

# THE EFFECT OF CONVOLVING FAMILIES OF $L$ -FUNCTIONS ON THE UNDERLYING GROUP SYMMETRIES

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ABSTRACT. Let  $\{\mathcal{F}_N\}$  and  $\{\mathcal{G}_M\}$  be families of primitive automorphic  $L$ -functions for  $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$  and  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ , respectively, such that, as  $N, M \rightarrow \infty$ , the statistical behavior (1-level density) of the low-lying zeros of  $L$ -functions in  $\mathcal{F}_N$  (resp.,  $\mathcal{G}_M$ ) agrees with that of the eigenvalues near 1 of matrices in  $G_1$  (resp.,  $G_2$ ) as the size of the matrices tend to infinity, where each  $G_i$  is one of the classical compact groups (unitary  $U$ , symplectic  $\mathrm{Sp}$ , or orthogonal  $O, \mathrm{SO}(\text{even}), \mathrm{SO}(\text{odd})$ ). Assuming that the convolved families of  $L$ -functions  $\mathcal{F}_N \times \mathcal{G}_M$  are automorphic, we study their 1-level density. (We also study convolved families of the form  $f \times \mathcal{G}_M$  for a fixed  $f$ .) Under natural assumptions on the families (which hold in many cases) we can associate to each family  $\mathcal{L}$  of  $L$ -functions a symmetry constant  $c_{\mathcal{L}}$  equal to 0 (resp., 1 or  $-1$ ) if the corresponding low-lying zero statistics agree with those of the unitary (resp., symplectic or orthogonal) group. Our main result is that  $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$ : the symmetry type of the convolved family is the product of the symmetry types of the two families. A similar statement holds for the convolved families  $f \times \mathcal{G}_M$ . We provide examples built from Dirichlet  $L$ -functions and holomorphic modular forms and their symmetric powers. An interesting special case is to convolve two families of elliptic curves with positive rank. In this case the symmetry group of the convolution is independent of the ranks, in accordance with the general principle of multiplicativity of the symmetry constants (but the ranks persist, before taking the limit  $N, M \rightarrow \infty$ , as lower-order terms).

## 1. INTRODUCTION

### 1.1. Preliminaries.

Many questions in number theory, such as the study of the density of the primes or properties of class numbers, can be related to understanding the distribution of zeros of  $L$ -functions. In the early 1970's Dyson and Montgomery [Mon] discovered the agreement between the pair correlation of zeros of the Riemann zeta-function  $\zeta(s)$  and of eigenvalues of matrices in the

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Gaussian Unitary Ensemble (GUE). Two decades later Katz and Sarnak [KaSa1, KaSa2] offered deeper insight into the connection between zero and eigenvalue statistics by studying *families* of  $L$ -functions. Ever since, random matrix theory [CFKRS, KaSa2, KeSn, ILS] has enjoyed remarkable success at modeling and predicting the behavior of  $L$ -functions.

Various pairs of eigenvalue/zero statistics can be shown, or at least are conjectured, to be in perfect agreement. Among the early such statistics studied were  $n$ -level correlations and nearest-neighbor spacings [Hej, Mon, Od1, Od2, RS]. These statistics pertain to the whole (infinite) sequence of critical zeros of a *single*  $L$ -function, and are shown to agree with the corresponding statistic of  $N \times N$  GUE matrices in the limit as  $N \rightarrow \infty$ . Neither of these statistics reveals anything about the behavior of low-lying critical zeros of  $L$ -functions; that is, of zeros near the arithmetically-crucial central point. The reason is that those statistics are defined by averaging quantities defined using a large (but finite) subset of the zeros, most of which will lie high up on the critical line —and thus the behavior of those few zeros that lie near the central point is irrelevant in the limit as the number of zeros used to compute the statistic tends to infinity.

The Katz-Sarnak philosophy has shifted the emphasis to the study of families of  $L$ -functions and their low-lying zeros, whose statistics (upon averaging over the family) are well modeled by the statistics of eigenvalues close to 1 of random matrices from the classical compact groups. In the function field case these classical group statistics are explained by the monodromy group of the family. For families of automorphic  $L$ -functions of number fields the connection is quite a bit more mysterious. Usually, the corresponding classical compact group is identified only by explicitly computing zero statistics. Our goal in this paper is to allow predicting the group attached to a “convolved” family assuming only knowledge of the groups describing the zero statistics of the two families being convolved. The relation turns out to be very simple to describe and it will hopefully shed some light into the properties of the (conjectural) correspondence between families of (number-field) automorphic  $L$ -functions and classical groups.

We first describe the main statistic studied in this paper. In order to break away from the universal global GUE statistics of the zeros of a single  $L$ -function, and to understand the neighborhood of the central point, we study the  $n$ -level density in a family of  $L$ -functions; the latter is a local statistic involving only critical zeros near the central point. Let  $\mathcal{F} = \cup \mathcal{F}_N$  be a family of  $L$ -functions ordered by their conductors (for example,  $\mathcal{F}_N$  might be Dirichlet  $L$ -functions with conductor  $N$  or cuspidal newforms of weight 2 and level  $N$ ) and write the zeros of  $L(s, f)$  as  $1/2 + i\gamma_{j;f}$ ; assuming the General Riemann Hypothesis (GRH), each  $\gamma_{j;f} \in \mathbb{R}$ . Given an  $n$ -variable test function  $\phi(t_1, \dots, t_n) = \phi_1(t_1) \cdots \phi_n(t_n)$  (where each  $\phi_k$  is a Schwartz function on  $\mathbb{R}$ ), the  $n$ -level density of  $\mathcal{F}$  is (by a slight abuse of language) the measure on  $\mathbb{R}^n$  with respect to which the integral of  $\phi$  is

$$D_{n,\mathcal{F}}(\phi) = \lim_{N \rightarrow \infty} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \phi_1 \left( \frac{\log Q_f}{2\pi} \gamma_{j_1;f} \right) \cdots \phi_n \left( \frac{\log Q_f}{2\pi} \gamma_{j_n;f} \right), \quad (1.1)$$

provided the limit exists, where  $Q_f$  is the analytic conductor of  $L(s, f)$ .<sup>1</sup> For many families of  $L$ -functions [DM, FI, Gü, HR, HM, ILS, KaSa2, Mil2, Ro, Rub, Yo2] (and, conjecturally at least, for any natural such family, in accordance with the Katz-Sarnak philosophy), the  $n$ -level density coincides with that of the normalized eigenvalues near 1 of matrices in one of the infinite families of classical compact Lie groups, in the limit as the size  $N$  of the matrix goes to infinity. In the context of matrices from classical Lie groups, the averaging over  $\mathcal{F}_N$  in equation (1.1) is replaced by averaging over the whole group with respect to its natural (Haar) probability measure —hence the terminology of “random matrices”. The  $n$ -level densities for different classical compact groups are distinct —it is this feature that allows “breaking” the universal GUE behavior observed when considering global statistics such as  $n$ -level correlations or neighbor spacings. This one-to-one correspondence between (infinite families of) classical groups and their  $n$ -level densities allows, at least conjecturally, to assign a definite “symmetry type” to each family of primitive  $L$ -functions. For families of zeta or  $L$ -functions of curves or varieties over finite fields, the corresponding classical compact group is determined by the monodromy group of the family [KaSa1]. However, for families of number-field automorphic  $L$ -functions there is no such thing as a monodromy group and the underlying symmetry only manifests itself (in our current understanding) through the zero statistics (although function field analogues of number-field families often suggest what the symmetry type should be). Our goal in this paper is to determine the symmetry group of the convolution of two families of number-field automorphic  $L$ -functions in terms of the symmetry groups of the families being convolved together.

For families where the signs of the functional equations are all even and there is not an obvious corresponding family with odd signs, a folklore conjecture (see for example page 2877 of [KeSn]) stated that the symmetry group should be symplectic. This was based on the observation that  $\mathrm{SO}(\text{even/odd})$  symmetries in all known examples arose from splitting orthogonal families according to the sign of the functional equation, while symplectic symmetries arose from a family with all even sign and no corresponding family with odd signs. *A priori* the symmetry type of a family with all functional equations even is either symplectic or  $\mathrm{SO}(\text{even})$ . In [DM] we studied the family  $\{L(s, \phi \times \mathrm{sym}^2 f)\}$ , where  $\phi$  is a fixed even Hecke-Maass eigenform on the modular group and  $f$  ranges over weight- $k$  full-level Hecke cusp forms; see [LS] for applications of this family. All  $L(s, \phi \times \mathrm{sym}^2 f)$  have even sign, and this family does not arise from splitting sign within an orthogonal family. In [DM] it is shown (via 1- and 2-level densities) that the symmetry type agrees only with  $\mathrm{SO}(\text{even})$ , thus disproving the folklore conjecture mentioned above.

