hydrogen and helium series is calculated for a very hot star under specified conditions. The equations of thermodynamic equilibrium are used to find approximately the distribution of H and He atoms in various quantum states, and hence to find during what proportion of the life of each atom it will be capable of absorbing the energy and momentum associated with the limits of the several series. It is concluded that electrons and H atoms will even in the nucleus of a planetary nebula be held back by gravitation against the maximum possible radiation-pressure. He atoms, on the other hand, will be subject to radiation-pressure up to 2700 times the attraction of gravity, and will therefore be ejected from the star until they reach a level when the concentration of atoms is too small for recombination and the He^{++} state becomes permanent, and, further absorption being impossible, they fall back.

The falling back of the electrons detached at the high levels will provide an energy from gravitation which will be given up in one or more hyperbolic collisions, and, finally, wholly transferred to atoms of H and He^{+} in capture at lower levels. Calculation of the energy available from this source suggests that these transfers will account for the occurrence, in these particularly rarefied regions, of the emission spectrum seen in place of an absorption spectrum in certain nebulæ beyond the Balmer limit.

Electrostatic considerations affecting the extension of the theory to heavier atoms are briefly discussed.

I am very grateful for the kindness of Professor E. A. Milne in giving me criticism and advice during the course of this investigation.

Physics Department, Birmingham University:
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The Effect of Compton Scattering by Free Electrons in a Stellar Atmosphere.

By P. A. M. Dirac.

(*Communicated by Prof. E. A. Milne.*)

§ 1. *Introduction.*—When radiation is scattered by free electrons, on Compton's theory of scattering * there is an increase in its wave-length, the energy of each light-quantum being reduced by an amount equal to the energy of recoil of a scattering electron. Compton puts forward the suggestion † that this effect may account for the observed displacement of the lines in the solar spectrum to the red near the limb of the sun, it being assumed that the light from the limb is on the whole more scattered than that from the centre. Actually it is the continuous background on either side of the absorption line that is supposed to be shifted to the red, this causing a shift in the line itself.

To apply the Compton theory of scattering to a stellar atmosphere one must, however, take into consideration the velocity of thermal agitation of the electrons, and this may be expected to modify Compton's

* Compton, *Phys. Rev.*, **21**, 483 (1923).
† Compton, *Phil. Mag.*, **46**, 908 (1923).

argument for two reasons. Firstly, there will be a considerable broadening of the line owing to the Doppler effect, electrons moving away from the incident radiation producing an additional increase in wave-length on scattering, whose magnitude can be shown to be much larger than the Compton effect, while electrons moving towards the incident radiation will produce a corresponding decrease. Secondly, there will be a reduction in the average wave-length of the scattered radiation owing to the fact that the incident radiation is more intense relative to electrons moving towards it, so that these electrons will scatter more than their proper share. The existence of this last effect has been shown by Milne,* who pointed out that if black-body radiation at a certain temperature is incident on free electrons in statistical equilibrium at that temperature, the scattered radiation must, according to the laws of thermodynamic equilibrium, have the same distribution in frequency as the incident radiation.

In the present paper a mathematical investigation is made of these effects. The principle of relativity is used to connect the process of scattering by an electron moving with any velocity to that of scattering by an electron initially at rest. In § 2 an expression is obtained for the spectral distribution of the scattered radiation when radiation of any given spectral distribution is incident on free electrons in statistical equilibrium at a given temperature. In § 3 the special case of an incident line spectrum is considered, the shift and broadening of a line on scattering being obtained. It is found that for the temperature of the sun's atmosphere the half width is about 10 Å., while the resultant shift is much smaller, of the order of the Compton shift 0·024 Å., and may be of either sign, depending on the frequency and the temperature of the electrons. These results are applied to a stellar atmosphere in § 4, it being shown that the effect is the production of wings to the lines, and not an unsymmetrical broadening, as would be required to give a measurable shift. A numerical estimate shows that the intensity of the observed wings of stellar lines cannot be accounted for in this way.

