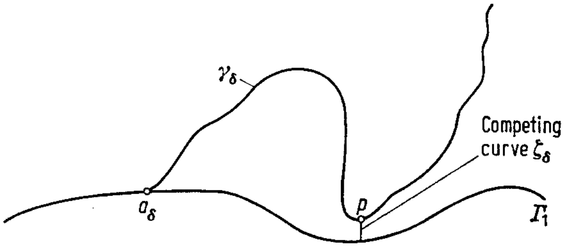

Fig. 6

Then, using (2.44) and the Mean Value Theorem, one has

$$L_\delta(\zeta_\delta) = \int_0^\eta (\tilde{T}(\zeta_\delta) + \delta) \left| \frac{d}{ds} M(\bar{u}_\delta, s) \right| ds$$

$$= \int_0^\eta (\tilde{T}(\bar{u}_\delta, s) + \delta) \, ds$$

$$= \int_0^\eta \tilde{T}_v(u_\delta, \xi(\bar{u}_\delta, s)) \, s \, ds + \delta\eta \qquad \text{for } 0 \leqq \xi \leqq \eta .$$

Hence

$$L_\delta(\zeta_\delta) \leqq \max_{s \in [0, c_2]} |\tilde{T}_v(u, s)| \cdot \tfrac{1}{2} \eta^2 + \delta\eta . \tag{2.77}$$

On the other hand, the parameter must have values t'_δ, t''_δ, with $0 < t'_\delta < t''_\delta < \bar{s}_\delta$, such that

$$\text{dist}\,(\gamma_\delta(t'_\delta), \Gamma_1) = v_\delta(t'_\delta) = \tfrac{1}{2} c_2 \tag{2.78}$$

and

$$\text{dist}\,(\gamma_\delta(t''_\delta), \Gamma_1) = v_\delta(t''_\delta) = c_2 . \tag{2.79}$$

Restricting γ_δ to $s \in [0, \bar{s}_\delta]$, we then infer that

$$L_\delta(\gamma_\delta) = \int_0^{\bar{s}_\delta} T(\gamma_\delta(s)) \, ds$$

$$\geqq \int_{t'_\delta}^{t''_\delta} T(\gamma_\delta(s)) \, ds$$

$$\geqq \min_{c_2/2 \leqq \text{dist}(y, \Gamma_1) \leqq c_2} T(y) \, (t''_\delta - t'_\delta) .$$

Thus, because of (2.78) and (2.79),

$$L_\delta(\gamma_\delta) \geqq \left(\min_{\frac{c_2}{2} \leqq \text{dist}(y, \Gamma_1) \leqq c_2} T(y) \right) \tfrac{1}{2} c_2$$

since the arc-length of γ_δ between $\gamma_\delta(t'_\delta)$ and $\gamma(t''_\delta)$ cannot be less than $v_\delta(t''_\delta) - v_\delta(t'_\delta)$.

Comparing (2.77) to this last inequality, one concludes that

$$\left(\min_{c_2/2 \leqq \operatorname{dist}(y, \Gamma_1) \leqq c_2} T(y)\right) \tfrac{1}{2} c_2 \leqq \max_{s \in [0, c_2]} |\tilde{T}_v(u, s)| \cdot \tfrac{1}{2} \eta^2 + \delta\eta$$

for all $\eta < c_2$, if γ_δ is to minimize L_δ. Finally, choosing η sufficiently small, one arrives at a contradiction and (2.50) is proved.

We can now establish a uniform bound on the arc-length of $\{\gamma_\delta\}$.