As a consequence of the counterexample to the folklore conjecture, the theory of low-lying zeros is more than just a theory of the signs of functional equations. By analyzing Rankin-Selberg convolutions of  $\mathrm{GL}_2$   $L$ -functions (and some of their lifts), we are led to attaching a

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<sup>1</sup>It is a simple consequence of the Riemann-von Mangoldt zero-counting formula that the density of the zeros near the central point  $s = 1/2$  is roughly  $(\log Q_f)/2\pi$ , so the rescaled (“normalized”) imaginary parts  $\gamma_{j_n;f} \cdot (\log Q_f)/2\pi$  have uniform (constant) density 1 in the large-conductor limit. Thus, for fixed  $A < B$ , each  $L(s, f)$  has roughly  $B - A$  normalized zeros with imaginary parts in  $[A, B]$ . Also, critical zeros not near  $s = 1/2$  (on a scale of  $(\log Q_f)/2\pi$ ) are “rescaled away to infinity” in the large-conductor limit.

symmetry constant  $c_{\mathcal{F}}$  to each family  $\mathcal{F}$  of  $L$ -functions. This constant depends only on the second moment (i. e., the average over the family) of the Satake parameters at each unramified prime. In all the cases investigated, the average is 0 (resp., 1 or  $-1$ ) if the family has unitary (resp., symplectic or orthogonal) symmetry. We are ready to set some notation and describe our main result, namely that in many cases the symmetry constant of the convolution of two families is the product of their symmetry constants.

## 1.2. $n$ -Level Densities and NT-good Families.

We list four desirable properties for a family of primitive  $L$ -functions to have; we call a family satisfying these properties NT-good. These properties are inspired by the families that have been successfully investigated to date, and codify the conditions for which we can calculate the 1-level (and sometimes even the  $n$ -level) densities for a family of  $L$ -functions. Though we could replace some of the bounds with slightly weaker conditions, these are the conditions that are met in practice.

**Definition 1.1** (NT-good). *Let  $\phi$  be an even Schwartz test function such that  $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$  for some  $\sigma > 0$ . A family  $\mathcal{F}$  of primitive automorphic  $L$ -functions for  $\text{GL}_n(\mathbb{A}_{\mathbb{Q}})$  is NT-good with symmetry constant  $c_{\mathcal{F}}$  if  $\mathcal{F}$  is a disjoint union of finite sets  $\mathcal{F}_N \subset \mathcal{F}$  such that, as  $N \rightarrow \infty$ :*

(1) Cardinality:

(i) Bounded multiplicities: *Members of a family can occur multiple times, say  $f \in \mathcal{F}_N$  occurs  $\mu_f$  times; however, we assume the multiplicities are bounded by a universal constant, independent of  $N$ :  $\mu_f \leq \mu_{\mathcal{F}}$ .*

(ii) Size of the family:  $|\mathcal{F}_N| \rightarrow \infty$ , *where each member is counted with its multiplicity:  $|\mathcal{F}_N| = \sum_{f \in \mathcal{F}_N} \mu_f$ .*

(2) Conductors: *The analytic log-conductors of  $f \in \mathcal{F}_N$  are essentially constant: say,  $\log Q_f = \log R_N + o(\log R_N)$  for all  $f \in \mathcal{F}_N$  and some sequence  $\{R_N\}$ . Further, there exists  $\delta_0, \delta'_0 > 0$  such that  $|\mathcal{F}_N|^{\delta_0} \ll R_N \ll |\mathcal{F}_N|^{\delta'_0}$ . (In particular,  $R_N \rightarrow \infty$  as  $N \rightarrow \infty$ .)*

(3) Sums over primes and squares of primes:<sup>2</sup>

(i) Prime sums: *For some  $r_{\mathcal{F}} \geq 0$ ,*

$$-2 \sum_p \frac{1}{\sqrt{p}} \frac{\log p}{\log R_N} \widehat{\phi} \left( \frac{\log p}{\log R_N} \right) \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} b_f(p) = r_{\mathcal{F}} \phi(0) + o(1); \quad (1.2)$$

<sup>2</sup>The numbers  $b_f(n)$  are the Dirichlet coefficients of the logarithmic derivative:  $L'(s, f)/L(s, f) = \sum_{n=1}^{\infty} b_f(n)/n^s$ , cf., Definition 2.1 below.

we call  $r_{\mathcal{F}}$  the rank of the family. Often (1.2) is satisfied because  $\exists \delta_1 > 0, \mu_1, r_{\mathcal{F}} \geq 0$  such that

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} b_f(p) = -\frac{r_{\mathcal{F}}}{\sqrt{p}} + O(|\mathcal{F}_N|^{-\delta_1} p^{\mu_1}). \quad (1.3)$$

It also suffices for this to hold for almost all primes, provided the contribution from the bad primes is negligible.

(ii) Prime-square sums: For some  $c_{\mathcal{F}} \in \{-1, 0, 1\}$ ,

$$-2 \sum_p \frac{1}{p} \frac{\log p}{\log R_N} \widehat{\phi} \left( 2 \frac{\log p}{\log R_N} \right) \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} b_f(p^2) = -c_{\mathcal{F}} \frac{\phi(0)}{2} + o(1). \quad (1.4)$$

Often (1.4) is satisfied because  $\exists \delta_2 > 0, \mu_2 \geq 0, c_{\mathcal{F}} \in \{-1, 0, 1\}$  such that

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} b_f(p^2) = c_{\mathcal{F}} + O(|\mathcal{F}_N|^{-\delta_2} p^{\mu_2}). \quad (1.5)$$

We call  $c_{\mathcal{F}}$  the symmetry constant of the family.

(4) Error terms: We have

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_p \sum_{\nu=3}^{\infty} \frac{b_f(p^\nu)}{p^{\nu/2}} \frac{\log p}{\log R_N} \widehat{\phi} \left( \nu \frac{\log p}{\log R_N} \right) = o(1). \quad (1.6)$$

Many estimates on  $b_f(p^\nu)$  imply (1.6); we give three natural ones:

- (i) Ramanujan Conjecture: For all  $p$ ,  $|\alpha_{f,j}(p)| \leq 1$  implying  $b_f(p^\nu) = O(1)$ .
- (ii)  $\exists \mu_3(\nu) > 0$  with  $\lim_{\nu \rightarrow \infty} \mu_3(\nu) = \mu_3 > 0$  such that

$$b_f(p^\nu) \ll \frac{p^{\nu/2}}{p^{1+\mu_3(\nu)\nu}}. \quad (1.7)$$

(iii)  $\exists \delta_3 > 0, \mu_3 \geq 0$  such that

$$\sum_{f \in \mathcal{F}_N} b_f(p^\nu) \ll |\mathcal{F}_N|^{1-\delta_3} p^{\mu_3}. \quad (1.8)$$

Condition (1) ensures we have enough  $L$ -functions for averaging, and when we convolve two families, it is a needed ingredient in controlling the contribution of imprimitive  $L$ -functions.<sup>3</sup> Condition (2) allows us to handle the conductors, and ensures that the number of  $L$ -functions is at least a power of the analytic conductor; this is often needed in averaging to show certain terms are small. Condition (4) allows us to ignore the contributions from  $\nu \geq 3$  in the explicit formula (2.10). The point of condition (4.ii) is that eventually (for  $\nu$  large) we have  $b(p^\nu)/p^{\nu/2} \ll 1/p^{1+\mu_3\nu}$ , and this will be summable over  $\nu$  and  $p$  (the small  $\nu \geq 3$  terms

<sup>3</sup>We could weaken our assumptions and allow  $\mu_f \leq |\mathcal{F}_N|^{1-\epsilon}$  for some  $\epsilon > 0$ . In the applications we have in mind, all multiplicities are bounded, so for ease of exposition we assume the multiplicities are bounded. We comment on this further in Corollary 1.7.

can be handled individually by our assumptions); an alternate bound where the cancelation comes not from each individual  $L(s, f)$  but rather from averaging over the family is given by condition (4.iii).