The effect of free electrons on the general opacity of a gas has been considered by Stewart † on the basis of the classical theory. The present paper, based on the light-quantum point of view, does not cover the same ground as Stewart's, being concerned with the effect of the electrons on light of particular wave-lengths near an absorption line, and not with any general effect produced throughout the spectrum.

§ 2. *Deduction of the Spectral Distribution of the Scattered Radiation.*—Consider a single scattering process in which the electron initially has the momentum (m_x, m_y, m_z) while the incident light-quantum has the momentum (g_x, g_y, g_z). Let m_t and g_t be the masses of the electron and light-quantum multiplied by c, the velocity of light. If the direction of the incident quantum is determined by the spherical polar coordinates θ and ϕ, and ν is its frequency, then

$$g_x=h\nu/c \,.\, \cos\theta,\; g_y=h\nu/c \,.\, \sin\theta\cos\phi,\; g_z=h\nu/c \,.\, \sin\theta\sin\phi,\; g_t=h\nu/c \qquad (1)$$

* Milne, *Proc. Phys. Soc. London*, **36**, 100 (1924).

† Stewart, *Phys. Rev.*, **22**, 324 (1923).

The quantities m_x, m_y, m_z, m_t and g_x, g_y, g_z, g_t are the components of two 4-dimensional vectors, the squares of whose "lengths" are given by

$$(m_u, m_u) \equiv m_t^2 - m_x^2 - m_y^2 - m_z^2 = m^2c^2 \quad . \quad . \quad (2)$$

and

$$(g_u, g_u) \equiv g_t^2 - g_x^2 - g_y^2 - g_z^2 = 0 \quad . \quad . \quad . \quad . \quad (3)$$

The suffix u is here used to denote any of the four suffixes x, y, z, and t, and the bracket notation is used to represent scalar products.

The laws of conservation of energy and momentum give

$$m_u + g_u = m_u' + g_u' \quad . \quad . \quad . \quad . \quad (4)$$

where dashed letters are used to denote the corresponding quantities after the scattering process. We can eliminate the quantities m_u' in the following way:

$$(m_u + g_u - g_u', m_u + g_u - g_u') = (m_u', m_u') = m^2c^2 = (m_u, m_u),$$

using (2) and the corresponding relation for dashed letters. Using (3), this reduces to

$$(m_u, g_u) - (m_u, g_u') = (g_u, g_u') \quad . \quad . \quad . \quad (5)$$

This equation determines the frequency ν' of the scattered quantum in terms of that of the incident quantum, the initial momentum of the electron, and the directions of the incident and scattered quanta.

There are only two independent invariants connected with the scattering process, which we may take to be (m_u, g_u) [$= m^2c^2\alpha$ say] and (g_u, g_u') [$= m^2c^2\beta$ say]. Any other invariant that one can form from the four vectors m_u, m_u', g_u, g_u' can be expressed at once in terms of these two by means of the equations (4) and (5).

If we take the incident radiation to be in the direction of the x axis and the scattered radiation to be in the xy plane, equation (5) reduces to

$$m_t - m_x \cos\theta' - m_y \sin\theta' = \nu/\nu' \, . \, (m_t - m_x) - h\nu/c \, . \, (1 - \cos\theta') \quad (6)$$

while α and β have the values

$$m^2c^2\alpha = h\nu/c \, . \, (m_t - m_x), \quad m^2c^2\beta = h^2\nu\nu'/c^2 \, . \, (1 - \cos\theta') \quad . \quad (7)$$

θ' is now simply the angle of scattering.

Suppose now that there is a continuous spectrum of radiation of intensity I_ν per unit frequency range, confined to a small solid angle $d\omega$, incident on dn scattering electrons all with the same momentum (g_x, g_y, g_z). Let $R(\nu')d\nu'd\omega'$ be the number of quanta scattered per unit time into the frequency range ν' to $\nu' + d\nu'$ and into a specified solid angle $d\omega'$. Then we can put

$$R(\nu')d\nu'd\omega' = dnI_\nu d\omega S dg_x' dg_y' dg_z',$$

where S is a certain scattering coefficient whose form is to be determined, and ν is to be regarded as a function of g_x', g_y', g_z' and m_x, m_y, m_z, being the frequency of that incident light-quantum which can be scattered by an $(m_x m_y m_z)$ electron into a $(g_x' g_y' g_z')$ light-quantum. The differentials dg_x', dg_y', dg_z' are determined by $d\nu'$ and $d\omega'$.

