THE EFFECT OF CONSTITUTIVE LAW PERTURBATIONS ON FINITE ANTIPLANE SHEAR DEFORMATIONS OF A SEMI-INFINITE STRIP

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Abstract. This paper is concerned with assessing the effects of small perturbations in the constitutive laws on antiplane shear deformation fields arising in the theory of nonlinear elasticity. The mathematical problem is governed by a second-order quasilinear partial differential equation in divergence form. Dirichlet (or Neumann) boundary-value problems on a semi-infinite strip, with nonzero data on one end only, are considered. Such problems arise in investigation of Saint-Venant end effects in elasticity theory. The main result provides a comparison between two solutions, one of which is a solution to a simpler equation, for example Laplace's equation. Three examples involving perturbations of power-law material models are used to illustrate the results.

1. Introduction. The equilibrium equations governing finite antiplane shear deformations of some homogeneous isotropic compressible or incompressible nonlinearly elastic materials have been shown to reduce to a single second-order quasilinear partial differential equation in two independent variables for the out-of-plane displacement (see e.g. [1–4]). In particular, for the generalized neo-Hookean incompressible materials for which the strain-energy depends only on the first invariant, the governing equation, in the absence of body forces is (1.1) below. This equation also governs finite antiplane shear deformations for a certain class of compressible materials [4]. Considerable attention has been paid to the analysis of solutions of (1.1) on rectangular domains whose lengths greatly exceed their widths. In particular, for such long thin domains (or for semi-infinite strips), with traction-free lateral sides, the asymptotic behavior of solutions of (1.1), as the axial variable increases, is of interest in connection with Saint-Venant's principle (see e.g. [5–8] and the references cited therein. Recent reviews on Saint-Venant's principle are given in [9, 10]. See also [11] for consideration of a fourth-order analog of (1.1).) The spatial evolution of solutions of (1.1) for both Dirichlet and Neumann boundary conditions on the

lateral sides is also relevant to the study of Phragmén-Lindelöf type principles (for such results, in two or three dimensions, see e.g. [12, 13] and the references cited therein). Recently, the present authors have investigated the asymptotic behavior of solutions of *inhomogeneous* equations which are generalizations of (1.1). In [14], inhomogeneous equations where a constant term is added to the left-hand side of (1.1) have been studied. In the antiplane shear context, this would correspond to a constant body-force. It was shown in [14] that solutions to Dirichlet problems for such equations are well approximated, away from the ends of a finite rectangle, by solutions to the corresponding one-dimensional problem for an ordinary differential equation on the cross section of the rectangle. Such results are of interest in assessing the approximate nature of one-dimensional theories compared to exact two-dimensional theories, and have played an important role, for example, in establishing plate and shell theories in solid mechanics. Applications to problems in geometry, for example, to the equation of a surface of constant mean curvature, have also been discussed in [14]. Generalizations of these results to the equations governing capillary surfaces and extensible films are described in [15].

In this paper, we return to the homogeneous equation (1.1) on a semi-infinite strip, subject to nonzero Dirichlet or Neumann boundary conditions on the near end only, and examine the effect of *perturbations of the coefficient* ρ on solutions. In the context of antiplane shear [1–4], this coefficient is the derivative of the strain-energy density and so our concern is with the effect of *constitutive law perturbations* on solutions. Such results are of interest given the practical difficulty in constructing constitutive models that provide an *exact description of material behavior*.

Specifically, we are concerned with second-order quasilinear partial differential equations in two independent variables of the form

$$[\rho(q^2)u_{,\alpha}]_{,\alpha} = 0, \qquad q \equiv (u_{,\beta}u_{,\beta})^{1/2}, \qquad (1.1)$$

where the usual summation convention is employed with subscripts preceded by a comma denoting partial differentiation with respect to the corresponding Cartesian coordinate. As mentioned above, in the context of antiplane shear, ρ is determined by the constitutive model governing material behavior; and in (1.1) $u = u(x_1, x_2)$ is the displacement field. A commonly used constitutive law gives rise to functions ρ of power-law form,

$$\rho = \mu(1 + bq^2/n)^{n-1}, \qquad \mu, b, n > 0, \qquad (1.2)$$

where μ is the shear modulus for infinitesimal deformations, and n is a material hardening parameter. The case $n = 1$ in (1.2) corresponds to the neo-Hookean material for which ρ is a constant, and (1.1) is Laplace's equation. Letting $\rho' = \partial\rho/\partial q^2$, we see from (1.2) that $\rho' > 0$ or $\rho' < 0$ according as $n > 1$ or $n < 1$ respectively. The material is said to harden or soften in shear in these situations [5]. When $n = 1/2$ in (1.2), equation (1.1) is reducible, by a change of scale, to the minimal surface equation

$$(1 + u_{,2}^2)u_{,11} - 2u_{,1}u_{,2}u_{,12} + (1 + u_{,1}^2)u_{,22} = 0. \qquad (1.3)$$

We consider both Dirichlet and Neumann problems for (1.1) on the semi-infinite strip $R = \{(x_1, x_2)|0 < x_2 < h, \; x_1 > 0\}$. For ease of exposition, we confine attention in what follows to the Dirichlet problem. The modifications necessary to treat the corresponding Neumann problem are described in Sec. 4. We assume the existence of classical solutions $u \in C^2(R) \cap C^1(\overline{R})$ satisfying (1.1) on R subject to the boundary conditions

$$u(x_1, 0) = 0, \quad u(x_1, h) = 0, \qquad x_1 \geq 0, \tag{1.4}$$

$$u, \; u_{,1} \to 0 \quad \text{(uniformly in } x_2\text{) as } x_1 \to \infty, \tag{1.5}$$

$$u(0, x_2) = f(x_2), \qquad 0 \leq x_2 \leq h, \tag{1.6}$$

where the prescribed function f is sufficiently smooth and satisfies $f(0) = f(h) = 0$.

For a rather general class of functions ρ it was established in [7] that solutions of (1.1), (1.4)–(1.6) decay exponentially with the distance x_1 from the end $x_1 = 0$. The exponential decay rate was characterized explicitly in [7] in terms of the function ρ and the strip width h. The hypotheses made in [7] concerning ρ, which we shall also assume here, are conveniently separated into two cases. It is assumed that there exist positive constants m_α, M_α and nonnegative constants K_α ($\alpha = 1, 2$) such that, for all solutions u of (1.1), (1.4)–(1.6) on R, we have either
Case 1.

$$0 < m_1 \leq \rho \leq M_1 + K_1 q^2 \rho, \tag{1.7}$$

or
Case 2.

$$0 < m_2 \leq \rho^{-1} \leq M_2 + K_2 q^2 \rho, \tag{1.8}$$

respectively. As pointed out in [7], if ρ were a bounded function of its arguments, then the K_1 in (1.7) could be taken to be zero. Roughly speaking, the first term on the right in (1.7) provides a bound on ρ as $q \to 0$, while the second term gives a bounding function for ρ as $q \to \infty$. A function ρ, for which (1.7) holds, is

$$\rho = \mu(1 + q^2), \tag{1.9}$$

which may be viewed as a special case of (1.2) with $n = 2$, $b = 2$. For this ρ, we can take $m_1 = \mu$, $M_1 = \mu$, $K_1 = 1$ in (1.7). For

$$\rho = \mu(1 + 2bq^2)^{-1/2}, \tag{1.10}$$

(corresponding to the value $n = 1/2$ in (1.2)), in which case (1.1) is reducible to the minimal surface equation (1.3), then (1.8) is satisfied with $m_2 = \mu^{-1}$, $M_2 = \mu^{-1}$, and $K_2 = 2b\mu^{-2}$. We observe, as in [7], that neither (1.7) nor (1.8) requires that equation (1.1) be *elliptic*, that is, for all solutions u and at all points of R,

$$\rho + 2\rho' q^2 \geq 0 \qquad (\rho' \equiv \partial \rho / \partial q^2), \tag{1.11}$$

although the results obtained in [7] and here are primarily of interest for elliptic equations. It should also be noted that, in choosing the constants m_α, M_α in (1.7), (1.8) for a given ρ, it is desirable to choose m_α as *large* as possible and M_α as *small* as possible.

In this paper, we are concerned with comparing solutions u of (1.1), (1.4)–(1.6) to solutions v of a differential equation arising from a *small perturbation* of the constitutive function ρ. The comparison solution v satisfies the *same boundary conditions* as u. In particular, such a comparison is of interest where the problem for v is much simpler than that for u, for example, v may be the solution to a *linear* problem. Indeed, for two of the illustrative examples considered in this paper, v is a harmonic function. We assume the existence of $v \in C^2(R) \cap C^1(\overline{R})$ satisfying

$$[\tilde{\rho}(p^2)v_{,\alpha}]_{,\alpha} = 0, \qquad p \equiv (v_{,\beta}v_{,\beta})^{1/2}, \tag{1.12}$$

on R, subject to the boundary conditions

$$v(x_1, 0) = 0, \quad v(x_1, h) = 0, \qquad x_1 \geq 0, \tag{1.13}$$

$$v, \; v_{,1} \to 0 \quad \text{(uniformly in } x_2\text{) as } x_1 \to \infty, \tag{1.14}$$

$$v(0, x_2) = f(x_2), \qquad 0 \leq x_2 \leq h. \tag{1.15}$$

The perturbed constitutive function $\tilde{\rho}$ is such that it satisfies conditions analogous to (1.7), (1.8), namely,

$$0 < \tilde{m}_1 \leq \tilde{\rho} \leq \widetilde{M}_1 + \widetilde{K}_1 p^2 \tilde{\rho} \tag{1.16}$$

or

$$0 < \tilde{m}_2 \leq (\tilde{\rho})^{-1} \leq \widetilde{M}_2 + \widetilde{K}_2 p^2 \tilde{\rho}. \tag{1.17}$$

Thus, in Case 1, both (1.7) *and* (1.16) are assumed to hold while in Case 2, both (1.8) *and* (1.17) are to hold. (As was remarked in connection with (1.7), (1.8), it is desirable to choose $\tilde{m}_\alpha$ as *large* as possible and $\widetilde{M}_\alpha$ as *small* as possible.) For $\varepsilon \ll 1$, it is also assumed that there exist positive functions $\gamma_1(p, \varepsilon)$, $\gamma_2(p, \varepsilon)$, with

$$\gamma_1(p, \varepsilon) \leq c_1(\varepsilon) < 1, \tag{1.18}$$

$$\gamma_2(p, \varepsilon) \leq c_2, \tag{1.19}$$

where c_2 is a constant such that

$$p|\rho(q^2) - \tilde{\rho}(p^2)| \leq \gamma_1(p, \varepsilon)\rho(q^2)|q - p| + \varepsilon^2 p \gamma_2(p, \varepsilon)[\rho(q^2)\tilde{\rho}(p^2)]^{1/2}. \tag{1.20}$$