Condition (3) is the interesting one, especially (3.ii). It is here that we see family-dependent behavior. In order to use the Explicit Formula (2.10) successfully, we need to be able to determine family averages of  $b_f(p)$  and  $b_f(p^2)$ . Condition (3) holds in all the families of  $L$ -functions studied to date [DM, FI, Gao, Gü, HM, HR, ILS, Mil1, Mil2, RR, Ro, Rub, Yo2]; further  $r_{\mathcal{F}}$  is zero except for families of elliptic curves with positive rank. The main term of the family averages of  $b_f(p^2)$  do not depend on  $r_{\mathcal{F}}$ , which surfaces only in the averages of  $b_f(p)$ . Condition (3) holds in the form (1.5) in all families studied to date except for one-parameter families of elliptic curves with constant  $j$ -invariant. (Michel [Mic] proved that (1.5) holds for one-parameter families of elliptic curves with non-constant  $j$ -invariant. If the  $j$ -invariant is constant one can often show by direct computation that either (1.4) holds or (1.5) holds on average; see [Mil1, Mil2].)

We conclude this subsection by listing some NT-good families with *constant* analytic conductors in each  $\mathcal{F}_N$ .

#### Unitary

- $\{L(s, \chi) : \chi \text{ a non-trivial Dirichlet character of prime conductor } m\}, m \rightarrow \infty$  (see [HR]).

#### Symplectic

- $\{L(s, \chi_d) : d \text{ ranges over subsets of fundamental discriminants in } [N, 2N]\}, N \rightarrow \infty$  (see [Gao, HR, Mil5, Rub]).
- $\{L(s, \text{sym}^r f) : r \text{ even and } f \text{ ranges over weight-}k \text{ full-level cusp forms}\}, k \rightarrow \infty$  (see [Gü, ILS]).
- $\{L(s, \phi \times f) : \phi \text{ a fixed Maass form and } f \text{ ranges over weight-}k \text{ full-level cusp forms}\}, k \rightarrow \infty$  (see [DM]).
- $\{L(s, \psi) : \psi \text{ a character of the ideal class group of the imaginary quadratic field } \mathbb{Q}(\sqrt{-D}) \text{ with } D > 3 \text{ square-free and congruent to } 3 \text{ modulo } 4\}$  (see [FI]).

#### Orthogonal

- $\{L(s, f) : f \text{ ranges over weight-}k \text{ level-}N \text{ cuspidal newforms with } k, N \text{ or both tending to infinity}\};$  if we split by sign of the functional equations we get SO(even) or SO(odd) symmetry ([ILS, Mil7, RR, Ro] for the 1-level and [HM] for the  $n$ -level density).
- $\{L(s, \phi \times \text{sym}^2 f) : \phi \text{ a fixed Maass form and } f \text{ ranges over weight-}k \text{ full-level cusp forms}\}, k \rightarrow \infty$  (see [DM]).
- $\{L(s, \text{sym}^r f) : r \text{ odd and } f \text{ ranges over weight-}k \text{ full level cusp forms}\},$  with O symmetry if  $r \equiv 1, 5 \pmod{8}$ , SO(even) symmetry for  $r \equiv 7 \pmod{8}$  and SO(odd) symmetry for  $r \equiv 3 \pmod{8}$ ,  $k \rightarrow \infty$  (see [Gü]).

With some more work, families with monotone increasing conductors can be handled. This allows us to add an entry to each list. For unitary families we may consider non-primitive Dirichlet characters with square-free conductor (see [Mil8]). For symplectic we may consider primitive quadratic Dirichlet characters (see [Rub]). For orthogonal families we may consider one-parameter (see [Mil2]) or two-parameter (see [Yo2]) families of elliptic curves. A generic one-parameter family should have rank 0 and equidistribution of signs of functional equations, giving O symmetry; however there are numerous families with positive rank, as well as constant sign families (see [Mil2] for exact statements and details).

### 1.3. Main Results.

*We adopt the following convention throughout this paper: If  $\mathcal{F}$  and  $\mathcal{G}$  are two families of unitary automorphic cuspidal representations of  $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$  and  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$  with trivial central character, then by  $\mathcal{F} \times \mathcal{G}$  we mean the set of all the (conjectural) Rankin-Selberg automorphic representations  $f \times g$  of  $\mathrm{GL}_{mn}(\mathbb{A}_{\mathbb{Q}})$ , where  $f \in \mathcal{F}$ ,  $g \in \mathcal{G}$ . (Here the  $f \times g$  are counted with multiplicity  $\mu_f \mu_g$ .) For every purpose in this paper, this is equivalent to considering  $\mathcal{F}$  and  $\mathcal{G}$  to be families of automorphic  $L$ -functions and  $\mathcal{F} \times \mathcal{G}$  consists of the Rankin-Selberg convolution  $L$ -functions  $L(s, f \times g)$ .*

We occasionally remind the reader of this convention. We need some control over the number of pairs  $(f, g)$  where  $g$  is the contragredient of  $f$ . In this case the convolved  $L$ -function  $L(f \times g, s)$  is imprimitive, and thus its zeros are the superposition of the zeros of at least two primitive  $L$ -functions. In most cases the number and contribution of these imprimitive  $L$ -functions to the 1-level density is negligible.

**Definition 1.2** (Symmetry constant, family constant). *We denote the symmetry constant of the family  $\mathcal{F}$  by  $c_{\mathcal{F}}$ . It equals 0 (resp., 1 or  $-1$ ) if the 1-level density of the family agrees with unitary (resp., symplectic or any of the three orthogonal groups: O, SO(even) or SO(odd)). As the three orthogonal groups all have  $c_{\mathcal{F}} = 1$ , to distinguish them we set  $\epsilon_{\mathcal{F}}$  equal to 0 (resp., 1 or  $-1$ ) if  $\mathcal{F}$  has half of the signs of its functional equation even (resp., all signs even or odd); if  $\mathcal{F}$  is not associated to an orthogonal group, then  $c_{\mathcal{F}}$  alone determines the group and we simply put  $\epsilon_{\mathcal{F}} = 0$ . Finally,  $r_{\mathcal{F}}$  denotes the rank of  $\mathcal{F}$ ; except for families of elliptic curves, all other known families have  $r_{\mathcal{F}} = 0$ . We call  $\tilde{c}_{\mathcal{F}} = (c_{\mathcal{F}}, \epsilon_{\mathcal{F}}, r_{\mathcal{F}})$  the family constant of  $\mathcal{F}$ .*

**Theorem 1.3.** *Let  $\mathcal{F}$  and  $\mathcal{G}$  be NT-good families of unitary automorphic cuspidal representations of  $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$  and  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$  with trivial central character, with symmetry constants  $c_{\mathcal{F}}$  and  $c_{\mathcal{G}}$ . Assume  $\mathcal{F} \times \mathcal{G}$  is an NT-good family. Then the family  $\mathcal{F} \times \mathcal{G}$  (which is the limit of  $\mathcal{F}_N \times \mathcal{G}_M$ , where  $N$  and  $M$  tend to infinity together) has symmetry constant*

$$c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}. \tag{1.9}$$

*If the family constants are  $\tilde{c}_{\mathcal{F}} = (c_{\mathcal{F}}, \epsilon_{\mathcal{F}}, r_{\mathcal{F}})$  and  $\tilde{c}_{\mathcal{G}} = (c_{\mathcal{G}}, \epsilon_{\mathcal{G}}, r_{\mathcal{G}})$  then the new family constant is  $\tilde{c}_{\mathcal{F} \times \mathcal{G}} = (c_{\mathcal{F}} \cdot c_{\mathcal{G}}, \epsilon_{\mathcal{F} \times \mathcal{G}}, 0)$ .*

**Remark 1.4.** *Note that the ranks of the two families do not enter in the determination of the classical compact group associated to  $\mathcal{F} \times \mathcal{G}$ ; the new family has rank 0 (in the sense of Definition 1.1). Determining the distribution of signs of the functional equations of  $\mathcal{F} \times \mathcal{G}$  is often an involved calculation depending on fine properties of the two families; however, it is not needed if we merely wish to classify the symmetry as unitary, symplectic or (non-specific) orthogonal.*

**Remark 1.5.** *It is worthwhile to emphasize the meaning of the above theorem. Our notion of a symmetry constant arises from 1-level density expansions, though we expect to see the same correspondence for any statistic ( $n$ -level correlations, central values, moments, ...). Thus, an alternate way to phrase our results is that the 1-level density of the convolution (as the conductors tend to infinity) agrees with the scaling limit of either unitary, symplectic or orthogonal matrices (depending on the value of the constant).*

**Remark 1.6.** *In the proof of Theorem 1.3, our goal is to calculate the  $n$ -level densities for sufficiently large support to uniquely determine the corresponding symmetry group. A more involved argument could increase the support in some of our examples, but not far enough to see new features; we therefore content ourselves with giving the more general argument. We describe two examples in great detail which illustrate the technicalities that must be surmounted to prove the convolved families are NT-good; the first is symmetric powers of modular forms in §6, and the second is families of elliptic curves in §7.*

**Corollary 1.7.** *The results of Theorem 1.3 still hold if we weaken the Bounded Multiplicities condition (1.i) in Definition 1.1: Instead of assuming  $\mu_f$  and  $\mu_g$  are uniformly bounded, it suffices to assume that*

$$\#\{(f, g) : f \in \mathcal{F}_N, g \in \mathcal{G}_M, f = \tilde{g}\} = O(|\mathcal{F}_N|^{1-\delta}|\mathcal{G}_M| + |\mathcal{F}_N||\mathcal{G}_M|^{1-\delta}) \quad (1.10)$$

for some  $\delta > 0$  (in other words, that there is a power savings in the number of pairs where a  $g$  is the contragredient of an  $f$  —these lead to imprimitive  $L(s, f \times g)$ ).

