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Topics: Vibration and Bar (music)

The Effect of an Axial Force on the Vibration of Hinged Bars

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The Effect of an Axial Force on the Vibration of Hinged Bars

By S. WOINOWSKY-KRIEGER[1]

It can be shown that the vibration of an extensible bar, carrying no transverse load and having the ends fixed at the supports, causes an axial tensile force with a period equal to the half-period of the vibration of the bar. This force modifies the process of the vibration to a nonlinear one and produces an increase of the frequency of vibration according to the increase of the amplitude.

Nomenclature

The following nomenclature is used in the paper:

l = length of bar
y = instantaneous deflection of any point x, of bar
E = Young's modulus of material
A = cross-sectional area of bar
B = EI = flexural rigidity of bar
r = $\sqrt{I/A}$ = radius of gyration
β = spring constant of supports of bar relative to axial displacement
S_0 = initial axial tensile force of bar
S_1 = axial tensile force due to deflection
q = transverse load per unit length
t = time
ψ = a function of t alone
μ = vibration mass per unit length of bar
a = half amplitude of vibration
α = a/r
ω = frequency in radians per sec
n = positive integers
P_n = $\frac{n^2\pi^2 B}{l^2}$ = Euler's load of bar for buckling form with n half waves
P_0 = $-S_0$ = initial compressive load of bar

Vibration of Bars

The usual theory of vibration of the bars is based on the assumption that one end of the bar, being free to move in an axial direction, an extensionless deflection of the bar is obtained. In technical practice we often have to deal with immovable end hinges, or with hinges connected with supports in such a manner that, as the ends approach each other, a tensile force is produced in the bar which is proportional to the amount of that motion. In these cases the effect of the axial force on the process of vibration must be investigated. Further, we assume an initial tensile force and an extensibility of the bar. The deflection of the bar does not need to be small in comparison with its transverse dimensions; however, it must be small enough to represent the curvature of the deflected bar by the approximate expression $\partial^2 y/\partial x^2$.

In the absence of transverse load, the deflection of the vibrating bar, Fig. 1, is defined by the differential equation

$$B\frac{\partial^4 y}{\partial x^4} = -\mu\frac{\partial^2 y}{\partial t^2} + (S_0 + S_1)\frac{\partial^2 y}{\partial x^2} \qquad [1]$$

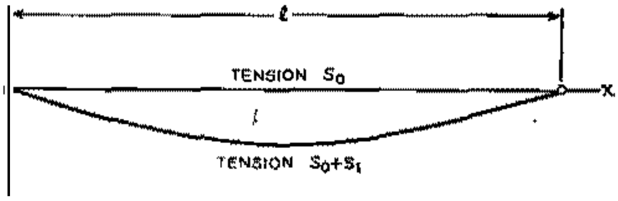

FIG. 1

The value S_0 being given, we will first express the unknown force S_1 through the deflection y. The amount of approach of both hinged ends of the bar due to the deflection is

$$\Delta l = \frac{1}{2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx \qquad [2]$$

Now, the axial force S_1 produces an elongation of the bar

$$\Delta l = S_1\left(\frac{l}{EA} - \frac{1}{\beta}\right) = \frac{S_1 l}{EA}\left(1 - \frac{EA}{l\beta}\right) \qquad [3]$$

It may be seen from the last expression that the constant $1/\beta$ can be omitted in our further investigation without any loss of generality. If, in fact, the actual constant $1/\beta \neq 0$, we should only replace the actual area A in our final results by the reduced value

$$A' = \frac{A}{1 - \frac{EA}{l\beta}} \qquad [4]$$

Equating now the Expressions [2] and [3], we get

$$S_1 = EA\frac{\Delta l}{l} = \frac{B}{2lr^2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx \qquad [5]$$

Substituting the last expression in Equation [1], we obtain

$$\frac{\partial^2 y}{\partial t^2} = -\frac{B}{\mu}\frac{\partial^4 y}{\partial x^4} + \frac{1}{\mu}\left[S_0 + \frac{B}{2lr^2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx\right]\frac{\partial^2 y}{\partial x^2} \qquad [6]$$

Putting

$$y = a\psi \sin\frac{n\pi x}{l} \qquad [7]$$

the geometrical conditions at the hinged ends of the bar are

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satisfied. Using Equation [6], we get the following equation for ψ

$$\frac{d^2\psi}{dt^2} = -\frac{n^4\pi^4\alpha^2 B}{4\mu l^4}\psi^3 - \left(\frac{n^4\pi^4 B}{\mu l^4} + \frac{n^2\pi^2 S_0}{\mu l^2}\right)\psi = 0 \ldots [8]$$

Multiplying this by $(d\psi)/(dt)$ and integrating the result between the maximum deflection ($\psi = 1$, $[d\psi]/[dt] = 0$) and any deflection ψ, we obtain

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{n^4\pi^4\alpha^2 B}{8\mu l^4}(1 - \psi^4) + \left(\frac{n^4\pi^4 B}{\mu l^4} + \frac{n^2\pi^2 S_0}{\mu l^2}\right)(1 - \psi^2) \ldots\ldots\ldots\ldots [9]$$