Proof of Lemma 6. Lemma 7 permits a reparametrization of $z_\delta(s)$ with v_δ as the new parameter. A uniform bound on the arc-length of this initial piece of the curves $\{\gamma_\delta\}$ for $0 \leqq v_\delta \leqq c_2$ is now immediate:

$$\begin{aligned} \int_0^{c_2} \sqrt{1 + \left|\frac{du_\delta}{dv_\delta}\right|^2}\, dv_\delta &= \int_0^{c_2} \sqrt{1 + |\lambda_\delta|^2}\, dv_\delta \\ &\leqq \int_0^{c_2} \sqrt{1 + (\tfrac{1}{2})^2}\, dv_\delta \\ &= \tfrac{1}{2}\sqrt{5}\, c_2 . \end{aligned}$$

The argument leading to Lemma 7 and Lemma 8 can be repeated without alteration to establish estimates analogous to (2.49) and (2.50), valid in a neighborhood of Γ_2.

This leads to the conclusion that for $\gamma_\delta \colon [0, s_\delta] \to \mathscr{D}$, parametrized by arc-length, there are values s_δ^* and s_δ^{**} of the parameter with $0 < s_\delta^* < s_\delta^{**} < s_\delta$ such that

$$s_\delta^* \leqq \frac{\sqrt{5}}{2} c_2, \tag{2.81}$$

$$s_\delta - s_\delta^{**} \leqq \frac{\sqrt{5}}{2} c_2, \tag{2.82}$$

$$\min\{\operatorname{dist}(\gamma_\delta(s), \Gamma_1), \operatorname{dist}(\gamma_\delta(s), \Gamma_2)\} \geqq c_3 \quad \text{for } s_\delta^* \leqq s \leqq s_\delta^{**}. \tag{2.83}$$

We have yet to obtain a uniform bound on $s_\delta^{**} - s_\delta^*$, which is the arclength of the middle piece of γ_δ. Let $l(t)$ be the parametrization of a line segment that minimizes the Euclidean distance between Γ_1 and Γ_2, and let d be its length. Then,

$$L_\delta(\ell) \leqq \left(\max_{y \in \overline{\mathscr{D}}} T(y) + \delta\right) d \leqq \left(2 \max_{y \in \overline{\mathscr{D}}} T(y)\right) d.$$

On the other hand, by (2.83),

$$\begin{aligned} L_\delta(\gamma_\delta) &\geqq \int_{s_\delta^*}^{s_\delta^{**}} T(\gamma_\delta(s))\, ds \\ &\geqq \left(\min_{\substack{\operatorname{dist}(y, \partial\mathscr{D}) \geqq c_3 \\ y \in \overline{\mathscr{D}}}} T(y)\right) (s_\delta^{**} - s_\delta^*). \end{aligned}$$

Thus $L_\delta(\gamma_\delta) \leqq L_\delta(l)$ implies the uniform bound

$$s_\delta^{**} - s_\delta^* \leqq \frac{\left(2 \max\limits_{y \in \overline{\mathscr{D}}} T(y)\right) d}{\left(\min\limits_{\substack{\operatorname{dist}(y,\partial\mathscr{D}) \geqq c_3 \\ y \in \mathscr{D}}} T(y)\right)}.$$

Writing $s_\delta = s_\delta^* + (s_\delta^{**} - s_\delta^*) + (s_\delta - s_\delta^{**})$ and using the last inequality together with (2.81), (2.82), we obtain the desired uniform bound on the arc-length of γ_δ. This completes the proof of Lemma 6.

One can now pass to the limit as $\delta \to 0$.

Lemma 9. *There exists a subsequence* $\{\gamma_{\delta_j}\}$ *converging uniformly to a limit* $\underline{\gamma}$, *which is a minimizer of* (2.9) *and satisfies the Euler-Lagrange equation*:

$$|\dot{\underline{\gamma}}| \, \nabla T(\underline{\gamma}) = \frac{d}{dt}\left(T(\underline{\gamma}) \frac{\dot{\gamma}}{|\dot{\underline{\gamma}}|}\right) \quad \text{in } \mathscr{D}.$$

Proof. Reparametrizing γ_δ by setting $t = \dfrac{s}{s_\delta}$, we obtain a sequence of curves $\gamma_\delta \colon [0, 1] \to \mathscr{D}$ which according to Lemma 6, obeys $|d\gamma_\delta/dt| = s_\delta < c_1$. Applying the Arzelà-Ascoli Theorem, one infers the uniform convergence of a subsequence γ_{δ_j}, to a Lipschitz-continuous limit $\underline{\gamma}$.