Instead of convolving  $\mathcal{F}$  and  $\mathcal{G}$ , we can instead fix an  $f \in \mathcal{F}$  and consider the family  $f \times \mathcal{G}$  obtained by taking the limit as  $M \rightarrow \infty$  of  $f \times \mathcal{G}_M$ .

**Theorem 1.8.** *Assume  $\mathcal{G}$  and  $f \times \mathcal{G}$  are NT-good and that  $\mathcal{G}$  satisfies (1.3) and (1.5). The symmetry type of  $f \times \mathcal{G}$  is controlled by the following two pieces of input:  $c_{\mathcal{G}}$  and  $b_f(p^2)$ . If  $f$  is a Dirichlet character, holomorphic cusp form or Maass form then we may associate a symmetry constant  $c_f$  to  $f$  such that  $c_{f \times \mathcal{G}} = c_f \cdot c_{\mathcal{G}}$ . In particular, we have*

- (1) *if  $f$  is a quadratic Dirichlet character then  $f \times \mathcal{G}$  has the same symmetry as  $\mathcal{G}$ , and if  $f$  is a non-quadratic Dirichlet character then  $f \times \mathcal{G}$  has unitary symmetry;*
- (2) *if  $\mathcal{G}$  has unitary (resp., symplectic, orthogonal) symmetry, then  $f \times \mathcal{G}$  has unitary (resp., orthogonal, symplectic) symmetry if  $f$  is a Hecke holomorphic or Maass form.*

**Remark 1.9.** *If instead (1.2) and (1.4) hold then the result is probably still true (it can be shown in special cases by partial summation); in general, though, more detailed knowledge about sums of the coefficients of  $L(s, f)$  will be needed.*



**Remark 1.10.** *The universality in Theorems 1.3 and 1.8 is reminiscent of that found by Rudnick and Sarnak [RS], where the universality in the  $n$ -level correlations of primitive automorphic cuspidal  $L$ -functions is related to universality in the second moments of the Fourier coefficients  $a_\pi(p)$ .*

An especially interesting case is when at least one of the two families is a one-parameter family of elliptic curves over  $\mathbb{Q}(T)$  with positive rank  $r_{\mathcal{F}}$ . Miller [Mil2] showed that the 1- and 2-level densities of zeros of these families agree with those of *subgroups* of the orthogonal group (in many cases unconditionally, in other cases assuming standard conjectures; see [Yo2] for similar results involving special two-parameter families). As the conductors tend to infinity, the random matrix ensemble modeling this situation (as  $N \rightarrow \infty$ ) is

$$\left\{ \left( \begin{array}{c} I_{r_{\mathcal{F}} \times r_{\mathcal{F}}} \\ g \end{array} \right), g \in \mathcal{C} \right\}, \quad (1.11)$$

where  $I_{r_{\mathcal{F}} \times r_{\mathcal{F}}}$  is the  $r_{\mathcal{F}} \times r_{\mathcal{F}}$  identity matrix and  $\mathcal{C}$  is  $O(N)$  (resp.,  $SO(2N)$  or  $SO(2N + 1)$ ) if half the signs of the functional equation are even (resp., all or none); the correct model is not known for finite conductors (but see [Mil4] for numerical investigations of zeros near the central point). Indeed, by Silverman's specialization theorem and the Birch and Swinnerton-Dyer conjecture, for all  $t$  sufficiently large each elliptic curve has at least  $r_{\mathcal{F}}$  zeros at the central point; moreover, the ensemble (1.11) models these zeros as independent from the remaining others. This independence is in agreement with function-field analogues. We shall see in Theorem 7.3 that if one convolves two families of elliptic curves with positive rank then, to first order, one does not see any effects of this rank in the symmetry group of the new family! What this means is that the rank parameter  $r$  of the new family is zero as defined by Condition (3) of Definition 1.1. The ranks of the convolved families appear *only* as a lower-order correction term that is unfortunately difficult to isolate since it is smaller than the bounds we can prove for the error terms (though, conjecturally, it is larger than the actual bounds for these terms and should, in principle, be detectable). In this regard our results are similar to Goldfeld's [Go]; he considered twists of a fixed elliptic curve by quadratic Dirichlet characters and conjectured that the new family's rank is independent of the rank of the fixed elliptic curve.

In §2 and §3 we review the needed results from number theory and random matrix theory. We discuss the properties and consequences of being an NT-good family of  $L$ -functions in §4 and then in §5 prove Theorems 1.3 and 1.8. We then give examples of families where these conditions are met: Convolving families of holomorphic cusp forms in Example 5.4, symmetric powers of holomorphic cusp forms in §6, and one-parameter families of elliptic curves in §7; these examples are all independent of each other and may be read in any order.

## 2. NUMBER THEORY REVIEW

We quickly review the notion of automorphic  $L$ -functions. These are the  $L$ -functions attached to automorphic representations of  $GL_n(\mathbb{A}_{\mathbb{Q}})$ . Our examples are built out of objects in

$\mathrm{GL}_1$  (Dirichlet  $L$ -functions) and  $\mathrm{GL}_2$  (Maass forms and holomorphic modular forms, including those attached to elliptic curves). We build more complicated  $L$ -functions by taking Rankin-Selberg convolutions and other natural functorial operations (e. g., forming the symmetric square  $L$ -functions). These constructions take us beyond  $\mathrm{GL}_2$ . It is impossible to cover here but the barest facts about automorphic  $L$ -functions; see [Bor, Jac, JPS, RS] for more details. We will focus on primitive  $L$ -functions; these are attached to cuspidal representations and cannot be further factored as products of other  $L$ -functions, hence their critical zeros form an irreducible set in this sense.

Let  $\pi = \hat{\otimes}_v \pi_v$  be a unitary irreducible cuspidal automorphic representation of  $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$  with trivial central character. Here  $v$  is either a prime  $p$  or  $\infty$ , and each  $\pi_v$  is an irreducible admissible representation of  $\mathbb{Q}_v$  (where  $\mathbb{Q}_{\infty} := \mathbb{R}$ ). The finite part of the  $L$ -function attached to  $\pi$  is an Euler product

$$L(s, \pi) = \prod_p L(s, \pi_p). \quad (2.1)$$

Outside a finite set of primes,  $\pi_p$  is unramified and

$$L(s, \pi_p) = \det(I - p^{-s} A_{\pi}(p))^{-1} = \prod_{j=1}^n (1 - \alpha_{\pi,j}(p) p^{-s}), \quad (2.2)$$

where  $\{A_{\pi}(p)\} \in \mathrm{GL}_n(\mathbb{C})$  is a semi-simple conjugacy class parametrized by the eigenvalues  $\alpha_{\pi,j}(p)$ . The Satake correspondence is the bijection  $A_{\pi}(p) \leftrightarrow \pi_p$  between semi-simple conjugacy classes in  $\mathrm{GL}_n(\mathbb{C})$  and unramified irreducible admissible representations of  $\mathrm{GL}_n(\mathbb{Q}_p)$ .

The complex numbers  $\{\alpha_{\pi,j}(p)\}_{j=1}^n$  are called the Satake parameters of  $\pi_p$ . In the context at hand, the generalized Ramanujan conjecture is the statement that  $|\alpha_{\pi,j}| = 1$  at the unramified places; at a ramified prime  $p$  some of the  $\alpha_{\pi,j}(p)$  may vanish.