$R(\nu')d\nu'd\omega'$ is the frequency of occurrence of a certain event, so it transforms under a Lorentz transformation according to the same law as the frequency of a clock moving with the electrons, which is the same law as m_t^{-1}. Hence $R(\nu')d\nu'd\omega'm_t$ is invariant. Now $I_\nu d\omega$ transforms according to the same law as ν, and the differential $dg_x'dg_y'dg_z'$ according to the same law as g_t' * or ν'. We thus find, since dn is invariant, that $\nu\nu'm_tS$ is invariant, so that it is a function of α and β, as we have seen that these are the only independent invariants connected with the process.

Putting

$$\nu\nu'm_tS = F(\alpha, \beta)/m^2h,$$

which makes F a dimensionless function of the dimensionless quantities α and β, we get

$$R d\nu' d\omega' = dnI_\nu d\omega F/m^2h\nu\nu'm_t \, . \, dg_x'dg_y'dg_z'.$$

From the relations corresponding to (1) for dashed letters, we find

$$\frac{\partial(g_x'g_y'g_z')}{\partial(\nu'\theta'\phi')} = \left(\frac{h}{c}\right)^3 \nu'^2 \sin\theta' = \left(\frac{h}{c}\right)^3 \nu'^2 \frac{d\omega'}{d\theta'd\phi'}.$$

Hence

$$R(\nu') = h^2/m^2c^3 \, . \, dnI_\nu d\omega\nu' F/\nu m_t \quad . \quad . \quad . \quad (8)$$

The form of the function F cannot be determined without some special theory of the process of scattering, such as, for example, Compton's assumption that the intensity of the scattered radiation is the same as that given by the classical theory for an electron recoiling with the appropriate velocity,† or Jauncey's corpuscular quantum theory.‡ For visible radiation, these theories give values for the amount of scattered radiation that do not differ appreciably from the classical theory value, which is

$$h\nu R = \tfrac{1}{2}e^4/m^2c^4 \, . \, nI_\nu d\omega(1+\cos^2\theta')$$

for n electrons at rest. This gives, with the help of equations (7), in which m_x, m_y, and m_z have been put equal to zero,

$$F(\alpha, \beta) = \frac{e^4}{h^2c^2} \cdot \frac{mc^2}{h\nu}\tfrac{1}{2}(1+\cos^2\theta') = \frac{e^4}{h^2c^2\alpha}\left(1 - \frac{\beta}{\alpha^2} + \frac{\beta^2}{2\alpha^4}\right) \quad . \quad (9)$$

approximately. For the present, however, we shall keep F arbitrary.

To generalise equation (8) for the case when there are N scattering electrons in statistical equilibrium at the temperature T, we must replace dn by

$$N(2\pi mkT)^{-\frac{3}{2}}e^{-(m_x^2+m_y^2+m_z^2)/2mkT}dm_xdm_ydm_z,$$

and integrate with respect to m_x, m_y, and m_z. The use of the Newtonian kinetic energy $(m_x^2+m_y^2+m_z^2)/2m$ instead of the relativity one (m_tc-mc^2) in the exponential is an approximation which is quite justifiable even for stellar temperatures.

* Dirac, *Proc. Roy. Soc.*, A, **106**, 583 (1924).
† Compton, *Phys. Rev.*, **21**, 491 (1923).
‡ Jauncey, *Ibid.*, **22**, 233 (1923).