The hypothesis (1.20), though somewhat complicated, serves to define the constitutive law perturbation of concern here. The form of (1.20) arises from the following considerations. We write

$$\rho(q^2) - \tilde{\rho}(p^2) = (1/2)[\rho(q^2) - \rho(p^2) + \rho(p^2) - \tilde{\rho}(p^2)]$$
$$+ (1/2)[\rho(q^2) - \tilde{\rho}(q^2) + \tilde{\rho}(q^2) - \tilde{\rho}(p^2)] \tag{1.21}$$

so that

$$|\rho(q^2) - \tilde{\rho}(p^2)| \leq (1/2)[|\rho(q^2) - \rho(p^2)| + |\tilde{\rho}(q^2) - \tilde{\rho}(p^2)|]$$
$$+ (1/2)[|\rho(q^2) - \tilde{\rho}(q^2)| + |\rho(p^2) - \tilde{\rho}(p^2)|]. \tag{1.22}$$

The second two terms on the right in (1.22) measure the difference between $\rho(s)$ and $\tilde{\rho}(s)$ *at the same value of their arguments* while the first pair of terms measure the difference in $\rho(s)$, $\tilde{\rho}(s)$ respectively, *at different values of their arguments*. The

results we establish in the sequel make use of energy inequality techniques developed in our previous work [7, 8, 14]. In particular, in [14] (see [14], equation (2.4)), an inequality of the form

$$p|\rho(q^2) - \rho(p^2)| \le c_1(p)\rho(q^2)|q - p|, \tag{1.23}$$

where $0 < c_1(p) < 1$, was used. Using a similar inequality, we are led to assume here that the first pair of terms on the right in (1.22), multiplied by p, are bounded by the first term on the right in (1.20). The second two terms on the right in (1.22), multiplied by p, are then assumed to be bounded by the second term on the right in (1.20), with ε $(0 < \varepsilon \ll 1)$ being a measure of the perturbation in ρ generated by $\tilde{\rho}$. As an example, suppose that

$$\rho(q^2) = \mu(1 + \varepsilon^2 q^2)^{-1/2}, \qquad \tilde{\rho}(p^2) = \mu, \tag{1.24}$$

so that u satisfies an equation of minimal surface type while v is a harmonic function. The constitutive function ρ in (1.24) is a special case of the power-law material (1.2) with $n = 1/2$ and $b = \varepsilon^2/2$. Thus, (1.24) may be viewed as a perturbation away from $b = 0$ in (1.2), with the value of n fixed at $n = 1/2$. It is shown in Appendix A that (1.20) is satisfied by the ρ, $\tilde{\rho}$ given in (1.24) with

$$\gamma_1(p, \varepsilon) = \varepsilon p, \qquad \gamma_2(p, \varepsilon) = p^2/2. \tag{1.25}$$

In Appendix A and in Sec. 3 it is also shown that the choices for $\gamma_1(p, \varepsilon)$, $\gamma_2(p, \varepsilon)$ given by (1.25) do indeed satisfy (1.18), (1.19). Another example we shall consider is the power-law material (1.2) with hardening exponent $n = 1 + \varepsilon^2$, so that the material is close to being neo-Hookean. Thus, we have

$$\rho(q^2) = \mu\left(1 + \frac{bq^2}{1 + \varepsilon^2}\right)^{\varepsilon^2}, \qquad \tilde{\rho}(p^2) = \mu, \tag{1.26}$$

and so we wish to compare u with the harmonic function v satisfying (1.13)–(1.15). It is shown in Appendix A that (1.20) is satisfied in this case with the choice

$$\gamma_1(p, \varepsilon) = \frac{pb^{1/2}\varepsilon^2}{1 + \varepsilon^2}\{[\varepsilon^2/(1 + \varepsilon^2)]^{\varepsilon^2/2} + pb^{1/2}\}, \qquad \gamma_2(p, \varepsilon) = \frac{bp^2\varepsilon^2}{1 + \varepsilon^2}. \tag{1.27}$$

In Sec. 3 it is shown that (1.18), (1.19) again hold for suitably chosen $c_1(\varepsilon)$ and c_2.

The plan of the paper is as follows: In the next section, we establish our main result; namely, we derive an exponential decay estimate for a quadratic functional defined on the difference between u and v. We show that this "weighted energy" is bounded above by an exponential that decays with the axial variable x_1 and we obtain an estimate (lower bound) for the decay rate. Furthermore, we show that the energy is of order ε^4, *for all* x_1. We give three illustrative examples in Sec. 3, two of which involve comparison of u with *harmonic* functions, while the third example compares solutions for two *softening* power-law materials. The extension to Neumann boundary conditions is described in Sec. 4.

2. An energy comparison. In this section, we establish our main comparison result between u and v. Let $w(x_1, x_2)$ be defined by

$$w(x_1, x_2) = u(x_1, x_2) - v(x_1, x_2).\tag{2.1}$$

We show that the energy measure

$$E(z) = \int_{R_z} \rho(q^2) w_{,\alpha} w_{,\alpha}\, dA, \qquad (q^2 = u_{,\beta} u_{,\beta}),\tag{2.2}$$

contained in the subdomain

$$R_z = \{(x_1, x_2) | 0 \le z < x_1 < \infty,\ 0 < x_2 < h\}$$

has exponential decay in z and is of order ε^4 as $\varepsilon \to 0$. In fact, we shall show that

$$E(z) \le \varepsilon^4 C_\alpha e^{-2k_\alpha z}, \qquad z \ge 0,\tag{2.3}$$

where $\alpha = 1$ or 2 in Case 1 or 2 respectively, and the estimated decay rate $2k_\alpha$ and the amplitude C_α are constants which can be explicitly determined. Note that the summation convention is *not* used in (2.3).

The result (2.3) is established in several stages. First, for $\alpha = 1$ or 2 we derive the differential inequality

$$F'(z) + 2d\kappa_\alpha(z)F(z) \le D\varepsilon^4 \kappa_\alpha(z)\tilde{E}(z), \qquad z \ge 0,\tag{2.4}$$

where the prime denotes differentiation with respect to z. Here

$$F(z) = E(z) + \varepsilon^4 c_2^2 (1 + c_1)^{-1} \tilde{E}(z)/2,\tag{2.5}$$

where $c_1(\varepsilon)$ and c_2 are the quantities introduced in (1.18), (1.19) respectively, and

$$\tilde{E}(z) = \int_{R_z} \tilde{\rho}(p^2) v_{,\alpha} v_{,\alpha}\, dA, \qquad (p^2 = v_{,\beta} v_{,\beta})\tag{2.6}$$

is an energy measure defined on solutions v of the comparison problem (1.12)–(1.15). The quantity $\kappa_\alpha(z)$ in (2.4) is

$$\kappa_\alpha = \frac{m_\alpha \pi}{B_\alpha(z)} \qquad (\alpha = 1, 2),\tag{2.7}$$

where

$$B_\alpha(z) = \begin{cases} \int_0^h \rho(q^2)\, dx_2 & \text{(Case 1)}, \\ \int_0^h \rho^{-1}(q^2)\, dx_2 & \text{(Case 2)}. \end{cases}\tag{2.8}$$

The constants d and D in (2.4) are given by

$$d = \frac{1 - c_1}{4(1 + c_1)}, \qquad D = \frac{(3 + c_1^2)c_2^2}{4(1 + c_1)^2(1 - c_1)},\tag{2.9}$$

respectively. The differential inequality (2.4) is integrated once. The next step uses an exponential decay estimate for $\tilde{E}(z)$, established in [7] (see (2.1) of [7]). The estimate is

$$\tilde{E}(z) \le \tilde{G}_\alpha e^{-2\tilde{\nu}_\alpha z}, \qquad z \ge 0,\tag{2.10}$$

where

$$\tilde{\nu}_\alpha = \tilde{m}_\alpha \pi / \widetilde{M}_\alpha h \qquad (\alpha = 1, 2), \tag{2.11}$$

and

$$\tilde{G}_\alpha = \tilde{E}_0 \exp\left[\frac{2\pi \tilde{m}_\alpha \tilde{K}_\alpha \tilde{E}_0}{\widetilde{M}_\alpha^2 h^2}\right]. \tag{2.12}$$

In (2.12) $\tilde{E}_0 \equiv \tilde{E}(0)$ denotes the total energy (assumed finite) contained in the semi-infinite strip R. Bounds on $\tilde{E}(0)$ in terms of the boundary data (1.15) are obtained in [7]. When (2.10) is inserted in the integrated form of (2.4), the resulting differential inequality may again be integrated. This introduces the total weighted energy $E_0 \equiv E(0)$ which, in turn, can be bounded in terms of $\tilde{E}(0)$ (see (2.50) below). The final step makes use of the hypotheses (1.7), (1.8) in Cases 1, 2 respectively, to obtain the desired result (2.3).

To establish (2.4) we proceed as follows. If L_z denotes the line segment $x_1 = z$, $0 \le x_2 \le h$, we find, by using the divergence theorem and (1.1), (1.4), (1.5), (1.12)–(1.14) that

$$\begin{aligned}
E(z) = &-\int_{L_z} \rho(q^2) w w_{,1} \, dx_2 - \int_{L_z} [\rho(q^2) - \tilde{\rho}(p^2)] w v_{,1} \, dx_2 \\
&- \int_{R_z} [\rho(q^2) - \tilde{\rho}(p^2)] v_{,\beta} w_{,\beta} \, dA,
\end{aligned} \tag{2.13}$$

where we use the notation

$$\int_{L_z} \psi \, dx_2 \equiv \int_0^h \psi(z, x_2) \, dx_2. \tag{2.14}$$

Thus, from (2.13) and the definition of p in (1.12), we have

$$\begin{aligned}
E(z) \le &\int_{L_z} \rho(q^2)|w|\,|w_{,1}| \, dx_2 + \int_{L_z} |\rho(q^2) - \tilde{\rho}(p^2)|\,|w|\,|v_{,1}| \, dx_2 \\
&+ \int_{R_z} |\rho(q^2) - \tilde{\rho}(p^2)|\,p(w_{,\beta} w_{,\beta})^{1/2} \, dA.
\end{aligned} \tag{2.15}$$

Denote the third integral on the right in (2.15) by I_3. Then we can use the hypothesis (1.20), together with (1.18) and (1.19) to obtain

$$\begin{aligned}
I_3 \le &c_1 \int_{R_z} |q - p| \rho(q^2)(w_{,\beta} w_{,\beta})^{1/2} \, dA \\
&+ c_2 \varepsilon^2 \int_{R_z} [\rho(q^2)^{1/2}(w_{,\beta} w_{,\beta})^{1/2} \tilde{\rho}(p^2)^{1/2} p] \, dA.
\end{aligned} \tag{2.16}$$

By virtue of the definitions of p, q, and w we have

$$p^2 = v_{,\beta} v_{,\beta}, \qquad pq = (v_{,\beta} v_{,\beta})^{1/2}(u_{,\alpha} u_{,\alpha})^{1/2} \ge v_{,\beta} u_{,\beta}, \tag{2.17}$$

and so

$$|q - p| = (q^2 - 2pq + p^2)^{1/2} \le (v_{,\beta} v_{,\beta} - 2v_{,\beta} u_{,\beta} + u_{,\beta} u_{,\beta})^{1/2} = (w_{,\beta} w_{,\beta})^{1/2}. \tag{2.18}$$