**Definition 2.1.** For  $\pi$  an automorphic representation,  $p$  a prime and  $\pi_p$  with Satake parameters  $\alpha_{\pi,1}(p), \dots, \alpha_{\pi,n}(p)$ , we define, for  $\nu = 1, 2, 3, \dots$ ,

$$b_{\pi}(p^{\nu}) := \alpha_{\pi,1}(p)^{\nu} + \dots + \alpha_{\pi,n}(p)^{\nu}. \quad (2.3)$$

With this definition one has  $b_{\pi}(p^{\nu}) = \mathrm{Trace}(A_{\pi}(p)^{\nu})$  for unramified  $p$ , and

$$\frac{L'(s, \pi)}{L(s, \pi)} = \sum_p \sum_{\nu=1}^{\infty} \frac{b_{\pi}(p^{\nu})}{p^{\nu s}}. \quad (2.4)$$

The archimedean  $L$ -factor associated to  $\pi_{\infty}$  is of the form

$$L(s, \pi_{\infty}) = \prod_{j=1}^n \Gamma_{\mathbb{R}}(s + \mu_{\pi,j}), \quad \text{where } \Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right). \quad (2.5)$$

The numbers  $\{\mu_{\pi,j}\}_{j=1}^n$  are analogs of the Satake parameters, and the analog of the Ramanujan conjecture is in this case Selberg's (generalized) eigenvalue conjecture, namely the statement that the  $\mu_{\pi,j}$  have non-negative real part.

We define the completed  $L$ -function by

$$\Lambda(s, \pi) := N_\pi^{s/2} L(s, \pi_\infty) L(s, \pi), \quad (2.6)$$

where  $N_\pi$  is a positive integer called the arithmetic conductor. We have the functional equation

$$\Lambda(s, \pi) = \epsilon(\pi) \Lambda(1 - s, \tilde{\pi}), \quad (2.7)$$

where  $\tilde{\pi}$  is the contragredient of  $\pi$  and  $\epsilon(\pi)$  is a complex constant such that  $|\epsilon(\pi)| = 1$ . In the self-dual case, when  $\pi \simeq \tilde{\pi}$ , the functional equation relates  $L(\cdot, \pi)$  to itself, and  $\epsilon(s, \pi)$  equals  $\pm 1$ .

For our applications, it is the analytic conductor (not the arithmetic conductor) that is important for understanding the behavior of the zeros near the central point. The two are related, and the analytic conductor may be taken as

$$Q_\pi = \mu_{\pi,1} \cdots \mu_{\pi,n} N_\pi. \quad (2.8)$$

We use the analytic conductor to rescale the low lying zeros, and then apply the explicit formula to convert sums of an even Schwartz test function over the zeros of the  $L$ -function to sums of the Fourier transform of the test function evaluated at prime powers. For such calculations, it is the logarithm of the analytic conductor that normalizes the zeros; see for example section 4 of [ILS]. In some other papers our factors of  $\mu_{\pi,j}$  are replaced with  $\mu'_{\pi,j}/2$ . As we shall always be interested in situations where the analytic conductors tend to infinity, both normalizations lead to the same results. Note that we have  $N_\pi^{s/2}$  in our functional equation —other authors sometimes write this factor as  $(N'_\pi)^s$ , which would lead to a factor of  $(N'_\pi)^2$  in the analytic conductor.

Throughout the paper we make the following two assumptions, unless specified.

- *We assume the Generalized Riemann Hypothesis for all automorphic  $L$ -functions.* Thus we may write the non-trivial zeros as  $\frac{1}{2} + i\gamma$  with  $\gamma \in \mathbb{R}$ , and the correct scaling for zeros near the central point is  $\gamma \mapsto \tilde{\gamma} = \gamma \frac{\log Q_\pi}{2\pi}$  (low-lying  $\tilde{\gamma}$ 's have natural uniform density 1 —see footnote 1). However, the results on 1-level densities may be interpreted, and remain true, even when the  $\gamma$  are allowed to be complex; see for example [ILS]. In other instances (such as in §5.16), GRH is used to bound error terms and thus enters in the argument in a more essential manner.
- *We assume the Langlands functoriality conjectures for the automorphic representations under consideration.* This is necessary in order to ensure that their attached automorphic  $L$ -functions have good analytic properties. In some cases the analytic properties of an  $L$ -function are known even without knowledge of its automorphicity (e.g., for some symmetric-power  $L$ -functions attached to holomorphic modular forms [KiSh1, KiSh2, K]). On the other hand, the automorphicity of all symmetric powers of an automorphic representation  $f$  implies the Ramanujan-Selberg conjectures

for  $L(s, f)$ , though bounds towards this goal are in some cases available unconditionally [K] (e.g., Deligne’s proof of Ramanujan for holomorphic modular forms).

**Remark 2.2.** *Automorphic  $L$ -functions associated to cuspidal representations are primitive in the sense that one cannot write  $\Lambda(s, \pi)$  as  $\Lambda(s, \pi_1)\Lambda(s, \pi_2)$ . However, a general non-cuspidal automorphic  $L(s, \pi)$  factors as a product of primitive ones and its critical zeros are clearly a union of the zeros of its primitive factors. While the low lying zeros of each primitive factor will reveal a specific underlying symmetry (at least conjecturally), the low lying zeros of the imprimitive function will in general not correspond to a definite symmetry. Here we are using the word “symmetry” in the sense of §3. However, even the assumption of functoriality does not ensure that lifts of cuspidal forms are cuspidal. The simplest counterexample is the imprimitive  $L$ -function  $L(s, \pi \times \tilde{\pi})$  where  $\pi$  is a cuspidal automorphic representation of  $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ ,  $n > 1$ , and  $\tilde{\pi}$  is its contragredient.<sup>4</sup> We will need to deal with this possibility on occasion.*

While we consider quite general families of  $L$ -functions, the building blocks for examples which we can prove satisfy the necessary conditions are (the automorphic representations attached to) either Dirichlet characters or modular forms. For their corresponding  $L$ -functions, classical summation formulas for Fourier coefficients are available that make our approach tractable.

Let  $\phi$  be an even Schwartz test function on  $\mathbb{R}$  whose Fourier transform

$$\widehat{\phi}(y) = \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i xy} dx \quad (2.9)$$

has compact support. Let  $\mathcal{F}$  be a finite family of  $L$ -functions satisfying GRH. For example,  $\mathcal{F}$  might be the set of all  $L(s, \chi)$  with  $\chi$  a non-trivial Dirichlet character of conductor  $m$ , and we would then investigate the limit as  $m \rightarrow \infty$ . Other examples include weight- $k$  level- $N$  cuspidal newforms (and let either  $k$ , or  $N$ , or both tend to infinity), as well as one-parameter families of elliptic curves (where the parameter  $t$  varies over an interval  $[N, 2N]$ , and then we let  $N \rightarrow \infty$ ).

Consider a family  $\mathcal{F}$  of  $L$ -functions  $L(s, f)$  and denote by  $Q_f$  the analytic conductor of  $L(s, f)$ . Let  $\mathcal{F}_N$  be the finite subfamily of  $\mathcal{F}$  consisting of those functions with  $Q_f = N$ . Thus,  $\mathcal{F} = \cup_N \mathcal{F}_N$ . To study the zeros of the functions in the family  $\mathcal{F}$ , we use the Explicit Formula to convert sums over zeros to sums over primes. For any  $L$ -function  $L(s, f)$  [ILS, RS]:

$$\sum_{\ell} \phi\left(\gamma_{j;f} \frac{\log R}{2\pi}\right) = \frac{A_f}{\log R} \widehat{\phi}(0) - 2 \sum_p \sum_{\nu=1}^{\infty} \widehat{\phi}\left(\frac{\nu \log p}{\log R}\right) \frac{b_f(p^{\nu}) \log p}{p^{\nu/2} \log R}, \quad (2.10)$$

<sup>4</sup>If  $\pi_1, \pi_2$  are automorphic unitary cuspidal representations as above, but not necessarily normalized to have trivial central character, then  $L(s, \pi_1 \times \pi_2)$  is imprimitive when  $\pi_2 \simeq \tilde{\pi}_1 \otimes |\det(\cdot)|^s$  is a twist of the contragredient of  $\pi_1$ .

where  $A_f$  is an integral of gamma factors coming from the functional equation of  $L(s, f)$ . We have

$$A_f = \log Q_f + o(\log Q_f), \quad (2.11)$$

and the little-oh implicit constant often depends only on  $\mathcal{F}$  and not the individual  $f$ . We shall also consider variations of the above family; for example, we may let  $\mathcal{F}_N$  be the set of  $f$  in  $\mathcal{F}$  with  $N \leq Q_f \leq 2N$ . While the subject is considerably simplified if the conductors in  $\mathcal{F}_N$  are constant, monotonically increasing conductors can be handled with additional work (see [Mil2] for details for families of elliptic curves).