This gives

$$R(\nu')=\frac{h^2}{m^2c^3}N(2\pi mkT)^{-\frac{3}{2}}\iiint_{-\infty}^{\infty}\frac{I_\nu d\omega\nu' F}{\nu m_t}e^{-(m_x^2+m_y^2+m_z^2)/2mkT}dm_x dm_y dm_z$$

$$=h^2/m^2c^3 \,.\, N(2\pi mkT)^{-\frac{3}{2}}\int_0^{\infty} I_\nu d\omega\psi(\nu,\nu')d\nu \quad . \quad . \quad . \quad . \quad (10)$$

where

$$\psi(\nu,\nu')=\iint_{-\infty}^{\infty}\frac{\nu' F}{\nu m_t}e^{-(m_x^2+m_y^2+m_z^2)/2mkT}\Big/\frac{\partial\nu}{\partial m_x}\,.\,dm_y dm_z \quad . \quad . \quad . \quad (11)$$

Here $\partial\nu/\partial m_x$ is to be evaluated from (6) and expressed in terms of m_y, m_z, and ν.

Equation (10) gives the contribution of each spectral component of the incident radiation to the scattered radiation of frequency ν'. These contributions are plainly additive, so if the incident radiation consists of a continuous background with an absorption line, the shift and broadening of the scattered line will be just the same as if the incident radiation is simply a bright line of the same frequency.

§ 3. *Broadening and Shift produced by the Scattering.*—Suppose the incident radiation to be monochromatic of intensity $I_\nu d\nu$. The number of quanta of scattered radiation in the frequency interval ν' to $\nu'+d\nu'$ is

$$h^2/m^2c^3 \,.\, N(2\pi mkT)^{-\frac{3}{2}}I_\nu d\nu d\omega\psi(\nu,\nu') \quad . \quad . \quad (12)$$

We may approximate to the value of ψ by eliminating m_x from (11) by means of (6), and expanding the integrand (excluding the exponential part of it) in powers of the dimensionless variables m_y/mc and m_z/mc. On integrating term by term, we shall obtain ψ as a series of ascending powers of the small quantity $(kT/mc^2)^{\frac{1}{2}}$, whose value for $T=6000°$ C. is about 10^{-3}. $h\nu$ may be counted of the same order of magnitude as kT, this being the case when the incident frequency is of the order of magnitude of the frequency of maximum black-body intensity for the temperature T.

Working to the 1st order only, we have from (6)

$$m_x(1-\cos\theta')=-mc(\nu'-\nu)/\nu+m_y\sin\theta'$$

and

$$\partial\nu/\partial m_x=\nu/mc \,.\, (1-\cos\theta').$$

It is easily verified that

$$m_x^2+m_y^2=\frac{2}{1-\cos\theta'}\left[m_y-\frac{mc(\nu'-\nu)\sin\theta'}{2\nu(1-\cos\theta')}\right]^2+\frac{m^2c^2(\nu'-\nu)^2}{2\nu^2(1-\cos\theta')}.$$

Hence, putting

$$a\nu^2=4kT/mc^2 \,.\, \nu^2(1-\cos\theta')$$

we get from (11)

$$\psi(\nu,\nu')=\frac{F_0}{\nu(1-\cos\theta')}e^{-(\nu'-\nu)^2/a_\nu^2}\iint_{-\infty}^{\infty}$$

$$\exp.\left\{-\left[\frac{2}{1-\cos\theta'}\left(m_y-\frac{mc(\nu'-\nu)\sin\theta'}{2\nu(1-\cos\theta')}\right)^2+m_z^2\right]\Big/2mkT\right\}dm_y dm_z$$

$$=\frac{\sqrt{2\pi mkT}F_0}{\nu(1-\cos\theta')^{\frac{1}{2}}}e^{-(\nu'-\nu)^2/a_\nu^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where F_0 denotes the value of $F(\alpha, \beta)$ when m_x, m_y, and m_z are put equal to zero.

Expression (12) and equation (13) give the ordinary exponential distribution of intensity for the scattered radiation about the primary frequency, as might be expected from general physical considerations. The intensity falls off to $1/e$ of its maximum value at a distance $\nu'-\nu=a_\nu$, and we may call this the half-width of the line. For $T=6000°$ C., $\theta'=\frac{1}{2}\pi$, a_ν/ν is about 2×10^{-3}. For a wave-length of 5000 Å., this gives a half-width of 10 Å. The greatness of this value is really due to the smallness of the mass of an electron.