By inserting (2.18) in the first integral on the right in (2.16), using Schwarz's inequality in the second integral, and recalling the definitions of $E(z)$, $\tilde{E}(z)$ from (2.2) and (2.6) respectively, we obtain

$$I_3 \leq c_1 E(z) + c_2 \varepsilon^2 E^{1/2}(z) \tilde{E}^{1/2}(z). \tag{2.19}$$

The second integral on the right in (2.15), denoted by I_2, is bounded in a similar fashion. Thus, since $|v_{,1}| \leq (v_{,\beta} v_{,\beta})^{1/2} = p$, on using (1.20), (1.18), (1.19) we find that

$$I_2 \leq c_1 \int_{L_z} |q - p| \rho(q^2) |w| \, dx_2 + c_2 \varepsilon^2 \int_{L_z} \rho(q^2)^{1/2} |w| \tilde{\rho}(p^2)^{1/2} p \, dx_2 \tag{2.20}$$
$$\equiv J_1 + J_2.$$

Making use of (2.18) to bound J_1, we obtain

$$J_1 \leq c_1 \int_{L_z} \rho(q^2) |w| (w_{,\beta} w_{,\beta})^{1/2} \, dx_2$$
$$\leq c_1 \left(\int_{L_z} \rho(q^2) w_{,\beta} w_{,\beta} \, dx_2 \right)^{1/2} \left(\int_{L_z} \rho(q^2) w^2 \, dx_2 \right)^{1/2}. \tag{2.21}$$

The last step in (2.21) follows from Schwarz's inequality. Using a scheme involving a change of variable introduced in [7] (see pp. 314–315 of [7]) (and also used in [8], [14]), we can show that

$$\int_{L_z} \rho(q^2) w^2 \, dx_2 \leq \frac{B_1^2(z)}{\pi^2} \int_{L_z} [\rho(q^2)]^{-1} w_{,2}^2 \, dx_2 \quad \text{(Case 1)},$$
$$\int_{L_z} [\rho(q^2)]^{-1} w^2 \, dx_2 \leq \frac{B_2^2(z)}{\pi^2} \int_{L_z} \rho(q^2) w_{,2}^2 \, dx_2 \quad \text{(Case 2)}, \tag{2.22}$$

where $B_\alpha(z)$ $(\alpha = 1, 2)$ is defined in (2.8). By using (2.22) and the left-hand sides of (1.7), (1.8), we find from (2.21) that

$$J_1 \leq \frac{c_1 B_\alpha(z)}{m_\alpha \pi} \int_{L_z} \rho(q^2) w_{,\alpha} w_{,\alpha} \, dx_2, \tag{2.23}$$

which, in view of (2.2) and (2.7), can be written as

$$J_1 \leq c_1 \kappa_\alpha^{-1}(z)[-E'(z)]. \tag{2.24}$$

A bound for J_2, defined in (2.20), follows similarly. Thus, by using Schwarz's inequality, we have

$$J_2 \leq c_2 \varepsilon^2 \left(\int_{L_z} \rho(q^2) w^2 \, dx_2 \right)^{1/2} \left(\int_{L_z} \tilde{\rho}(p^2) p^2 \, dx_2 \right)^{1/2}. \tag{2.25}$$

Again using (2.22), the left-hand sides of (1.7), (1.8), the definition (2.6) of $\tilde{E}$, and the obvious inequality

$$\int_{L_z} \rho(q^2) w_{,2}^2 \, dx_2 \leq \int_{L_z} \rho(q^2) w_{,\alpha} w_{,\alpha} \, dx_2, \tag{2.26}$$

we find from (2.25) that

$$J_2 \leq c_2 \varepsilon^2 \kappa_\alpha^{-1}(z)\{[-E'(z)]^{1/2}[-\tilde{E}'(z)]^{1/2}\}. \tag{2.27}$$

Thus, on combining (2.24), (2.27), we obtain from (2.20) that

$$I_2 \leq c_1 \kappa_\alpha^{-1}(z)[-E'(z)] + c_2 \varepsilon^2 \kappa_\alpha^{-1}(z)[-E'(z)]^{1/2}[-\tilde{E}'(z)]^{1/2}. \tag{2.28}$$

It remains to obtain an upper bound for I_1, the first integral on the right in (2.15). By using Schwarz's inequality, we have

$$I_1 \leq \left(\int_{L_z} \rho(q^2)w^2\,dx_2\right)^{1/2}\left(\int_{L_z} \rho(q^2)w_{,1}^2\,dx_2\right)^{1/2}. \tag{2.29}$$

Again employing (2.22), the left-hand sides of (1.7), (1.8), and the arithmetic-geometric mean inequality, we find that

$$I_1 \leq \frac{B_\alpha(z)}{2m_\alpha\pi}\int_{L_z}\rho(q^2)w_{,\alpha}w_{,\alpha}\,dx_2 = \kappa_\alpha^{-1}(z)[-E'(z)]/2. \tag{2.30}$$

Combining the results (2.19), (2.28), and (2.30) on the right-hand side of (2.15), we find that

$$(1-c_1)E(z) \leq c_2\varepsilon^2 E^{1/2}(z)\tilde{E}(z)^{1/2} + \kappa_\alpha^{-1}(z)(1+2c_1)[-E'(z)]/2 \\ + c_2\varepsilon^2\kappa_\alpha^{-1}(z)[-E'(z)]^{1/2}[-\tilde{E}'(z)]^{1/2}. \tag{2.31}$$

The final step in establishing (2.4) from (2.31) is carried out by using the weighted arithmetic-geometric mean inequality

$$2ab \leq \delta a^2 + \frac{1}{\delta}b^2 \qquad (\delta > 0) \tag{2.32}$$

in the first and third terms on the right in (2.31). Thus, we obtain

$$(1-c_1-\delta_1/2)E(z) + 1/2(1+2c_1+\delta_2)\kappa_\alpha^{-1}(z)E'(z) \\ + c_2^2\varepsilon^4\kappa_\alpha^{-1}(z)\tilde{E}'(z)/(2\delta_2) - c_2^2\varepsilon^4\tilde{E}(z)/(2\delta_1) \leq 0 \tag{2.33}$$

for arbitrary δ_1, $\delta_2 > 0$. Choosing $\delta_1 = 1 - c_1$ and rearranging (2.33) we get

$$E'(z) + (1+2c_1+\delta_2)^{-1}(1-c_1)\kappa_\alpha(z)E(z) + \varepsilon^4 c_2^2(1+2c_1+\delta_2)^{-1}\delta_2^{-1}\tilde{E}'(z) \\ - \varepsilon^4 c_2^2(1-c_1)^{-1}(1+2c_1+\delta_2)^{-1}\kappa_\alpha(z)\tilde{E}(z) \leq 0 \tag{2.34}$$

for arbitrary $\delta_2 > 0$. For algebraic simplicity in what follows, we take $\delta_2 = 1$ in (2.34). (Other choices for δ_1, δ_2 may also be made; see the discussion in Sec. 3.) On defining $F(z)$ as in (2.5), it is then readily verified that (2.34) has precisely the desired form (2.4).

The differential inequality (2.4) can be written as

$$\left[F(z)\exp\left\{2d\int_0^z \kappa_\alpha(\eta)\,d\eta\right\}\right]' \leq D\varepsilon^4\kappa_\alpha(z)\tilde{E}(z)\exp\left\{2d\int_0^z \kappa_\alpha(\eta)\,d\eta\right\}, \tag{2.35}$$

and so, on integrating, we find that

$$F(z) \leq F(0) \exp \left\{ -2d \int_0^z \kappa_\alpha(\eta) \, d\eta \right\}$$
$$+ D\varepsilon^4 \int_0^z \kappa_\alpha(\sigma) \widetilde{E}(\sigma) \exp \left\{ -2d \int_\sigma^z \kappa_\alpha(\eta) \, d\eta \right\} d\sigma . \tag{2.36}$$

Now, by using Schwarz's inequality, we can easily show (cf. pp. 315–316 of [7]) that

$$\frac{1}{m_\alpha \pi} \int_\sigma^z \kappa_\alpha(\eta) d\eta \geq (z - \sigma)^2 \bigg/ \int_\sigma^z B_\alpha(s) \, ds . \tag{2.37}$$

Making use of the hypotheses (1.7) or (1.8), in Cases 1 or 2 respectively, we can show (see, e.g., p. 316 of [7]) that (2.37) leads to

$$\frac{1}{m_\alpha \pi} \int_\sigma^z \kappa_\alpha(\eta) \, d\eta \geq \frac{z - \sigma}{M_\alpha h} - \frac{K_\alpha}{M_\alpha^2 h^2} [\widehat{E}(\sigma) - \widehat{E}(z)], \tag{2.38}$$

where

$$\widehat{E}(z) = \int_{R_z} \rho(q^2) q^2 \, dA \qquad (q^2 = u_{,\alpha} u_{,\alpha}) \tag{2.39}$$

is the energy associated with the problem (1.1), (1.4)–(1.6). Since $\widehat{E}(\sigma)$ is decreasing in σ (as is shown in [7]), and discarding the last positive term in (2.38), we obtain

$$\frac{1}{m_\alpha \pi} \int_\sigma^z \kappa_\alpha(\eta) \, d\eta \geq \frac{z - \sigma}{M_\alpha h} - \frac{K_\alpha \widehat{E}_0}{M_\alpha^2 h^2} , \tag{2.40}$$

where $\widehat{E}_0 \equiv \widehat{E}(0)$ is the total energy (assumed finite) in R. By inserting (2.40) (when $\sigma = 0$) into the first-term on the right in (2.36), we obtain

$$F(z) \leq F(0) \exp \left[\frac{dm_\alpha K_\alpha \pi}{M_\alpha^2 h^2} \right] \exp \left[\frac{-2dm_\alpha \pi z}{M_\alpha h} \right]$$
$$+ D\varepsilon^4 \int_0^z \kappa_\alpha(\sigma) \widetilde{E}(\sigma) \exp \left\{ -2d \int_\sigma^z \kappa_\alpha(\eta) \, d\eta \right\} d\sigma . \tag{2.41}$$