After averaging over the family, the resulting sums are often evaluated using the following consequence of the Prime Number Theorem:

**Theorem 2.3.** *Let  $\widehat{F}$  be an even Schwartz function of compact support. Then for any positive integer  $\nu$ ,*

$$\sum_p \widehat{F} \left( \nu \frac{\log p}{\log R} \right) \frac{\log p}{\log R} \frac{1}{p} = \frac{1}{2\nu} F(0) + O \left( \frac{1}{\log R} \right). \quad (2.12)$$

### 3. RANDOM MATRIX THEORY REVIEW

Katz and Sarnak conjecture that to any infinite family  $\mathcal{F}$  of  $L$ -functions one can associate one of the infinite families of classical compact matrix groups (unitary, orthogonal, or symplectic), say  $G(\mathcal{F})$ , such that the large-conductor statistics of zeros near the central point for  $L(s, f)$ ,  $f \in \mathcal{F}$ , agree with those of the eigenvalues near 1 of matrices in  $G(\mathcal{F})$  (as the matrix size  $N \rightarrow \infty$ ). Specifically, the  $n$ -level density of rescaled critical zeros for the family  $\mathcal{F}$  is the function  $W_{n, \mathcal{F}}$  (more precisely, the important object is the measure  $W_{n, \mathcal{F}} dx_1 \dots dx_n$  on  $\mathbb{R}^n$ ) defined by its action on any test function  $\phi(x_1, \dots, x_n) = \phi_1(x_1) \cdots \phi_n(x_n)$  (where  $\phi_1, \dots, \phi_n$  are Schwartz functions) by:

$$\begin{aligned} D_{n, \mathcal{F}}(\phi) &= \lim_{N \rightarrow \infty} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \phi_1 \left( \gamma_{j_1; f} \frac{\log Q_f}{2\pi} \right) \cdots \phi_n \left( \gamma_{j_n; f} \frac{\log Q_f}{2\pi} \right) \\ &= \int \cdots \int \phi_1(x_1) \cdots \phi_n(x_n) W_{n, \mathcal{F}}(x_1, \dots, x_n) dx_1 \cdots dx_n \\ &= \int \cdots \int \widehat{\phi}_1(u_1) \cdots \widehat{\phi}_n(u_n) \widehat{W}_{n, \mathcal{F}}(u_1, \dots, u_n) du_1 \cdots du_n. \end{aligned} \quad (3.1)$$

(The Fourier transform  $\widehat{W}_{n, \mathcal{F}}$  is most often used in proofs for technical reasons, e. g., to use the Explicit Formula (2.10).) The Katz-Sarnak philosophy posits that the  $n$ -level densities  $W_{n, \mathcal{F}}$  agree with the  $n$ -level densities  $W_{n, G(\mathcal{F})}$  of the matrix group  $G(\mathcal{F})$  associated to the family  $\mathcal{F}$ . This philosophical correspondence has been proved for many families when the Schwartz test

functions  $\phi_i$  have Fourier transforms supported in a sufficiently small neighborhood of zero. The  $n$ -level densities for the classical compact groups are (see [KaSa1]):

$$\begin{aligned}
W_{n,\text{SO}(\text{even})}(x) &= \det(K_1(x_i, x_j))_{i,j \leq n} \\
W_{n,\text{SO}(\text{odd})}(x) &= \det(K_{-1}(x_i, x_j))_{i,j \leq n} + \sum_{k=1}^n \delta(x_k) \det(K_{-1}(x_i, x_j))_{i,j \neq k} \\
W_{n,\text{O}}(x) &= \frac{1}{2}W_{n,\text{SO}(\text{even})}(x) + \frac{1}{2}W_{n,\text{SO}(\text{odd})}(x) \\
W_{n,\text{Sp}}(x) &= \det(K_{-1}(x_i, x_j))_{i,j \leq n} \\
W_{n,\text{U}}(x) &= \det(K_0(x_i, x_j))_{i,j \leq n},
\end{aligned} \tag{3.2}$$

where  $K(y) = \frac{\sin \pi y}{\pi y}$ ,  $K_\epsilon(x, y) = K(x - y) + \epsilon K(x + y)$  for  $\epsilon = 0, \pm 1$ , and  $\delta(u)$  is the Dirac Delta functional.<sup>5</sup> The Fourier transforms of the 1-level densities are

$$\begin{aligned}
\widehat{W}_{1,\text{SO}(\text{even})}(u) &= \delta(u) + \frac{1}{2}\eta(u) \\
\widehat{W}_{1,\text{SO}(\text{odd})}(u) &= \delta(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W}_{1,\text{O}}(u) &= \delta(u) + \frac{1}{2} \\
\widehat{W}_{1,\text{Sp}}(u) &= \delta(u) - \frac{1}{2}\eta(u) \\
\widehat{W}_{1,\text{U}}(u) &= \delta(u),
\end{aligned} \tag{3.3}$$

where

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1. \end{cases} \tag{3.4}$$

When working with test functions  $\phi$  whose Fourier transform  $\widehat{\phi}$  is supported in a small neighborhood of 0, it is still possible to distinguish between unitary, symplectic and orthogonal  $n$ -level densities; however, as long as  $\widehat{\phi}$  is supported in  $(-1, 1)$ , all three flavors of orthogonal symmetry (even, odd, or full) agree:

$$\begin{aligned}
\int \widehat{\phi}(u) \widehat{W}_{1,\text{SO}(\text{even})}(u) du &= \widehat{\phi}(u) + \frac{1}{2}\phi(0) \\
\int \widehat{\phi}(u) \widehat{W}_{1,\text{SO}(\text{odd})}(u) du &= \widehat{\phi}(u) + \frac{1}{2}\phi(0) \\
\int \widehat{\phi}(u) \widehat{W}_{1,\text{O}}(u) du &= \widehat{\phi}(u) + \frac{1}{2}\phi(0) \\
\int \widehat{\phi}(u) \widehat{W}_{1,\text{Sp}}(u) du &= \widehat{\phi}(u) - \frac{1}{2}\phi(0) \\
\int \widehat{\phi}(u) \widehat{W}_{1,\text{U}}(u) du &= \widehat{\phi}(u).
\end{aligned} \tag{3.5}$$

Let  $\text{sign}(G) = 0$  (resp.,  $\frac{1}{2}$ , 1) for  $G = \text{SO}(\text{even})$  (resp.,  $\text{O}$ ,  $\text{SO}(\text{odd})$ ). For even functions  $\phi(x_1, x_2) = \phi_1(x_1)\phi_2(x_2)$  such that  $\widehat{\phi}(u_1, u_2) = \widehat{\phi}_1(u_1)\widehat{\phi}_2(u_2)$  is supported in  $|u_1| + |u_2| < 1$ ,

$$\begin{aligned}
\int \int \widehat{f}_1(u_1) \widehat{f}_2(u_2) \widehat{W}_{2,\mathcal{G}}(u) du_1 du_2 &= \left[ \widehat{f}_1(0) + \frac{1}{2}f_1(0) \right] \left[ \widehat{f}_2(0) + \frac{1}{2}f_2(0) \right] \\
&\quad + 2 \int |u| \widehat{f}_1(u) \widehat{f}_2(u) du - 2\widehat{f}_1 f_2(0) - f_1(0) f_2(0) \\
&\quad + \text{sign}(\mathcal{G}) f_1(0) f_2(0).
\end{aligned} \tag{3.6}$$

<sup>5</sup>While these determinant formulas hold for arbitrary support, in practice the resulting formulas for  $n \geq 3$  require some combinatorics when the support is large before agreement is seen with number theory. Hughes and Miller [HM] derive an alternate formula for  $n$ -level statistics; while their formula holds for smaller support, in the range where it is applicable it facilitates comparisons with number theory.

Thus, for arbitrarily small support, the 2-level density distinguishes the three orthogonal groups; see [Mil1] for the calculation.