To see that $\underline{\gamma}$ does indeed minimize (2.9), let $\xi \colon [t_1, t_2] \to \overline{\mathscr{D}}$ be any Lipschitz-continuous curve with $\xi(t_1) \in \Gamma_1$ and $\xi(t_2) \in \Gamma_2$. Since γ_δ minimizes L_δ, it follows that

$$L_\delta(\gamma_\delta) \leqq L_\delta(\xi) = L(\xi) + \delta \int_{t_1}^{t_2} |\dot{\xi}(t)| \, dt$$

so that

$$\limsup_\delta L_\delta(\gamma_\delta) \leqq L(\xi).$$

On the other hand, the uniform convergence of $\{\gamma_{\delta_j}\}$ to $\underline{\gamma}$ implies that

$$\liminf_{j \to \infty} s_{\delta_j} \geqq |\dot{\underline{\gamma}}(t)| \qquad \text{a.e.}$$

Thus, appealing to Fatou's Lemma, one has

$$\begin{aligned} \liminf_{j \to \infty} L_{\delta_j}(\gamma_{\delta_j}) &= \liminf_j \int_0^1 T(\gamma_{\delta_j}) \, |\dot{\gamma}_{\delta_j}(t)| \, dt \\ &\geqq \int_0^1 \liminf_j T(\gamma_{\delta_j}) \, s_{\delta_j} \, dt \qquad (2.84) \\ &\geqq \int_0^1 T(\underline{\gamma}) \, |\dot{\underline{\gamma}}(t)| \, dt = L(\underline{\gamma}). \end{aligned}$$

We conclude that $L(\underline{\gamma}) \leqq L(\xi)$, and so $\underline{\gamma}$ is a minimizer of (2.9).

The regularity of $\underline{\gamma}$ now follows easily upon introduction of geodesic polar coordinates. Invoking the L-minimizing property of $\underline{\gamma}$ just established, one shows that between any two points of $\underline{\gamma}$ the curve must in fact coincide with the geodesic joining these points ([8], pg. 292). Thus $\underline{\gamma}$ must satisfy the Euler-Lagrange equation for L.

Corollary.

$$\lim_{j\to\infty} L(\gamma_{\delta_j}) = L(\underline{\gamma}). \tag{2.85}$$

Proof. Since $\underline{\gamma}$ minimizes L among curves joining Γ_1 to Γ_2, it is immediate that

$$\liminf_{j\to\infty} L(\gamma_{\delta_j}) \geqq L(\underline{\gamma}).$$

But, since γ_{δ_j} minimizes L_{δ_j} and $\underline{\gamma}$ is Lipschitz-continuous, it follows that

$$\limsup_{j\to\infty} L(\gamma_{\delta_j}) \leqq \limsup L_{\delta_j}(\gamma_{\delta_j}) \leqq \limsup L_{\delta_j}(\underline{\gamma}) = L(\underline{\gamma}).$$

Hence (2.85) holds.

Having established the existence of a geodesic in the d_T metric that joins Γ_1 and Γ_2, we turn to two final results which were needed in Part A. The first is a uniform estimate of the angle $\dot{\gamma}_\delta$ makes with $\nabla T(\gamma_\delta)$.

Lemma 10. *There are positive numbers $\bar{s}$ and $\overline{m}$, independent of δ, such that γ_δ: $[0, 1] \to \mathscr{D}$ satisfies*

$$|\langle \nabla T(\gamma_\delta(s)), \dot{\gamma}_\delta(s)\rangle| \geqq \overline{m} \quad \text{if } 0 \leqq s \leqq \bar{s} \quad \text{or} \quad 1 - \bar{s} \leqq s \leqq 1. \tag{2.86}$$

Proof. This assertion follows from (2.39) together with Lemma 7, which supplies a uniform bound on the amount by which γ_δ can stray from the normal direction.