The exponentially decaying term in the first expression on the right in (2.41) has a decay rate $2\nu_\alpha$ ($\alpha = 1, 2$) given by

$$\nu_\alpha = \frac{dm_\alpha \pi}{M_\alpha h} , \tag{2.42}$$

where d is given in $(2.9)_1$. We now use the decay estimate for $\widetilde{E}(\sigma)$, given by (2.10), in the integrand in the integral in (2.41). This substitution will then introduce the decay rate $\widetilde{\nu}_\alpha$, given by (2.11), into the final estimate. By using integration by parts, we find that the entire second expression on the right in (2.41), denoted by J, is

such that

$$J \le \frac{D\widetilde{G}_\alpha \varepsilon^4}{2d} \left\{ \left[e^{-2\tilde{\nu}_\alpha \sigma} \exp\left\{ -2d \int_\sigma^z \kappa_\alpha(\eta)\, d\eta \right\} \right]_0^z \right.$$

$$\left. + 2\tilde{\nu}_\alpha \int_0^z e^{-2\tilde{\nu}_\alpha \sigma} \exp\left\{ -2d \int_\sigma^z \kappa_\alpha(\eta)\, d\eta \right\} d\sigma \right\}$$

$$\le \frac{D\widetilde{G}_\alpha \varepsilon^4}{2d} \left[e^{-2\tilde{\nu}_\alpha z} - \exp\left\{ -2d \int_0^z \kappa_\alpha(\eta)\, d\eta \right\} \right]$$

$$+ \frac{D\widetilde{G}_\alpha \varepsilon^4 \tilde{\nu}_\alpha}{2d} \int_0^z e^{-2\tilde{\nu}_\alpha \sigma} \exp\left[\frac{dm_\alpha K_\alpha \widehat{E}_0}{M_\alpha^2 h^2} \right] \exp\left[\frac{-2dm_\alpha(z-\sigma)}{M_\alpha h} \right] d\sigma, \tag{2.43}$$

where we have used (2.40) to obtain the final inequality in (2.43). By discarding the second (negative) term in (2.43) and inserting the resulting bound in (2.41), we find that

$$F(z) \le F(0) \exp\left[\frac{2\nu_\alpha K_\alpha}{M_\alpha h} \right] e^{-2\nu_\alpha z} + \frac{D\widetilde{G}_\alpha \varepsilon^4}{2d} e^{-2\tilde{\nu}_\alpha z}$$

$$+ \frac{D\widetilde{G}_\alpha \varepsilon^4 \tilde{\nu}_\alpha}{2d} \exp\frac{[2\nu_\alpha K_\alpha \widehat{E}_0]}{M_\alpha h} \frac{[e^{-2\nu_\alpha z} - e^{-2\tilde{\nu}_\alpha z}]}{\tilde{\nu}_\alpha - \nu_\alpha}, \tag{2.44}$$

where if $\tilde{\nu}_\alpha = \nu_\alpha$, then the limit is understood in the last term of (2.44).

The desired decay estimate (2.3) follows from (2.44) by observing that the exponentially decaying terms on the right-hand side of (2.44) can be bounded by an exponentially decaying term of the form $\exp(-2k_\alpha z)$, where

$$k_\alpha = \min(\nu_\alpha, \tilde{\nu}_\alpha) = \frac{\pi}{h} \min\left(d\frac{m_\alpha}{M_\alpha}, \frac{\tilde{m}_\alpha}{\widetilde{M}_\alpha} \right). \tag{2.45}$$

We recall from $(2.9)_1$ that $d = (1 - c_1)/[4(1 + c_1)]$ and $c_1 = c_1(\varepsilon)$ is the quantity defined by (1.18), (1.20). To obtain the sharpest possible estimated decay rate k_α, one should choose m_α, $\tilde{m}_\alpha$ as *large* as possible and M_α, $\widetilde{M}_\alpha$ as *small* as possible. In the illustrative examples discussed in Sec. 3, we have

$$\nu_\alpha < \tilde{\nu}_\alpha. \tag{2.46}$$

In fact, it is shown in Appendix C that (2.46) holds if the constitutive functions ρ and $\tilde{\rho}$ satisfy a condition suggested by (1.20) (see equation (C.1)). On assuming that (2.46) holds, and observing from (2.5) that $F(z) \ge E(z)$, we find from (2.44) that

$$E(z) \le H_\alpha e^{-2\nu_\alpha z}, \qquad z \ge 0, \tag{2.47}$$

where

$$H_\alpha = F(0) \exp\left(\frac{2\nu_\alpha K_\alpha}{M_\alpha h} \right) + \frac{\varepsilon^4 D\widetilde{G}_\alpha \tilde{\nu}_\alpha}{2d(\tilde{\nu}_\alpha - \nu_\alpha)} \exp\left(\frac{2\nu_\alpha K_\alpha \widehat{E}_0}{M_\alpha h} \right). \tag{2.48}$$

Observe that the quantity $\widetilde{G}_\alpha$ defined in (2.12) contains $\widetilde{E}_0$, and that $F(0)$, given by (2.5) with $z = 0$, contains both E_0 and $\widetilde{E}_0$. We can easily obtain an upper bound for E_0 in terms of $\widetilde{E}_0$. By virtue of the definition (2.1) of w and by using (1.6),

(1.15), we see that $w = 0$ on $x_1 = 0$. Thus, (2.15) evaluated at $z = 0$ yields $E_0 \leq I_3(0)$ and so, by (2.19), we find that

$$(1 - c_1)E_0 \leq c_2 \varepsilon^2 E_0^{1/2} \widetilde{E}_0^{1/2} , \tag{2.49}$$

and so

$$E_0 \leq \frac{c_2^2 \varepsilon^4}{(1 - c_1)^2} \widetilde{E}_0 . \tag{2.50}$$

Thus, we have from (2.5) that

$$F(0) = E_0 + c_2^2 \varepsilon^4 (1 + c_1)^{-1} \widetilde{E}_0/2 \leq c_2^2 \varepsilon^4 (3 + c_1^2)(1 + c_1)^{-1}(1 - c_1)^{-2} \widetilde{E}_0/2 . \tag{2.51}$$

Upper bounds for the total energies $\widetilde{E}_0$ and $\widehat{E}_0$ in terms of the boundary data have been established in [7], and these bounds can be used in (2.50), (2.51), and (2.48) to obtain, from (2.47), the desired result

$$E(z) \leq \varepsilon^4 C_\alpha e^{-2\nu_\alpha z} , \qquad z \geq 0 \tag{2.52}$$

for a computable constant C_α.

If the hypothesis (C.1) of Appendix C is not made (so that (2.46) is no longer assured) one can still proceed from (2.44), and use (2.50), (2.51) to obtain an estimate of the form

$$E(z) \leq \varepsilon^4 C_\alpha e^{-2k_\alpha z} , \qquad z \geq 0 \tag{2.53}$$

for a computable constant C_α, where k_α is given by (2.45). This completes the derivation of (2.3).

3. Illustrative examples. In this section we illustrate by three examples how to make explicit the comparison (2.52) of the solution of problem (1.1), (1.4)–(1.6) with the solution of the perturbed problem (1.12), (1.13)–(1.15). Two of these examples were mentioned in Sec. 1, i.e., the ρ and $\tilde{\rho}$ given by (1.24) and (1.26). We consider first the problem defined by (1.24).

Problem 1. Here $\rho(q^2)$ and $\tilde{\rho}(p^2)$ are given by (1.24), i.e.,

$$\rho(q^2) = \mu(1 + \varepsilon^2 q^2)^{-1/2} , \qquad \tilde{\rho}(p^2) = \mu . \tag{3.1}$$

We observe, in connection with (3.1), that one might think of ρ as a perturbation of $\tilde{\rho}$ rather than $\tilde{\rho}$ being a perturbation of ρ, but it is immaterial which we designate as the perturbation of the other. At any rate, as mentioned in Sec. 1, it is shown in Appendix A that (1.20) is satisfied with $\gamma_1(p, \varepsilon)$ and $\gamma_2(p, \varepsilon)$ given by (1.25), i.e.,

$$\gamma_1(p, \varepsilon) = \varepsilon p , \qquad \gamma_2(p, \varepsilon) = p^2/2 . \tag{3.2}$$

In order to find the constants $c_1(\varepsilon)$ and c_2 defined by (1.18) and (1.19) respectively, it is necessary to find a bound for the maximum value of $p \equiv |\operatorname{grad} v|$ on $\overline{R}$. Since p is the gradient of a harmonic function this is not difficult to accomplish with the conditions on $f(x_2)$ implied by the fact that $v \in C^2(R) \cap C^1(\overline{R})$. The derivation of an appropriate bound is carried out in Appendix B where it is shown that, with the notation

$$\max_{R} p^2 = P^2 , \tag{3.3}$$

then

$$P^2 \leq \max_{x_2 \in [0,h]} \left\{ \frac{h^2}{\pi^2} [f''(x_2)]^2 + [f'(x_2)]^2 \right\}. \tag{3.4}$$

It follows that $c_1(\varepsilon)$ and c_2 can be taken as

$$c_1(\varepsilon) = \varepsilon P, \qquad c_2 = P^2/2. \tag{3.5}$$

Thus, for sufficiently small ε, the last inequality in (1.18) is satisfied.

The ρ and $\tilde{\rho}$ given by (3.1) belong to Case 2 (see (1.8) and (1.17)). Clearly (1.17) is satisfied with the choices

$$\tilde{m}_2 = \mu^{-1}, \quad \widetilde{M}_2 = \mu^{-1}, \quad \tilde{K}_2 = 0. \tag{3.6}$$

Since

$$(1 + \varepsilon^2 q^2)^{1/2} = \frac{1 + \varepsilon^2 q^2}{[1 + \varepsilon^2 q^2]^{1/2}} \leq 1 + \frac{\varepsilon^2 q^2}{[1 + \varepsilon^2 q^2]^{1/2}}, \tag{3.7}$$

it follows that (1.8) is satisfied for the choices

$$m_2 = \mu^{-1}, \quad M_2 = \mu^{-1}, \quad K_2 = \varepsilon^2 \mu^{-2}, \tag{3.8}$$

(cf. [7], p. 325). Recalling the definitions of $\tilde{\nu}_2$, ν_2 from (2.11), (2.42) respectively, we have

$$\tilde{\nu}_2 = \frac{\pi}{h}, \qquad \nu_2 = \frac{\pi d}{h} \tag{3.9}$$

with d given, by virtue of (2.9) and (3.5), by

$$d = \frac{(1 - \varepsilon P)}{4(1 + \varepsilon P)}. \tag{3.10}$$

Thus, it is clear that $\nu_2 < \tilde{\nu}_2$ so that (2.46) holds and the decay estimate (2.47), with $\alpha = 2$, holds for Problem 1 with

$$\nu_2 = \frac{\pi}{4h} \frac{(1 - \varepsilon P)}{(1 + \varepsilon P)}. \tag{3.11}$$

Since $\tilde{K}_2 = 0$, the expression (2.12), with $\alpha = 2$, simplifies to

$$\tilde{G}_2 = \tilde{E}_0. \tag{3.12}$$

Also, since by (3.8) the constant K_2 is of order ε^2, the decay estimate (2.47), (2.48) can be written as

$$E(z) \leq \varepsilon^4 L \tilde{E}_0 [1 + O(\varepsilon^2)] e^{-2\nu_2 z} \tag{3.13}$$

where ν_2 is given by (3.11). Here the constant L is given by

$$2L = \left[\frac{P^4}{4} (3 + P^2 \varepsilon^2)(1 + P\varepsilon)^{-1}(1 - P\varepsilon)^{-2} + D/(d(1 - d)) \right], \tag{3.14}$$

where D is defined by (2.9). For Problem 1, on using (3.5), (3.6), (3.8), and (3.10), we have

$$D = \frac{(3 + \varepsilon^2 P^2) P^4}{16(1 + \varepsilon P)^2 (1 - \varepsilon P)}, \qquad d = \frac{(1 - \varepsilon P)}{4(1 + \varepsilon P)}. \tag{3.15}$$