The shift $\delta\nu$ of the centre of gravity of the line vanishes to the order of accuracy of equation (13). If it is worked out correct to the second order in $(kT/mc^2)^{\frac{1}{2}}$, it is found to be, assuming F to have its classical theory value given by (9),

$$\delta\nu/\nu=(4kT-h\nu)/mc^2 \,.\, (1-\cos\theta') \quad . \quad . \quad . \quad (14)$$

The second term in this expression, $-h\nu/mc^2 \,.\, (1-\cos\theta')$, represents just the Compton effect for stationary electrons, while the first, $4kT/mc^2 \,.\, (1-\cos\theta')$, is due to a slightly unsymmetrical broadening caused by the motion of the electrons. The two effects are of the same order of magnitude and in opposite directions, so they may just cancel. In any case the shift is of the order of magnitude of the Compton shift 0·024 Å., and since this is of a smaller order of magnitude than the breadth of the line, we should get a different value for the shift if we considered the intensity plotted on a wave-length base than on a frequency base. Such a small shift obscured by such a large broadening could certainly not be detected observationally.

§ 4. *Application to a Stellar Atmosphere.*—The intensity of any spectral component of the radiation from a star is determined by the temperature of the average depth from which radiation of that particular frequency comes. This is given by the mean free path of a light-quantum in the stellar atmosphere, meaning the average magnitude of the vector distance travelled by a quantum without absorption. Any scattering electrons present in the atmosphere will reduce this vector distance. This will cause a general reduction of intensity throughout the spectrum, as worked out by Stewart,* the change in wave-length on scattering not mattering in this case.

There will be an additional effect in the neighbourhoods of absorption lines, owing to the chance of a light-quantum whose wave-length is near an absorption line having its wave-length shifted into the absorption line on scattering, in which case it will be absorbed practically at once. The electrons will thus cause an increase in the effective absorption coefficient of the atmosphere for frequencies near an absorption line, and this will manifest itself by wings attached to the spectral lines. The width of the wings will be given by the maximum change of wave-length on scattering, which has been found to be of the order of 10 Å. No effect of the nature of an unsymmetrical broadening of the lines could be produced, as this would require a large value for the

* Stewart, *loc. cit.*

increase in the absorption coefficient for a short wave-length range just outside the line, falling off rapidly to zero, instead of a fairly uniform value extending for a comparatively large range. The displacement of solar lines to the red near the limb of the sun cannot therefore be accounted for by the Compton effect.

To obtain an estimate of the intensity of the wings produced in this way, we may assume the classical theory value for F given by equation (9). This gives, from (10) and (13), for monochromatic incident radiation of unit intensity,

$$\mathrm{R}(\nu') = \frac{\mathrm{N}e^4}{4mc^3h\nu^2(\pi mk\mathrm{T})^{\frac{1}{2}}} \frac{1+\cos^2\theta'}{(1-\cos\theta')^{\frac{1}{2}}} e^{-(\nu'-\nu)^2/a_\nu{}^2}.$$

If we let N denote the number of electrons per unit volume, $h\nu'\mathrm{R}(\nu')$ will be the intensity of the scattered radiation (per unit frequency range) per unit depth of the scattering layer. The chance of a light-quantum being scattered into the frequency range ν_0 to $\nu_0+d\nu_0$ per unit length of its path is now

$$d\nu_0\iint h\nu_0\mathrm{R}(\nu_0) \,.\, \sin\theta' d\theta' d\phi',$$

provided $d\nu_0$ is compared with a_ν. Assuming that the quantum is immediately absorbed when this happens, we find for the length absorption coefficient when ν is very nearly equal to ν_0, the value

$$d\nu_0 \frac{22\sqrt{2\pi}}{15} \frac{\mathrm{N}e^4}{mc^3\nu_0(mk\mathrm{T})^{\frac{1}{2}}},$$

corresponding to a mass absorption coefficient

$$k = d\nu_0 \frac{22\sqrt{2\pi}}{15} \frac{\mathrm{N}e^4}{\rho mc^3\nu_0(mk\mathrm{T})^{\frac{1}{2}}}$$

where ρ is the density of the stellar atmosphere.