Now consider the function measuring distance to Γ_1 in the d_T metric, $h: \mathscr{D} \to \boldsymbol{R}$ given by

$$h(y) = \inf_{y_0\in\Gamma_1} d_T(y_0, y) \quad \text{(see (2.13))}.$$

Lemma 11. *The function h is a Lipschitz-continuous function on $\mathscr{D}$ satisfying*

$$|\nabla h(y)| = T(y) \text{ a.e.} \tag{2.87}$$

Proof. Let $y \in \mathscr{D}$. With the aid of the Hopf-Rinow Theorem one obtains a sequence of curves $\{\beta_\delta\}$ minimizing

$$\inf_{\substack{\gamma(t_1)\in\Gamma_1\\ \gamma(t_2)=y}} L_\delta(\gamma).$$

Minor modifications of the argument employed to prove Lemmas 6–8 enable one to conclude that $\int_{t_1}^{t_2} |\dot{\beta}_\delta|\, dt$ is uniformly bounded in δ, which ensures the compactness necessary to obtain a limiting geodesic $\beta_y(t)$, as in Lemma 9. Thus, for all $y \in \mathscr{D}$, there is a geodesic between Γ_1 and y that minimizes distance in the d_T metric.

Now let y_1 and y_2 lie in $\mathscr{D}$ and let $\beta_{y_1}(t)$ and $\beta_{y_2}(t)$ be the corresponding geodesics. Also, let $l(t) = (1-t)y_1 + ty_2$ for $0 \leqq t \leqq 1$. We suppose y_1 and y_2 are sufficiently close together that $l(t) \in \mathscr{D}$ for all $t \in [0, 1]$. Then,

$$h(y_1) = L(\beta_{y_1}) \leqq L(\beta_{y_2}) + L(l) = h(y_2) + L(l).$$

Similarly, $h(y_2) \leqq h(y_1) + L(l)$. Thus,

$$\begin{aligned} |h(y_2) - h(y_1)| &\leqq L(l) \\ &= \int_0^1 T(l(t))\, |y_2 - y_1|\, dt \\ &\leqq |T|_{L^\infty(\mathscr{D})}\, |y_2 - y_1|, \end{aligned}$$

so that h is (locally) Lipschitz-continuous and therefore differentiable almost everywhere in $\mathscr{D}$ ([9]).

Now let y be a point of differentiability of h and let $\beta_y : [0, 1] \in \bar{\mathscr{D}}$ be a geodesic that minimizes distance. Let $\{x_n\} \in \mathscr{D}$ be any sequence converging to y and set $l_n(t) = (1-t)y + tx_n$ for $t \in [0, 1]$. Repeating the argument above, we find that

$$\frac{|h(x_n) - h(y)|}{|x_n - y|} \leqq \int_0^1 T((1-t)y + tx_n)\, dt;$$

consequently,

$$\lim_{x_n \to y} \frac{|h(x_n) - h(y)|}{|x_n - y|} \leqq T(y). \tag{2.88}$$

Further, defining the sequence of points $y_n = \beta_y\left(1 - \frac{1}{n}\right)$, we see that β_y: $\left[0, 1 - \frac{1}{n}\right] \to \bar{\mathscr{D}}$ must be a geodesic between Γ_1 and y_n that minimizes distance. Therefore, for some $t^* \in \left(1 - \frac{1}{n}, 1\right)$ using a generalization of the Mean Value Theorem (see *e.g.* [3]), one has

$$\begin{aligned} h(y) - h(y_n) &= \int_0^1 T(\beta_y(t))\, |\dot{\beta}_y(t)|\, dt - \int_0^{1-\frac{1}{n}} T(\beta_y(t))\, |\dot{\beta}_y(t)|\, dt \\ &= \int_{1-\frac{1}{n}}^1 T(\beta_y(t))\, |\dot{\beta}_y(t)|\, dt \\ &= T(\beta_y(t^*)) \int_{1-\frac{1}{n}}^1 |\dot{\beta}_y(t)|\, dt, \end{aligned}$$

Hence

$$|h(y) - h(y_n)| \geqq T(\beta_y(t^*))\,|y - y_n|,$$

so that

$$\lim_{n\to\infty} \frac{|h(y) - h(y_n)|}{|y - y_n|} \geqq T(y).$$

This inequality, together with (2.88), implies (2.87).