We note that the decay rate given by (3.11) is much too conservative since a simple estimate based on the arithmetic-geometric mean inequality gives

$$
\begin{aligned}
E(z) &\leq 2\left\{\widehat{E}(z) + \int_{R_z} \rho(q^2)v_{,\alpha}v_{,\alpha}\,dA\right\} \\
&\leq 2\{\widehat{E}(z) + \widetilde{E}(z)\} \\
&\leq Ne^{-2\pi z/h}.
\end{aligned}
\tag{3.16}
$$

The last inequality follows from results established in [7]. Here N is a constant which is $O(1)$ in the parameter ε. We should point out, however, that in this paper we are emphasizing the feature that $E(z)$ is $O(\varepsilon^4)$ *for all* z and hence that u and v are close in energy measure *for all* z. At various steps in our proof we have used inequalities that preserve the $O(\varepsilon^4)$, but which result in a poorer estimate of the actual decay rate. For instance, in this problem $c_1(\varepsilon)$ is $O(\varepsilon)$. If we were interested in improving the estimate of the decay rate, then in (2.33) we could choose

$$
\delta_1 = \beta_1\varepsilon, \qquad \delta_2 = \beta_2\varepsilon
\tag{3.17}
$$

for arbitrary positive constants β_1 and β_2. Then (2.33) would become

$$
\begin{aligned}
&[1 - (P + \beta_1/2)\varepsilon]E(z) + (1/2)[1 + (2P + \beta_2)\varepsilon]\kappa_\alpha^{-1}(z)E'(z) \\
&+ P^4(8\beta_2)^{-1}\varepsilon^3\kappa_\alpha^{-1}(z)\widetilde{E}'(z) - P^4(8\beta_1)^{-1}\varepsilon^3\widetilde{E}(z) \leq 0.
\end{aligned}
\tag{3.18}
$$

By continuing the proof as above we would obtain, instead of (3.13),

$$
E(z) \leq \varepsilon^3 C_2^* e^{-2k_2^* z},
\tag{3.19}
$$

where C_2^* is a computable constant, but now

$$
k_2^* = \frac{\pi}{h}\left\{\frac{1 - \varepsilon[P + \beta_1/2]}{1 + \varepsilon[2P + \beta_2]}\right\}.
\tag{3.20}
$$

Thus by sacrificing a power of ε in the multiplicative factor in the decay estimate we are able to improve the estimated decay rate obtaining a new estimate which differs from the expected decay rate by a term of $O(\varepsilon)$. This trade off between the size of the multiplicative factor in the estimate and the decay rate is reminiscent of that observed by Horgan and Payne [7, 8] in related contexts.

Before proceeding to the next example, we give an indication of how Problem 1 might arise in applications. Let ϕ be the solution of the following minimal surface problem:

$$
\{[1 + |\nabla\phi|^2]^{-1/2}\phi_{,\alpha}\}_{,\alpha} = 0 \quad \text{on } R
\tag{3.21}
$$

with the boundary conditions

$$
\phi(x_1, 0) = 0, \quad \phi(x_1, h) = 0, \qquad x_1 \geq 0,
\tag{3.22}
$$

$$
\phi, \phi_{,1} \to 0 \quad \text{(uniformly in } x_2\text{) as } x_1 \to \infty,
\tag{3.23}
$$

$$
\phi(0, x_2) = \varepsilon f(x_2), \qquad 0 \leq x_2 \leq h.
\tag{3.24}
$$

Then if we set

$$
\phi = \varepsilon u,
\tag{3.25}
$$

the function u will be the solution of (1.1), (1.4)–(1.6) with the $\rho(q^2)$ of (1.24), the case we have just considered. Thus, we might think of approximating ϕ by εv where v is the harmonic function taking the value zero on the long sides and the value $f(x_2)$ on the end $x_1 = 0$. A measure of the merit of this approximation follows from the observation that

$$
\begin{aligned}
E^*(z) &\equiv \int_{R_z} [1 + |\nabla\phi|^2]^{-1/2} |\nabla(\phi - \varepsilon v)|^2 \, dA \\
&= \varepsilon^2 E(z) \\
&\leq \varepsilon^6 C_2 e^{-2\nu_2 z}, \qquad z \geq 0.
\end{aligned}
\tag{3.26}
$$

In the last step we make use of (2.52) with $\alpha = 2$.

Problem 2. Here ρ and $\tilde{\rho}$ are given by (1.26), i.e.,

$$
\rho(q^2) = \mu \left(1 + \frac{bq^2}{1 + \varepsilon^2}\right)^{\varepsilon^2}, \qquad \tilde{\rho}(p^2) = \mu.
\tag{3.27}
$$

As we mentioned in Sec. 1, the problem for u may be thought of as a problem for a material that is close to neo-Hookean while the problem for v is the neighboring neo-Hookean problem governed by Laplace's equation. As we observed in Sec. 1, γ_1 and γ_2 are now given by (see Appendix A)

$$
\gamma_1(p, \varepsilon) = \frac{pb^{1/2}\varepsilon^2}{1 + \varepsilon^2}\{[\varepsilon^2/(1 + \varepsilon^2)]^{\varepsilon^2/2} + pb^{1/2}\}, \qquad \gamma_2(p, \varepsilon) = \frac{bp^2\varepsilon^2}{1 + \varepsilon^2}.
\tag{3.28}
$$

Since the v in the perturbed problem is again a harmonic function we can use (3.4) as the bound for P, the maximum value of p in $\overline{R}$. We thus arrive at

$$
c_1(\varepsilon) = \frac{b^{1/2}\varepsilon^2}{1 + \varepsilon^2}\{P + b^{1/2}P^2\}, \qquad c_2 = \frac{bP^2\varepsilon^2}{1 + \varepsilon^2}.
\tag{3.29}
$$

Thus, for sufficiently small ε, the last inequality in (1.18) is satisfied.

The ρ and $\tilde{\rho}$ given by (3.27) belong to Case 1 (see (1.7) and (1.16)). Clearly (1.16) is satisfied with the choices

$$
\tilde{m}_1 = \mu, \quad \widetilde{M}_1 = \mu, \quad \tilde{K}_1 = 0.
\tag{3.30}
$$

Since

$$
1 \leq \left[1 + \frac{bq^2}{(1 + \varepsilon^2)}\right]^{\varepsilon^2} \leq 1 + \frac{bq^2\varepsilon^2}{1 + \varepsilon^2},
\tag{3.31}
$$

it follows that (1.7) is satisfied for the choices

$$
m_1 = \mu, \quad M_1 = \mu, \quad K_1 = \frac{b\varepsilon^2}{1 + \varepsilon^2}
\tag{3.32}
$$

(cf. [7], p. 324). Thus, we have

$$
\tilde{\nu}_1 = \frac{\pi}{h}, \qquad \nu_1 = \frac{\pi d}{h},
\tag{3.33}
$$

where d is given by (2.9) and $(3.29)_1$. Again we see that $\nu_1 < \tilde{\nu}_1$ so that (2.46) holds and so the decay estimate (2.52) (with $\alpha = 1$) holds for Problem 2 with ν_1 given by $(3.33)_2$. Since $\tilde{K}_1 = 0$ and $K_1 = O(\varepsilon^2)$, we can again obtain an estimate of the form (3.13), with ν_2 replaced by ν_1 for Problem 2. Also, since $c_1(\varepsilon)$ is $O(\varepsilon^2)$, it is again possible to improve the estimated decay rate at the expense of lowering the order of ε in the multiplicative factor by making the appropriate choices for δ_1 and δ_2 in (2.33).

Problem 3. As a third example we consider the following expressions for ρ and $\tilde{\rho}$:

$$\rho(q^2) = \mu \left[1 + \frac{q^2}{1+\varepsilon^2} \right]^{-1/4} , \qquad \tilde{\rho}(p^2) = \mu[1 + p^2]^{-1/4} . \qquad (3.34)$$

We observe that these values of ρ and $\tilde{\rho}$ are of the power-law form (1.2) with

$$n = \frac{3}{4} , \qquad b = \frac{3}{4(1+\varepsilon^2)} . \qquad (3.35)$$

Thus, we are considering softening materials with a fixed value of the softening parameter, and the perturbation may be viewed as a perturbation in the material parameter b away from the value $3/4$. We observe that this problem differs from the previous examples in that the differential equation for v is now also quasilinear.

It is shown in Appendix A that for this example

$$\gamma_1(p, \varepsilon) = \frac{(1+\varepsilon^2)^{-1/4} p}{(1+p^2)^{1/2} + [(1+\varepsilon^2)(1+p^2)]^{1/4}} \quad , \quad \gamma_2(p, \varepsilon) = (1+\varepsilon^2)^{-1} p^2/4. \qquad (3.36)$$

To find the constants $c_1(\varepsilon)$ and c_2, defined by (1.18), (1.19) respectively, one would require a bound for P, the maximum of $p = |\operatorname{grad} v|$ on $\overline{R}$. In contrast with the analogous situation for Problems 1 and 2 where such a bound was obtained in (3.3), (3.4), here one requires an estimate for the gradient of solutions of the quasilinear differential equation (1.12), with $\tilde{\rho}$ given by $(3.34)_2$. While techniques for obtaining such bounds are known, the detailed calculations required to obtain explicit estimates are beyond the scope of the present work.

From (3.36), it follows that $c_1(\varepsilon)$ and c_2 may be taken as

$$c_1(\varepsilon) = \frac{(1+\varepsilon^2)^{-1/4} P}{(1+P^2)^{1/2} + [(1+\varepsilon^2)(1+P^2)]^{1/4}} \quad , \quad c_2 = (1+\varepsilon^2)^{-1} P^2/4. \qquad (3.37)$$

For ε sufficiently small, the last inequality in (1.18) is satisfied.