In studying families of elliptic curves [Mil2, Yo2], often the corresponding classical compact group is a subgroup of one of the orthogonal groups. For one-parameter families of elliptic curves over  $\mathbb{Q}(T)$  with rank  $r_{\mathcal{F}}$ , as remarked in (1.11), the correct model as the conductors tend to infinity appears to be

$$\left\{ \left( \begin{array}{c} I_{r_{\mathcal{F}} \times r_{\mathcal{F}}} \\ g \end{array} \right), g \in \mathcal{C} \right\}, \quad (3.7)$$

where  $I_{r_{\mathcal{F}} \times r_{\mathcal{F}}}$  is the  $r_{\mathcal{F}} \times r_{\mathcal{F}}$  identity matrix and  $\mathcal{C}$  is O (resp., SO(even) or SO(odd)) if half the signs of the functional equation are even (resp., all or none), though see [Mil4] for a discussion of the behavior for finite conductors. These  $r_{\mathcal{F}}$  independent zeros replace  $\widehat{\phi}(u) + \frac{1}{2}\phi(0)$  with  $\widehat{\phi}(u) + \frac{1}{2}\phi(0) + r_{\mathcal{F}}\phi(0)$  in the 1-level density expansion, and there is a similar modification in the  $n$ -level density.

Because of this effect of rank, we attach a family constant to each family of  $L$ -functions  $\mathcal{F}$ :

$$\widetilde{c}_{\mathcal{F}} = (c_{\mathcal{F}}, \epsilon_{\mathcal{F}}, r_{\mathcal{F}}). \quad (3.8)$$

Here  $c_{\mathcal{F}}$  is the symmetry constant of the family, equal to 0 (resp., 1 or  $-1$ ) if the family is unitary (resp., symplectic or orthogonal); we call any subgroup of O, SO(even) or SO(odd) orthogonal. Since the three orthogonal groups all have  $c_{\mathcal{F}} = 1$ , we set  $\epsilon_{\mathcal{F}}$  equal to 0 (resp., 1 or  $-1$ ) if  $\mathcal{F}$  has half of the signs of its functional equation even (resp., all signs even or odd); if  $\mathcal{F}$  is not associated to an orthogonal group, then  $c_{\mathcal{F}}$  alone determines the precise group and we define  $\epsilon_{\mathcal{F}} = 0$ . Finally,  $r_{\mathcal{F}}$  denotes the rank of  $\mathcal{F}$ ; except for families of elliptic curves, all other known families have  $r_{\mathcal{F}} = 0$ .

Our main result (Theorem 1.3) is that in order to determine the symmetry of the Rankin-Selberg convolution of two NT-good families, all that matters is  $c_{\mathcal{F}}$  and  $c_{\mathcal{G}}$ . Thus we may interpret the symmetry constant as a convolution constant. Further, the new family has rank zero. This is unfortunate, since otherwise this would allow constructing families of  $L$ -functions with high central vanishing.

#### 4. NT-good FAMILIES AND $n$ -LEVEL DENSITIES

As a warm-up to proving our main theorems in §5, in this section we investigate some consequences of Definition 1.1 (NT-good).

It is worth commenting on the main terms in (1.2) and (1.3). Consider a one-parameter family of elliptic curves over  $\mathbb{Q}(T)$  with rank  $r_{\mathcal{F}}$ ;  $\mathcal{F}_N$  is essentially just  $\{E_t : t \in [N, 2N]\}$ . Then  $b_t(p) = a_t(p)/\sqrt{p}$ , where  $a_t(p)$  are the coefficients of the  $L$ -series of  $L(s, E_t)$  (with functional equation  $s \rightarrow 2 - s$ , so the critical strip is  $\Re s \in [0, 2]$ ). Rosen and Silverman [RoSi] prove a conjecture of Nagao's (unconditionally if the elliptic surface is rational; conditional

on Tate's conjecture otherwise):

$$\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{p \leq X} \frac{1}{p} \sum_{t=0}^{p-1} a_t(p) \log p = -r_{\mathcal{F}}. \quad (4.1)$$

Thus the  $a_t(p)$ 's give the rank of the family over  $\mathbb{Q}(T)$ . For many families of elliptic curves (see [ALM, Fe]), the main term of the average over  $E_t \in \mathcal{F}_N$  of  $a_t(p)/p$  is independent of  $p$ , and we have

$$\begin{aligned} -\frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \frac{b_t(p)}{\sqrt{p}} &= -\frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \frac{a_t(p)}{p} \\ &= \frac{1}{|\mathcal{F}_N|} \left[ -\frac{|\mathcal{F}_N|}{p} \sum_{t \bmod p} \frac{a_t(p)}{p} \right] + O\left(\frac{\sqrt{p}}{|\mathcal{F}_N|}\right) \\ &= \frac{r_{\mathcal{F}}}{p} + O\left(\frac{\sqrt{p}}{|\mathcal{F}_N|}\right). \end{aligned} \quad (4.2)$$

If we have such a family we use (1.3); if not, we need to do a little more work and use (1.2) and (2.12). The proofs follow similarly, the only real difference being a partial summation on the primes to handle the test functions.

*For ease of exposition we concentrate on cases where (1.3) holds, and remark that similar arguments handle the case when we have (1.2).*

**Theorem 4.1.** *Let  $\mathcal{F}$  be an NT-good family of automorphic  $L$ -functions for  $\mathrm{GL}_n$ . Then for even Schwartz test functions  $\phi$  such that  $\widehat{\phi}$  is supported in a sufficiently small (but explicitly computable in terms of the constants  $\delta_i, \mu_i$ ) neighborhood of 0, if  $r_{\mathcal{F}} = 0$  then the 1-level density of  $\mathcal{F}$  agrees with unitary (resp., symplectic or orthogonal) if  $c_{\mathcal{F}} = 0$  (resp., 1 or  $-1$ ); if  $r_{\mathcal{F}} > 0$  then the corresponding classical compact group is modified by having an  $r_{\mathcal{F}} \times r_{\mathcal{F}}$  identity matrix as in (1.11).*

*Proof.* Using the explicit formula to calculate the 1-level density, we have the expansion

$$D_{1, \mathcal{F}_N}(\phi) = \widehat{\phi}(0) - \frac{2}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\nu=1}^{\infty} \sum_{p=2}^{R_N^{\sigma}} \frac{b_f(p^{\nu}) \log p}{p^{\nu/2} \log R_N} \widehat{\phi}\left(\nu \frac{\log p}{\log R_N}\right) + o(1). \quad (4.3)$$

From (2.11), the factor of  $\widehat{\phi}(0)$  above comes from the constancy of the main term of the analytic conductors (and an analysis of the  $\Gamma$ -factor terms; in fact, this is what we use to determine  $R_N$ ); the  $o(1)$  term arises from the correction factor in  $\log Q_f = \log R_N + o(\log R_N)$ . As our family is NT-good, there is no contribution from  $b_f(p^{\nu})$  for  $\nu \geq 3$  (either for all support, or for support sufficiently small). Thus those terms may be absorbed into an error term.



We assume that (1.3) holds; the case when (1.2) holds follows similarly. The  $\nu = 1$  terms contribute

$$\begin{aligned}
S_1 &= -\frac{2}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{p=2}^{R_N^\sigma} \frac{b_f(p) \log p}{\sqrt{p} \log R_N} \widehat{\phi} \left( \frac{\log p}{\log R_N} \right) \\
&= 2 \sum_{p=2}^{R_N^\sigma} \left[ -\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} b_f(p) \right] \frac{\log p}{\sqrt{p} \log R_N} \widehat{\phi} \left( \frac{\log p}{\log R_N} \right) \\
&= 2 \sum_{p=2}^{R_N^\sigma} \left[ \frac{r_{\mathcal{F}}}{\sqrt{p}} + O(|\mathcal{F}_N|^{-\delta_1} p^{\mu_1}) \right] \frac{\log p}{\sqrt{p} \log R_N} \widehat{\phi} \left( \frac{\log p}{\log R_N} \right) \\
&= r_{\mathcal{F}} \sum_{p=2}^{R_N^\sigma} \frac{\log p}{p \log R_N} \widehat{\phi} \left( \frac{\log p}{\log R_N} \right) + O \left( \frac{1}{|\mathcal{F}_N|^{\delta_1}} \sum_{p=2}^{R_N^\sigma} p^{\mu_1 - \frac{1}{2}} \right) \\
&= r_{\mathcal{F}} \phi(0) + O \left( \frac{1}{\log R_N} \right) + O \left( \frac{R_N^{(\mu_1 + \frac{1}{2})\sigma}}{|\mathcal{F}_N|^{\delta_1}} \right), \tag{4.4}
\end{aligned}$$

where the main term in the last line is an immediate consequence of the Prime Number Theorem (see Theorem 2.3 for a proof); as  $|\mathcal{F}_N| \geq R_N^{\delta_0}$ , for  $\sigma$  sufficient small (in terms of  $\mu_1, \delta_1$  and  $\delta_0$ ), the last error term is negligible.