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References

1. ATTOUCH, H.: *Variational Convergence for Functions and Operators*, Appl. Math. Series, Pitman Adv. Publ. Program, (1984).
2. BERGER, M. S., & FRAENKEL, L. E.: *On the Asymptotic Solution of a Nonlinear Dirichlet Problem*, J. Math. Mech. **19**, (1970), 553–585.
3. BUCK, R.: *Advanced Calculus*, McGraw-Hill, (1965).
4. CAGINALP, G.: *An Analysis of a Phase Field Model of a Free Boundary*, Arch. Rational Mech. Anal., **92**, No. 3, (1986), 205–245.
5. CARR, J., GURTIN, M. & SLEMROD, M.: *Structured Phase Transitions on a Finite Interval*, Arch. Rational Mech. Anal., **86**, (1984), 317–351.
6. CODDINGTON, E., & LEVINSON, N.: *Theory of Ordinary Differential Equations*, McGraw-Hill, (1955).
7. DE GIORGI, E.: *Convergence Problems for Functionals and Operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis, Pitagoria Ed. Bologna, (1978).
8. DO CARMO, M.: *Differential Geometry of Curves and Surfaces*, Prentice-Hall, (1976).
9. FEDERER, H.: *Geometric Measure Theory*, Springer-Verlag, (1969).
10. GILBARG, D. & TRUDINGER, N. S.: *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, (1977).
11. GIUSTI, E.: *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, (1984).
12. GONZALEZ, E., MASSARI, U. & TAMANINI, I.: *On the Regularity of Boundaries of Sets Minimizing Perimeter with a Volume Constraint*, Indiana Univ. Math. Journal, **32**, No. 1, (1983), 25–37.
13. GURTIN, M. E.: *On a Theory of Phase Transitions with Interfacial Energy*, Arch. Rational Mech. Anal., **87**, (1984).
14. GURTIN, M. E.: *Some Results and Conjectures in the Gradient Theory of Phase Transitions*, I.M.A., University of Minn., preprint no. 156, (1985).
15. GURTIN, M. E., & MATANO, H.: *On the Structure of Equilibrium Phase Transitions within the Gradient Theory of Fluids*, Quarterly of App. Math., forthcoming.
16. HERMANN, R.: *Differential Geometry and the Calculus of Variations*, Academic Press, (1968).
17. HOWE, F.: *Perturbed Elliptic Problems and Essential Nonlinearities*, Comm. Partial Diff. Eqns., **8** (1983), 847–874.
18. KOHN, R. V., & STERNBERG, P.: *Local Minimizers and Singular Perturbations*, to appear.

19. KRANTZ, S., & PARKS, H.: *Distance to C^k Hypersurfaces*, J. Diff. Eqns., **40**, No. 1, (1981), 116–120.
20. MODICA, L.: *Gradient Theory of Phase Transitions and Minimal Interface Criterion*, Arch. Rational Mech. Anal., **98**, No. 2, (1987), 123.
21. MODICA, L.: *Gradient Theory of Phase Transitions with Boundary Contact Energy*, to appear.
22. MODICA, L., & MORTOLA, S.: *Un Esempio di Γ^--Convergenza*, Boll. Un. Mat. Ital., **B 14**, (1977), 285–299.
23. MODICA, L., & MORTOLA, S.: *Il Limite nella Γ-convergenza di una Famiglia di Funzionali Ellittici*, Boll. Un. Mat. Ital., **A 14**, (1977).
24. NOVICK-COHEN, A., & SEGEL, L. A.: *Nonlinear Aspects of the Cahn-Hilliard Equation*, Physica, **10 D**, (1984), 278–298.

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