Since

$$
\begin{aligned}
\rho^{-1}(q^2) &= \mu^{-1}(1+\varepsilon^2)^{-1/4}(1+q^2+\varepsilon^2)^{1/4} \\
&= \mu^{-1}(1+\varepsilon^2)^{-1/4}(1+q^2+\varepsilon^2)(1+q^2+\varepsilon^2)^{-3/4} \\
&= \mu^{-1}(1+\varepsilon^2)^{3/4}(1+q^2+\varepsilon^2)^{-3/4} + \mu^{-2}(1+\varepsilon^2)^{-1/2}(1+q^2+\varepsilon^2)^{-1/2} q^2 \rho(q^2),
\end{aligned}
\qquad (3.38)
$$

the conditions (1.8) of Case 2 hold with

$$m_2 = \mu^{-1} , \quad M_2 = \mu^{-1} , \quad K_2 = \mu^{-2}(1+\varepsilon^2)^{-1} . \qquad (3.39)$$

Also, as was shown in [7] (see pp. 324–325 of [7]) the conditions (1.17) of Case 2 hold with

$$\tilde{m}_2 = \mu^{-1}, \quad \widetilde{M}_2 = \mu^{-1}, \quad \tilde{K}_2 = \mu^{-2}. \tag{3.40}$$

Thus, we have

$$\tilde{\nu}_2 = \frac{\pi}{h}, \qquad \nu_2 = \frac{\pi d}{h}, \tag{3.41}$$

with d given by

$$d = \frac{1 - c_1}{4(1 + c_1)}, \tag{3.42}$$

and c_1 given by $(3.37)_1$. Again, we have $\nu_2 < \tilde{\nu}_2$ so that (2.46) holds and the decay estimate (2.52), with $\alpha = 2$, holds for Problem 3 with ν_2 given by (3.41), (3.42).

The decay behavior in this problem is somewhat different from that in the two previous examples. In the first two problems $\rho \to \tilde{\rho}$ as $q^2 \to 0$ whereas in this example that feature is no longer true. We observe also that since $c_1(\varepsilon)$ is $O(1)$ in ε as $\varepsilon \to 0$ we no longer have the possibility of a trade off between the decay coefficient and the order of ε in the multiplicative factor. At the same time, since $\tilde{K}_2$ is different from zero, the constant C_2 in (2.52) will in general be much larger than in the first two examples.

Finally, we provide another interpretation for the perturbation involved in Problem 3. Let ϕ be the solution of the problem

$$\{[1 + |\nabla\phi|^2]^{-1/4}\phi_{,\alpha}\}_{,\alpha} = 0 \quad \text{on } R \tag{3.43}$$

subject to the boundary conditions

$$\phi(x_1, 0) = 0, \quad \phi(x_1, h) = 0, \qquad x_1 \geq 0, \tag{3.44}$$

$$\phi, \phi_{,1} \to 0 \quad \text{(uniformly in } x_2\text{) as } x_1 \to \infty, \tag{3.45}$$

$$\phi(0, x_2) = (1 + \varepsilon^2)^{-1/2} f(x_2), \qquad 0 \leq x_2 \leq h. \tag{3.46}$$

Then, if we set

$$\phi = (1 + \varepsilon^2)^{-1/2} u, \tag{3.47}$$

u will be the solution of (1.1), (1.4)–(1.6) with the ρ of $(3.34)_1$. Thus we might think of approximating ϕ by $(1+\varepsilon^2)^{-1/2}v$ where v satisfies (1.11)–(1.14) with $\tilde{\rho}(p^2)$ given by $(3.34)_2$. A measure of the merit of this approximation can be assessed from the fact that

$$E^{**}(z) \equiv \int_{R_z} [1 + |\nabla\phi|^2]^{-1/4} |\nabla[\phi - (1 + \varepsilon^2)^{-1/2}v]|^2 \, dA = (1 + \varepsilon^2)^{-1} E(z)$$

$$\leq \frac{\varepsilon^4}{1 + \varepsilon^2} C_2 e^{-2\nu_2 z}, \qquad z \geq 0. \tag{3.48}$$

In the last step we use (2.52) with $\alpha = 2$. Note that, in contrast to the result (3.26), the right-hand side of (3.48) is of order ε^4 as $\varepsilon \to 0$ rather than ε^6, as in (3.26).

4. Concluding remarks. The preceding results are also valid when the Dirichlet boundary conditions (1.4), (1.6) are replaced by appropriate Neumann boundary conditions that correspond to traction boundary conditions in the antiplane shear context. Thus we now consider solutions of (1.1) on R subject to the boundary conditions

$$u_{,2}(x_1, 0) = 0, \quad u_{,2}(x_1, h) = 0, \quad x_1 \geq 0, \tag{4.1}$$

$$u, u_{,1} \to 0 \quad \text{(uniformly in } x_2\text{) as } x_1 \to \infty, \tag{4.2}$$

$$\rho(q^2)u_{,1} = g(x_2) \quad \text{on } x_1 = 0, \quad 0 \leq x_2 \leq h. \tag{4.3}$$

The prescribed function g, assumed sufficiently smooth, must also satisfy the "self-equilibration" condition

$$\int_0^h g(x_2)\, dx_2 = 0, \tag{4.4}$$

which follows from application of the divergence theorem to (1.1) and by using (4.1)–(4.3). In fact the same argument shows that

$$\int_{L_z} \rho(q^2)u_{,1}\, dx_2 = 0, \quad z \geq 0, \tag{4.5}$$

on *every* cross section L_z. The problem for v is again governed by (1.12), but now subject to the boundary conditions

$$v_{,2}(x_1, 0) = 0, \quad v_{,2}(x_1, h) = 0, \quad x_1 \geq 0, \tag{4.6}$$

$$v, v_{,1} \to 0 \quad \text{(uniformly in } x_2\text{) as } x_1 \to \infty, \tag{4.7}$$

$$\tilde{\rho}(p^2)v_{,1} = g(x_2) \quad \text{on } x_1 = 0, \quad 0 \leq x_2 \leq h. \tag{4.8}$$

As in the problem for u, we find that

$$\int_{L_z} \tilde{\rho}(p^2)v_{,1}\, dx_2 = 0, \quad z \geq 0. \tag{4.9}$$

By combining (4.5) and (4.9), we see that

$$\int_{L_z} [\rho(q^2)u_{,1} - \tilde{\rho}(p^2)v_{,1}]\, dx_2 = 0, \quad z \geq 0. \tag{4.10}$$

The proof of (2.3) given in Sec. 2 remains valid for the Neumann boundary conditions up to equation (2.13). We now introduce the functions $W_\alpha(x_1, x_2)$ ($\alpha = 1, 2$, in Cases 1, 2 respectively) by

$$W_\alpha = w - \overline{w}_\alpha(x_1), \tag{4.11}$$

where

$$\overline{w}_1(x_1) = \frac{1}{B_1} \int_0^h \rho(q^2)w\, dx_2, \qquad \overline{w}_2 = \frac{1}{B_2} \int_0^h [\rho(q^2)]^{-1} w\, dx_2. \tag{4.12}$$

By virtue of their definitions

$$\int_0^h \rho(q^2)W_1\, dx_2 = 0, \qquad \int_0^h [\rho(q^2)]^{-1} W_2\, dx_2 = 0. \tag{4.13}$$

By using (4.10), we can see that (2.13) can be written as

$$E(z) = -\int_{L_z} \rho(q^2) W_\alpha w_{,1} \, dx_2 - \int_{L_z} [\rho(q^2) - \tilde{\rho}(p^2)] W_\alpha v_{,1} \, dx_2$$
$$- \int_{R_z} [\rho(q^2) - \tilde{\rho}(p^2)] v_{,\beta} w_{,\beta} \, dA. \tag{4.14}$$

Thus, for the Neumann boundary conditions, (4.14) can be obtained from (2.13) by formally replacing w in the line integral terms by W_α ($\alpha = 1, 2$ in Cases 1, 2 respectively). One now proceeds from (4.14) exactly as was done in Sec. 2, except that in (2.22), the w^2 terms on the left are replaced by W_α^2 ($\alpha = 1, 2$). In deriving this version of (2.22), one makes use of the zero average conditions (4.13). (A similar procedure was followed in [7, 8].) In this way, one arrives at the estimates (2.47), (2.48), where the total energies E_0, $\widehat{E}_0$, and $\widetilde{E}_0$ are now understood to be those associated with the Neumann boundary conditions. With this understanding, one finds that (2.49)–(2.53) are also valid for Neumann boundary-value problems. The techniques described in [7, 8] may be used to obtain the required bounds for the total energies $\widehat{E}_0$, $\widetilde{E}_0$ in terms of the boundary data (4.3), (4.8).

Appendix A. Verification of (1.25), (1.27), and (3.36). If $\rho(q^2)$ and $\tilde{\rho}(p^2)$ are defined as in (1.24) we observe that

$$p|\rho(q^2) - \tilde{\rho}(p^2)| = \mu p[(1 + \varepsilon^2 q^2)^{-1/2} - 1]$$
$$= \frac{\mu p \varepsilon^2 q^2}{(1 + \varepsilon^2 q^2)^{1/2}[1 + (1 + \varepsilon^2 q^2)^{1/2}]} \tag{A.1}$$
$$= \mu \frac{p \varepsilon^2 [q(q-p) + p(q-p) + p^2]}{(1 + \varepsilon^2 q^2)^{1/2}[1 + (1 + \varepsilon^2 q^2)^{1/2}]}.$$

Clearly then

$$p|\rho(q^2) - \tilde{\rho}(p^2)| \le \frac{p\varepsilon \rho(q^2)[\varepsilon q + \varepsilon p]}{[1 + (1 + \varepsilon^2 q^2)^{1/2}]} |q - p| + \frac{\varepsilon^2 p^3 [\rho(q^2)\tilde{\rho}(p^2)]^{1/2}}{(1 + \varepsilon^2 q^2)^{1/4}[1 + (1 + \varepsilon^2 q^2)^{1/2}]}. \tag{A.2}$$

Now setting $\varepsilon q = s$ and $\varepsilon p = t$ we seek an upper bound for

$$\psi(s) = \frac{(s + t)}{1 + (1 + s^2)^{1/2}}. \tag{A.3}$$

For $t < 1$ it is easily checked that ψ is an increasing function of s and thus that

$$\psi(s) < 1. \tag{A.4}$$

This leads to

$$p|\rho(q^2) - \tilde{\rho}(p^2)| \le p\varepsilon \rho(q^2)|q - p| + \frac{\varepsilon^2 p^3}{2} [\rho(q^2)\tilde{\rho}(p^2)]^{1/2}. \tag{A.5}$$

By comparing with (1.20), we observe that

$$\gamma_1(p, \varepsilon) = p\varepsilon, \qquad \gamma_2(p, \varepsilon) = p^2/2, \tag{A.6}$$

which is (1.25). We observe further that $\varepsilon p < 1$ is required by condition (1.18). For p bounded, this merely imposes a restriction on the allowable magnitude of the parameter ε (see the discussion in Sec. 3).