For a temperature of 6000° C. this reduces to

$$k = \frac{170}{\mathrm{M}} \frac{\mathrm{P}e}{\mathrm{P}} \frac{d\nu_0}{\nu_0}$$

in C.G.S. units, where Pe is the partial electron pressure, P the total pressure of the atmosphere, and M its mean molecular weight. For a solar line $d\nu_0/\nu_0$ is about 10^{-4}, and we may take $\mathrm{P}e/\mathrm{P}=\frac{1}{2}$ and $\mathrm{M}=10$ roughly, giving $k=8{\cdot}5\times10^{-4}$.

This is very small compared with the general absorption coefficient in the sun's atmosphere, which, as worked out by Milne,* has the value 1140. The ratio of the reduction in intensity due to the wings to the intensity of the continuous background of the spectrum will be of the order of magnitude of the ratio of these two absorption coefficients, *i.e.* about 10^{-6}. The observed wings of the solar lines can therefore hardly be accounted for by this effect.

§ 5. *Summary.*—A calculation is made of the broadening and shift produced when a spectral line of radiation is scattered by free electrons in statistical equilibrium at a given temperature. It is found that for a temperature of 6000° C. the half-width of the scattered line is about 10 Å., while the shift is very much smaller.

* Milne, *M.N.R.A.S.*, June 1925, p. 770.

The effect of such scattering in a stellar atmosphere is shown to be the production of wings to the lines, and not a shift of the lines. Compton scattering cannot therefore account for the limb effect in the sun.

The intensity of the wings produced in this way is exceedingly small. The observed breadths of solar lines cannot be accounted for by electron scattering.

My best thanks are due to Professor E. A. Milne for much friendly discussion and assistance during the writing of this paper.

Determination of the Solar Parallax from Observations of Mars secured at the Royal Observatory, Cape of Good Hope, near the Opposition of 1924. *I.—Photographic Observations.* By H. Spencer Jones, M.A., D.Sc., and J. Halm, Ph.D.

Determinations of the solar parallax from observations of the planet Mars have been regarded with some suspicion, as peculiarly liable to be affected by errors of a systematic nature. It cannot be denied that suitable minor planets are preferable for the purpose and may be expected to provide a value of the solar parallax with a smaller probable error than will result from observations of Mars; on a photographic plate the star-like image of a minor planet is a much easier object to measure than the disc-image of a major planet, whilst in heliometer observing, the accuracy with which two star-images can be caused to swing through one another is greater than that with which the image of a star can be set in the centre of the disc of a planet. But apart from the increased accidental error to which the observations of any major planet are liable, there is the danger that the red colour of Mars may give rise to systematic errors, since at moderate zenith distances the effect of chromatic dispersion of the atmosphere cannot be separated from a true parallactic displacement. That the danger is a real one seems to be confirmed by the fact that all the determinations of the solar parallax by meridian observations of Mars have given a larger value than most of the other methods of determining this constant.

Such evidence as is available would appear to indicate that heliometer observations are free from systematic errors arising from atmospheric chromatic dispersion. Gill * was convinced that this was the case, at least in so far as star observations are concerned; as regards observations of Mars, in the discussion of his observations in 1877, he found on grouping the observations according to altitude no evidence of such an effect.† There is some force in Newcomb's argument: ‡

"It may be objected to the inclusion of Gill's Ascension result that it should be rejected for the same reason, since the colour of the planet would affect heliometer observations and meridian observations

* Gill, *History and Description of the Cape Observatory,* p. lxxvii.

† *Memoirs R.A.S.,* **46,** 157. ‡ *Fundamental Constants of Astronomy,* p. 154.