Using the abbreviations

$$p^2 = \frac{n^4\pi^4 B}{\mu l^4}\left(1 + \frac{\alpha^2}{4}\right) + \frac{n^2\pi^2 S_0}{\mu l^2} \ldots\ldots\ldots [10]$$

and

$$k^2 = \frac{1}{2 + \frac{8}{\alpha^2}\left(1 + \frac{S_0 l^2}{n^2\pi^2 B}\right)} \ldots\ldots\ldots\ldots [11]$$

Equation [9] becomes

$$\left[\frac{d\psi}{d(pt)}\right]^2 = (1 - \psi^2)(k^2\psi^2 + 1 - k^2) \ldots\ldots [12]$$

The solution of this equation is

$$\psi = cn[p(t + t_0), k] \ldots\ldots\ldots\ldots\ldots\ldots [13]$$

where k is the modulus of the elliptic function and t_0 a constant of integration, which we can make zero. Now, by Equation [7], each expression of the form

$$y = a \sin \frac{n\pi x}{l} cn\ (pt, k) \ldots\ldots\ldots\ldots [14]$$

with $n = 1, 2, 3, \ldots$ is a possible solution of Equation [1]. But this equation is nonlinear in y, and for this formal reason a superposition of any solutions of the form, Equation [14], is not practicable. The obvious mechanical reason for difficulty in obtaining a general solution of the problem is the coupling effect of axial forces resulting from each particular solution of the form of Equation [14].

The period of the function $cn(pt, k)$ is

$$4K = 4\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} \ldots\ldots\ldots\ldots [15]$$

and the corresponding frequency

$$\omega = \frac{\pi p}{2K} \ldots\ldots\ldots\ldots\ldots\ldots\ldots [16]$$

In the absence of axial forces (we can get this case assuming $S_0 = 0$, $r \to \infty$ and thus obtaining $\alpha = 0$, $k = 0$, $K = \pi/2$) the frequency becomes

$$\omega_0 = p_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots [17]$$

where $p_0^2 = n^4\pi^4 B/\mu l^4$. Comparing this well-known result with the case in which the ends of the bar are fixed ($S_1 \neq 0$), but the initial axial force is zero ($S_0 = 0$), we have by Equation [10]

$$\frac{\omega}{\omega_0} = \frac{\pi}{2K}\sqrt{1 + \frac{\alpha^2}{4}} \ldots\ldots\ldots\ldots [18]$$

The effect of the amplitude on the increase of the frequency is seen from Table 1.

TABLE 1 EFFECT OF AMPLITUDE ON INCREASE OF FREQUENCY

α	ω/ω_0	α	ω/ω_0
0	1	1.0	1.089
0.1	1.0008	1.5	1.190
0.2	1.0036	2	1.316
0.4	1.015	3	1.626
0.6	1.038	4	1.976
0.8	1.058	5	2.35

By Equation [5] the tensile force due to the deflection alone is

$$S_1 = \frac{B\psi^2}{2lr^2}\int_0^l \left(\frac{an\pi}{l}\right)^2 \cos^2\frac{n\pi x}{l}\,dx = \frac{P_n\alpha^2}{4}\,cn^2(pt, k) \ldots [19]$$

Hence the frequency of this force is equal to 2ω and its maximum value is

$$\max S_1 = \frac{P_n\alpha^2}{4} \ldots\ldots\ldots\ldots\ldots\ldots [20]$$

P_n being that Euler load which corresponds to the orthogonal function $y = \sin (n\pi x)/l$.

We have now to consider the effect of the initial axial force S_0. At first let S_0 be positive, by Equations [10] and [16] the frequency ω then increases if the value S_0 is increasing. Now let S_0 become negative. While $|S_0| = P_0$ increases, the frequency ω decreases, and it vanishes if

$$P_0 = P_n\left(1 + \frac{\alpha^2}{4}\right) \ldots\ldots\ldots\ldots\ldots [21]$$

Consequently the critical value of the compressive force in a vibrating bar is larger than the Euler load P_n when the deflection is zero, and it decreases to the value

$$P = P_0 - \max S_1 = P_n \ldots\ldots\ldots\ldots [22]$$

at the instant of the maximum deflection ($\psi = 1$). Until now, no transverse load on the bar has been assumed. Treating the problem more rigorously, we should introduce at least a load commensurate to the mass of the bar. This load can be replaced with sufficient accuracy by a transverse load following the law

$$q = q_0 \sin \frac{\pi x}{l} \ldots\ldots\ldots\ldots\ldots\ldots [23]$$

Introducing the load q in Equation [1], we can reduce our problem to the evaluation of an elliptic integral of the first kind

$$t = \int_0^1 \frac{d\psi}{\sqrt{\Psi}} \ldots\ldots\ldots\ldots\ldots\ldots [24]$$

where Ψ denotes a quadric in ψ. The inverse of this integral is again an elliptic function. Contrary to the result given in Equation [13], it would represent a nonsymmetrical vibration about the axis $y = 0$ of the undeflected bar.

Solution by Bessel Functions

A solution analogous to Equation [14] can be obtained, by means of Bessel functions, for the case of a vibrating circular plate with fixed edge, if the deflected surface of the plate is assumed a surface of revolution. However, such a solution would be of little practical interest. In fact, if the amplitude of the vibration of the plate remains small, as compared with its thickness, the influence of the external forces acting in the middle plane can be practically neglected. However, if the deflections of the plate are comparable with its thickness, the well-known theory of bending of plates with large deflections must be taken as the basis for further investigation, and the whole problem of vibrations then becomes much more complicated.