Next we establish (1.27) for $\rho(q^2)$ and $\tilde{\rho}(p^2)$ given by (1.26). Now

$$p|\rho(q^2) - \tilde{\rho}(p^2)| = \mu p \left| \left(1 + \frac{bq^2}{1 + \varepsilon^2}\right)^{\varepsilon^2} - 1 \right|, \tag{A.7}$$

and we bound the expression on the right making use of an inequality from [16, p. 356] (stated in a somewhat different form), i.e., for any real number a and $0 \le \alpha \le 1/2$,

$$a^\alpha - 1 \le \alpha(a - 1)a^{-(1-\alpha)/2} \tag{A.8}$$

(cf. (A.12), p. 131 of [8]). Thus

$$p|\rho(q^2) - \tilde{\rho}(p^2)| \le \mu b \varepsilon^2 p^2 q^2 (1 + \varepsilon^2)^{-1}[1 + b^2/(1 + \varepsilon^2)]^{-(1-\varepsilon^2)/2}$$
$$= \frac{\mu b \varepsilon^2 p[(q - p)(q + p) + p^2]}{(1 + \varepsilon^2)[1 + bq^2/(1 + \varepsilon^2)]^{(1-\varepsilon^2)/2}}. \tag{A.9}$$

It follows then that

$$p|\rho(q^2) - \tilde{\rho}(p^2)| \le |q - p|\frac{\rho(q^2)bp\varepsilon^2}{(1 + \varepsilon^2)}\left\{ \frac{p + q}{[1 + bq^2/(1 + \varepsilon^2)]^{(1+\varepsilon^2)/2}} \right\}$$
$$+ \frac{bp^3\varepsilon^2}{1 + \varepsilon^2}\frac{[\rho(q^2)\tilde{\rho}(p^2)]^{1/2}}{[1 + b^2q^2/(1 + \varepsilon^2)]}. \tag{A.10}$$

By comparing with (1.20), we observe that we may choose

$$\gamma_1(p, \varepsilon) = \frac{bp\varepsilon^2}{1 + \varepsilon^2} \max_q \left\{ \frac{(p + q)}{[1 + bq^2/(1 + \varepsilon^2)]^{(1+\varepsilon^2)/2}} \right\}, \qquad \gamma_2(p, \varepsilon) = bp^2\varepsilon^2/(1 + \varepsilon^2). \tag{A.11}$$

This maximization leads to a complicated expression; however, if we maximize $q/\mathscr{D}$ and $p/\mathscr{D}$ separately (where $\mathscr{D}$ is the denominator in the expression to be maximized) we find

$$\gamma_1(p, \varepsilon) \le \frac{b\varepsilon^2}{(1 + \varepsilon^2)}\{\sigma p + p^2\}, \tag{A.12}$$

where

$$\sigma = \max_q \frac{q}{\mathscr{D}} = \max_q \left\{ \frac{q}{[1 + \tau q^2]^{(1+\varepsilon^2)/2}} \right\} \tag{A.13}$$

and

$$\tau = \frac{b}{1 + \varepsilon^2}. \tag{A.14}$$

Clearly the maximum occurs for $q^2 = [\tau\varepsilon^2]^{-1}$, and we obtain

$$\sigma = b^{-1/2}[\varepsilon^2/(1 + \varepsilon^2)]^{\varepsilon^2/2}. \tag{A.15}$$

Thus, we find from (A.12) that $\gamma_1(p, \varepsilon)$ can be chosen as

$$\gamma_1(p, \varepsilon) = pb^{1/2}\varepsilon^2/(1 + \varepsilon^2)\{[\varepsilon^2/(1 + \varepsilon^2)]^{\varepsilon^2/2} + pb^{1/2}\}. \tag{A.16}$$

Equations (A.16) and $(A.11)_2$ give precisely the expressions (1.27).

To establish (3.36) we start with

$$p|\rho(q^2) - \tilde{\rho}(p^2)| = \mu p \frac{|(1 + q^2 + \varepsilon^2)^{1/4} - (1 + p^2)^{1/4}(1 + \varepsilon^2)^{1/4}|}{(1 + q^2 + \varepsilon^2)^{1/4}(1 + p^2)^{1/4}}$$

$$= \mu \frac{p|(1 + q^2 + \varepsilon^2)^{1/2} - (1 + p^2)^{1/2}(1 + \varepsilon^2)^{1/2}|}{(1 + q^2 + \varepsilon^2)^{1/4}(1 + p^2)^{1/4}[(1 + q^2 + \varepsilon^2)^{1/4} + (1 + p^2)^{1/4}(1 + \varepsilon^2)^{1/4}]}. \tag{A.17}$$

Further simplication leads to

$$p|\rho(q^2) - \tilde{\rho}(p^2)|$$

$$= \frac{\mu p|q^2 - p^2 - \varepsilon^2 p^2|}{(1 + q^2 + \varepsilon^2)^{1/4}(1 + p^2)^{1/4}[(1 + q^2 + \varepsilon^2)^{1/4} + (1 + p^2)^{1/4}(1 + \varepsilon^2)^{1/4}][(1 + q^2 + \varepsilon^2)^{1/2} + (1 + p^2)^{1/2}(1 + \varepsilon^2)^{1/2}]}$$

$$= (1 + \varepsilon^2)^{-1/4}\rho(q^2)|q - p|$$

$$\times \frac{p(q + p)}{(1 + p^2)^{1/4}[(1 + q^2 + \varepsilon^2)^{1/4} + (1 + p^2)^{1/4}(1 + \varepsilon^2)^{1/4}][(1 + q^2 + \varepsilon^2)^{1/2} + (1 + p^2)^{1/2}(1 + \varepsilon^2)^{1/2}]}$$

$$+ \frac{(1 + \varepsilon^2)^{-1/2}p^3\varepsilon^2[\rho(q^2)\tilde{\rho}(p^2)]^{1/2}}{[(1 + q^2 + \varepsilon^2)(1 + p^2)]^{1/8}[(1 + q^2 + \varepsilon^2)^{1/4} + (1 + p^2)^{1/4}(1 + \varepsilon^2)^{1/4}][(1 + q^2 + \varepsilon^2)^{1/2} + (1 + p^2)^{1/2}(1 + \varepsilon^2)^{1/2}]}. \tag{A.18}$$

By comparing with (1.20), we see that we may choose

$$\gamma_1(p, \varepsilon)$$

$$= \max_q \frac{(1 + \varepsilon^2)^{-1/4}p(q + p)}{(1 + p^2)^{1/4}[(1 + q^2 + \varepsilon^2)^{1/4} + (1 + p^2)^{1/4}(1 + \varepsilon^2)^{1/4}][(1 + q^2 + \varepsilon^2)^{1/2} + (1 + p^2)^{1/2}(1 + \varepsilon^2)^{1/2}]}$$

$$< (1 + \varepsilon^2)^{-1/4}\frac{p}{(1 + p^2)^{1/4}}\left\{\max_q \frac{1}{[(1 + q^2 + \varepsilon^2)^{1/4} + (1 + p^2)^{1/4}]}\right\}\left\{\max_q \frac{p + q}{(1 + q^2)^{1/2} + (1 + p^2)^{1/2}}\right\}$$

$$\leq \frac{(1 + \varepsilon^2)^{-1/4}p}{(1 + p^2)^{1/2} + [(1 + \varepsilon^2)(1 + p^2)]^{1/4}}, \tag{A.19}$$

where the second quantity in braces has been bounded above by unity.

Clearly, also

$$\gamma_2(p, \varepsilon) = (1 + \varepsilon^2)^{-1/8}p^2$$

$$\times \max_q\{[(1 + q^2 + \varepsilon^2)(1 + p^2)]^{-1/8}$$

$$\times [(1 + q^2 + \varepsilon^2)^{1/4} + (1 + p^2)^{1/4}(1 + \varepsilon^2)^{1/4}]^{-1}$$

$$\times [(1 + q^2 + \varepsilon^2)^{1/2} + (1 + p^2)^{1/2}(1 + \varepsilon^2)^{1/2}]^{-1}\}$$

$$\leq \frac{(1 + \varepsilon^2)^{-1/4}p^2}{4(1 + \varepsilon^2)^{3/4}}$$

$$= (1 + \varepsilon^2)^{-1}p^2/4. \tag{A.20}$$

Inequalities (A.19) and (A.20) yield the desired expressions (3.36).

Appendix B. Proof of (3.4). If v is a harmonic function that vanishes on the lateral sides of R and vanishes at infinity, then it follows from Hopf's second principle that $p^2 = |\nabla v|^2$ must take its maximum value on the end $z = 0$. Furthermore, Payne and Philippin [17] have shown that if g is any other harmonic function in R, then the quantity $e^{ag}|\nabla v|^2$, for any real constant a, also takes its maximum value either at $x_1 = \infty$ or at some point on the boundary of R. We apply this result making the special choice $g = 2x_1$, that is, we define

$$\theta = e^{2ax_1}p^2, \qquad 0 < a < \frac{\pi}{h}. \tag{B.1}$$

We remark that since p^2 is $O(e^{-2\pi x_1/h})$ as $x_1 \to \infty$, for our choice of a we are assured that the maximum does not occur at infinity. It is also easily seen that the maximum cannot occur on $x_2 = 0$ or $x_2 = h$. To see this suppose that the maximum did occur at a point Q on $x_2 = h$, $x_1 > 0$. Then since $\theta \not\equiv 0$ in R, it follows from Hopf's second principle that

$$\frac{\partial \theta}{\partial x_2}(Q) > 0. \tag{B.2}$$

But, at each point on $x_2 = h$,

$$\frac{\partial \theta}{\partial x_2} = 2(v_{,1}v_{,12} + v_{,2}v_{,22})e^{2ax_1} = 2(v_{,1}v_{,12} - v_{,2}v_{,11})e^{2ax_1} = 0, \tag{B.3}$$

since both $v_{,1}$ and $v_{,11}$ vanish on $x_2 = h$. Thus, θ cannot take its maximum value on $x_2 = h$, $x_1 > 0$. A similar argument shows that the maximum value is not taken on $x_2 = 0$, $x_1 > 0$. Thus the maximum must occur on $x_1 = 0$. If the maximum occurs at $(0,0)$ or $(0,h)$, then the assumed continuity (recall that $v \in C^2(R) \cap C^1(\overline{R})$) assures us that

$$p^2 \leq \max\{[f'(0)]^2, [f'(h)]^2\}. \tag{B.4}$$

On the other hand, suppose that the maximum value of θ occurs at a point $Q_1 = (0, \hat{x}_2)$ for $0 < \hat{x}_2 < h$. Then at Q_1, we have (again by using Hopf's second principle)

$$v_{,1}v_{,11} + v_{,2}v_{,12} + ap^2 < 0, \tag{B.5}$$

$$v_{,1}v_{,12} + v_{,2}v_{,22} = 0. \tag{B.6}$$

By using the differential equation and the boundary condition (1.15) at $x_1 = 0$, we may rewrite these expressions as

$$-v_{,1}f'' + f'v_{,12} + ap^2 < 0, \tag{B.7}$$

$$v_{,1}v_{,12} + f'f'' = 0. \tag{B.8}$$

If $v_{,1}(Q_1) = 0$, then by (B.8) it follows that $f''(Q_1) = 0$ (since $f'(Q_1) \neq 0$, otherwise p would vanish at this point). This would again lead to

$$p^2 \leq [f'(Q_1)]^2. \tag{B.9}$$

However, if $v_{,1}(Q_1) \neq 0$, we may eliminate $v_{,12}$ between (B.7) and (B.8) to find that at Q_1

$$p^2 \{a - f''/v_{,1}\} < 0 \tag{B.10}$$

or

$$v^2_{,1}(Q_1) \leq [f''(Q_1)]^2/a^2. \tag{B.11}$$

Thus, we have shown that if the maximum occurs at any point Q_1 on $x_1 = 0$, then

$$p^2 \leq a^{-2}[f''(Q_1)]^2 + [f'(Q_1)]^2$$
$$\leq \max_{x_2 \in [0,h]} \{a^{-2}[f''(x_2)]^2 + [f'(x_2)]^2\}, \tag{B.12}$$

where the last inequality takes (B.4) into account. Taking the limit as $a \to \pi/h$, we arrive at (3.4). In our arguments we have used the obvious fact that if the maximum value of $p^2 e^{2ax_1}$ occurs at a point on $x_1 = 0$ then the computed bound for $p^2 e^{2ax_1}$ is also a bound for p^2.

It is perhaps worth mentioning the curious fact that if the portion of the boundary on which nonzero data are given were not the flat end $x_1 = 0$ but rather a curve that is convex outward to the left, e.g., a semicircular cap, then bounds for p^2 could be obtained by using the following result of Payne [18]:

$$p^2 \leq \max_s \left[\frac{f^2_{,ss}}{\kappa^2(s)} + f^2_{,s} \right], \tag{B.13}$$

where $f_{,s}$, $f_{,ss}$ are the first and second tangential derivatives of f on the convex curve, respectively, and $\kappa(s)$ is the curvature. For a flat end, however, $\kappa(s) = 0$, and the bound becomes meaningless.

Appendix C. Verification of (2.46). In the motivation for assumption (1.20) given in Sec. 1, it was pointed out that one way of looking at this hypothesis was to think of the first term on the right of (1.20) as a measure of the difference in $\rho(s)$, $\tilde{\rho}(s)$ $(s \geq 0)$ at different values of their arguments and the second term as a measure of the difference between $\rho(s)$ and $\tilde{\rho}(s)$ for the same value of their arguments. With this in mind, suppose that ρ and $\tilde{\rho}$ are such that

$$|\rho(s) - \tilde{\rho}(s)| \leq c_3 \varepsilon^2 \max[\rho(s), \tilde{\rho}(s)] \tag{C.1}$$

for some constant $c_3 > 0$. For a given ρ, $\tilde{\rho}$, it is desirable to choose c_3 as *small* as possible. (We recall from Sec. 1 a similar observation concerning the choices of m_α, M_α, $\tilde{m}_\alpha$, $\widetilde{M}_\alpha$.) It follows that if $\rho(s) \geq \tilde{\rho}(s)$, then

$$\tilde{\rho}(s) \leq \rho(s) \leq \tilde{\rho}(s)(1 - \varepsilon^2 c_3)^{-1}; \tag{C.2}$$

and if $\rho(s) \leq \tilde{\rho}(s)$, then

$$\rho(s) \leq \tilde{\rho}(s) \leq \rho(s)(1 - \varepsilon^2 c_3)^{-1}, \tag{C.3}$$

where we assume that $1 - \varepsilon^2 c_3 > 0$. Thus, in general

$$\rho(s) \leq \tilde{\rho}(s)(1 - \varepsilon^2 c_3)^{-1} \tag{C.4}$$

and

$$\tilde{\rho}(s) \le \rho(s)(1 - \varepsilon^2 c_3)^{-1}. \tag{C.5}$$

We look now at Case 1 and Case 2 separately.

Case 1. From (C.4), (1.16), and (C.5) we have

$$
\begin{aligned}
\rho(s) &\le \{\widetilde{M}_1 + \widetilde{K}_1 s \tilde{\rho}(s)\}(1 - \varepsilon^2 c_3)^{-1} \\
&\le \{\widetilde{M}_1 + \widetilde{K}_1 s[|\tilde{\rho}(s) - \rho(s)| + \rho(s)]\}(1 - \varepsilon^2 c_3)^{-1} \\
&\le \widetilde{M}_1(1 - \varepsilon^2 c_3)^{-1} + \widetilde{K}_1 s\{\varepsilon^2 c_3 \rho(s)(1 - \varepsilon^2 c_3)^{-1} + \rho(s)\}(1 - \varepsilon^2 c_3)^{-1} \\
&= \widetilde{M}_1(1 - \varepsilon^2 c_3)^{-1} + \widetilde{K}_1(1 - \varepsilon^2 c_3)^{-2} s \rho(s).
\end{aligned}
\tag{C.6}
$$

Similarly, making use of (C.5), (1.7), and (C.4) we have

$$\tilde{\rho}(s) \le M_1(1 - \varepsilon^2 c_3)^{-1} + K_1(1 - \varepsilon^2 c_3)^{-2} s \tilde{\rho}(s). \tag{C.7}$$

It is also easily seen from (C.4), (C.5) and (1.16), (1.17) that

$$\rho(s) \ge \tilde{m}_1(1 - \varepsilon^2 c_3), \tag{C.8}$$

$$\tilde{\rho}(s) \ge m_1(1 - \varepsilon^2 c_3). \tag{C.9}$$

Now clearly from (C.7), (C.9), and (1.16) it follows that we can take

$$\widetilde{M}_1 \le M_1(1 - \varepsilon^2 c_3)^{-1}, \qquad \tilde{m}_1 \ge m_1(1 - \varepsilon^2 c_3), \tag{C.10}$$

and thus we have

$$\tilde{\nu}_1 = \frac{\pi}{h} \frac{\tilde{m}_1}{\widetilde{M}_1} \ge \frac{\pi}{h} \frac{m_1}{M_1}(1 - \varepsilon^2 c_3)^2 > \nu_1 = \frac{\pi(1 - c_1)m_1}{2h(1 + c_1)M_1}, \tag{C.11}$$

provided

$$(1 - c_1) < 2(1 + c_1)(1 - \varepsilon^2 c_3)^2. \tag{C.12}$$

The inequality (C.12) clearly holds for sufficiently small ε.

Case 2. We now rewrite (C.4) and (C.5) as

$$[\tilde{\rho}(s)]^{-1} \le [\rho(s)(1 - \varepsilon^2 c_3)]^{-1}, \tag{C.13}$$

$$[\rho(s)]^{-1} \le [\tilde{\rho}(s)(1 - \varepsilon^2 c_3)]^{-1}. \tag{C.14}$$

Then, from (C.13), (1.8), and (C.14) we obtain

$$
\begin{aligned}
[\tilde{\rho}(s)]^{-1} &\le [M_2 + K_2 s \rho(s)](1 - \varepsilon^2 c_3)^{-1} \\
&\le M_2(1 - \varepsilon^2 c_3)^{-1} + K_2(1 - \varepsilon^2 c_3)^{-2} s \rho(s).
\end{aligned}
\tag{C.15}
$$

The last inequality in (C.15) follows as in the proof of (C.6). Similarly, from (C.14), (1.17), and (C.13) we obtain

$$[\rho(s)]^{-1} \le \widetilde{M}_2(1 - \varepsilon^2 c_3)^{-1} + \widetilde{K}_2(1 - \varepsilon^2 c_3)^{-2} s \tilde{\rho}(s), \tag{C.16}$$

and as before

$$[\rho(s)]^{-1} \ge \tilde{m}_2(1 - \varepsilon^2 c_3), \qquad [\tilde{\rho}(s)]^{-1} \ge m_2(1 - \varepsilon^2 c_3). \tag{C.17}$$

Thus, again we arrive at

$$\tilde{\nu}_2 \geq \frac{\pi}{h}\frac{m_2}{M_2}(1 - \varepsilon^2 c_3)^2 > \nu_2 \qquad (C.18)$$

provided (C.12) holds. We have thus shown that if ρ and $\tilde{\rho}$ satisfy (C.1) and ε is sufficiently small, then (2.46) will hold. We remark that, in the three examples discussed in Sec. 3, (C.1) is satisfied.

REFERENCES

[1] J. K. Knowles, *On finite anti-plane shear for incompressible elastic materials*, J. Austral. Math. Soc. Ser. B **19**, 400–415 (1976)

[2] J. K. Knowles, *A note on anti-plane shear for compressible materials in finite elastostatics*, J. Austral. Math. Soc. Ser. B **20**, 1–7 (1977)

[3] M. E. Gurtin, *Topics in finite elasticity*, NSF-CBMS Regional Conference Series in Appl. Math. vol. 35, SIAM, Philadelphia, 1981

[4] Q. Jiang and J. K. Knowles, *A class of compressible elastic materials capable of sustaining finite anti-plane shear*, J. Elasticity **25**, 193–201 (1991)

[5] C. O. Horgan and J. K. Knowles, *The effect of nonlinearity on a principle of Saint-Venant type*, J. Elasticity **11**, 271–291 (1981)

[6] C. O. Horgan and R. Abeyaratne, *Finite anti-plane shear of a semi-infinite strip subject to a self-equilibrated end traction*, Quart. Appl. Math. **40**, 407–417 (1983)

[7] C. O. Horgan and L. E. Payne, *Decay estimates for second-order quasilinear partial differential equations*, Adv. in Appl. Math. **5**, 309–332 (1984)

[8] C. O. Horgan and L. E. Payne, *On Saint-Venant's principle in finite anti-plane shear: An energy approach*, Arch. Rational Mech. Anal. **109**, 107–137 (1990)

[9] C. O. Horgan and J. K. Knowles, *Recent developments concerning Saint-Venant's principle*, Advances in Applied Mechanics (J. W. Hutchinson, ed.), vol. 23, Academic Press, New York, 1983, pp. 179–269

[10] C. O. Horgan, *Recent developments concerning Saint-Venant's principle: An update*, Appl. Mech. Rev. **42**, 295–303 (1989)

[11] C. O. Horgan and L. E. Payne, *A Saint-Venant principle for a theory of nonlinear plane elasticity*, Quart. Appl Math. **50**, 641–675 (1992)

[12] S. Breuer and J. J. Roseman, *Phragmén-Lindelöf decay theorems for classes of nonlinear Dirichlet problems in a circular cylinder*, J. Math. Anal. Appl. **113**, 59–77 (1986)

[13] C. O. Horgan and L. E. Payne, *Decay estimates for a class of nonlinear boundary value problems in two dimensions*, SIAM J. Math. Anal. **20**, 782–788 (1989)

[14] C. O. Horgan and L. E. Payne, *On the asymptotic behavior of solutions of inhomogeneous second-order quasilinear partial differential equations*, Quart. Appl. Math. **47**, 753–771 (1989)

[15] C. O. Horgan and L. E. Payne, *Exponential decay estimates for capillary surfaces and extensible films*, Stability Appl. Anal. Contin. Media **1**, 261–282 (1991)

[16] D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970

[17] L. E. Payne and G. A. Phillipin, *On some maximum principles involving harmonic functions and their derivatives*, SIAM J. Math. Anal. **10**, 96–104 (1979)

[18] L. E. Payne, *Isoperimetric inequalities, maximum principles and their applications*, Report of lectures given at the University, Newcastle-upon-Tyne (1972)