We are left with the contribution from the squares of the primes (the  $\nu = 2$  terms). As  $\sum_{f \in \mathcal{F}_N} b_f(p^2) = c_{\mathcal{F}} |\mathcal{F}_N| + O(|\mathcal{F}_N|^{1-\delta_2} p^{\mu_2})$ , for sufficiently small support, up to a negligible term by Theorem 2.3 the resulting sum over primes is  $\frac{\phi(0)}{2}$ . Thus the 1-level density satisfies

$$D_{1, \mathcal{F}}(\phi) = \widehat{\phi}(0) - c_{\mathcal{F}} \cdot \frac{1}{2} \phi(0) + r_{\mathcal{F}} \phi(0), \tag{4.5}$$

which for small support agrees with the 1-level densities of (3.5) (trivially modified if there are  $r_{\mathcal{F}}$  forced eigenvalues at 1).  $\square$

**Remark 4.2** (Support of the test functions). *The allowable support of  $\widehat{\phi}$  is determinable from the constants  $\delta_i, \mu_i$ . In general, the support will not be large enough to distinguish the three orthogonal densities, though it will suffice to distinguish unitary from symplectic from orthogonal.*

**Remark 4.3** (General  $n$ -level density). *It is natural to investigate the 2-level density to distinguish the orthogonal groups. To do so requires two additional pieces of information: (1) the distribution of signs of functional equations in the family; (2) being able to average over the family  $b_f(p_1)b_f(p_2^2)$  and  $b_f(p_1^2)b_f(p_2^2)$ . The presence of cross terms can seriously complicate matters, though fortunately in all families considered to date these terms can be converted to products of averages of single terms. For cuspidal newforms this follows from the Petersson formula; for one-parameter families of elliptic curves, if  $p_1, \dots, p_k$  are distinct primes and*

$r_1, \dots, r_k$  are integers, simple counting (see [Mil2]) shows that

$$\sum_{t \bmod p_1 \cdots p_k} a_t^{r_1}(p_1) \cdots a_t^{r_k}(p_k) = \prod_{i=1}^k \sum_{t \bmod p_i} a_t^{r_i}(p_i), \quad (4.6)$$

reducing the analysis to single product terms. In general (though see Remark 4.4 below), if we know the distribution of signs we can determine the 2-level densities, as all that matters is knowing the fraction of  $L(s, f)$  with even or odd sign; however, the story is markedly different for third- and higher-level densities. There we need to know significantly more; we need to know which of the  $L(s, f)$  have odd functional equation, and we need to execute sums over just those  $L(s, f)$ . Thus, for families of elliptic curves the third- and higher-level densities are beyond current techniques (except for constant-sign families); see [Mil1] for more details.

**Remark 4.4** (Variation in the analytic conductors). We concentrate on the 1-level density in this paper. There are two ways to normalize the zeros of an  $L$ -function in a family  $\mathcal{F}_N$ : we can use  $\log Q_f$  or  $\log R_N$ . For the 1-level density, it does not matter which one is used in the normalization; however, for the higher  $n$ -level densities we need to use the explicit formula multiple times. While the normalization  $\log R_N$  greatly simplifies the 1-level computations (because all test functions are scaled equally), in the general case we are forced to evaluate sums of  $\log Q_f$  against the coefficients of the  $L$ -functions. With additional work, these sums can often be handled (see [Mil2] for the case of one-parameter families of elliptic curves).

## 5. PROOF OF THE MAIN RESULTS

### 5.1. Preliminaries.

Assuming  $\mathcal{F}$  and  $\mathcal{G}$  are NT-good families, by Theorem 4.1 we can determine their 1-level densities, and associate a classical compact group to the family (uniquely in the case of unitary and symplectic symmetry; for orthogonal symmetries, for small support the 1-level density cannot distinguish  $\mathrm{SO}(\text{even})$  from  $\mathrm{O}$  from  $\mathrm{SO}(\text{odd})$ ). Assuming a few additional conditions, we can determine the symmetry group of the Rankin-Selberg convolution of the two families  $\mathcal{F}$  and  $\mathcal{G}$ .

The following lemma is key to the proofs of our main results.

**Lemma 5.1.** *Assume  $\pi_1, \pi_2$  are automorphic cuspidal representations of  $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ ,  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ , respectively, and further that the functorial lift  $\pi_1 \times \pi_2$  to  $\mathrm{GL}_{mn}(\mathbb{A}_{\mathbb{Q}})$  exists. Assume that, for some prime  $p$ , both  $\pi_{1,p}$  and  $\pi_{2,p}$  are unramified, and their corresponding Satake parameters are  $\{\alpha_{\pi_1}(i)\}_{1 \leq i \leq n}$  and  $\{\alpha_{\pi_2}(j)\}_{1 \leq j \leq m}$ . Then*

$$b_{\pi_1 \times \pi_2}(p^\nu) = b_{\pi_1}(p^\nu) \cdot b_{\pi_2}(p^\nu). \quad (5.1)$$

*Proof.* Let  $b_{\pi_1}(p^\nu)$  and  $b_{\pi_2}(p^\nu)$  be as in Definition 1.1. By the local Langlands correspondence, the Satake parameters for  $\pi_{1,p} \times \pi_{2,p}$  are

$$\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} = \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}, \quad (5.2)$$

which gives

$$\begin{aligned}
b_{\pi_1 \times \pi_2}(p^\nu) &= \sum_{k=1}^{nm} \alpha_{\pi_1 \times \pi_2}(k)^\nu \\
&= \sum_{i=1}^n \sum_{j=1}^m \alpha_{\pi_1}(i)^\nu \cdot \alpha_{\pi_2}(j)^\nu \\
&= \sum_{i=1}^n \alpha_{\pi_1}(i)^\nu \cdot \sum_{j=1}^m \alpha_{\pi_2}(j)^\nu \\
&= b_{\pi_1}(p^\nu) \cdot b_{\pi_2}(p^\nu).
\end{aligned} \tag{5.3}$$

□

**Remark 5.2.** *In Lemma 5.1 above the assumption that  $\pi_1 \times \pi_2$  is automorphic is not really necessary in order to define the Satake parameters at  $p$  nor the coefficients  $b_{\pi_1 \times \pi_2}(p^\nu)$  of a hypothetical  $\pi_1 \times \pi_2$ . Recall that our viewpoint is that the  $L$ -function  $L(s, \pi_1 \times \pi_2)$  should be automorphic and primitive, which is the case if  $\pi_1, \pi_2$  are cuspidal, except when  $\pi_1 \simeq \tilde{\pi}_2$  (assuming unitary and of trivial central character).*

If  $\pi$  is an automorphic cuspidal representation of  $\mathrm{GL}_n$ , then  $L(s, \pi)$  is primitive if and only if  $\pi$  is primitive. For two unitary cuspidal automorphic representations  $\pi$  and  $\pi'$  of  $\mathrm{GL}_n$  with trivial central character, their convolution  $L(s, \pi \times \pi')$  is primitive if and only if  $\pi'$  is *not* the contragredient of  $\pi$ . This is equivalent to the lift  $\pi \times \pi'$  being a cuspidal representation of  $\mathrm{GL}_{n^2}$ .

We now prove our main results.

## 5.2. Convolving two families.

*Proof of Theorem 1.3.* We first prove the theorem under the assumption that all convolved  $L$ -functions are primitive, and then handle the general case.

As we are assuming the convolved family is NT-good, in the new family  $\mathcal{F}_N \times \mathcal{G}_M$  the conductors are essentially constant, say  $\log Q_{f \times g} = \log R_{N,M} + o(\log R_{N,M})$ . By the multiplicativity assumptions, it will be relatively easy to evaluate

$$\begin{aligned}
&D_{1, \mathcal{F}_N \times \mathcal{G}_M}(\phi) \\
&= \widehat{\phi}(0) - 2 \cdot \frac{1}{|\mathcal{F}_N| \cdot |\mathcal{G}_M|} \sum_{f \times g \in \mathcal{F}_N \times \mathcal{G}_M} \sum_{\nu=1}^{\infty} \sum_{p=2}^{R_{N,M}^\sigma} \frac{b_{f \times g}(p^\nu) \log p}{p^{\nu/2} \log R_{N,M}} \widehat{\phi} \left( \nu \frac{\log p}{\log R_{N,M}} \right) + o(1).
\end{aligned} \tag{5.4}$$

Some care is required for the  $\nu \geq 3$  terms. We need to show that these give a negligible contribution. If condition (4.iii) holds for either of the NT-good families (namely, that if we sum over the family, we have a power savings in the family cardinality), then this follows

immediately. If not, we need  $\delta_3(\mathcal{F}), r_3(\mathcal{F})$  (and similarly for the family  $\mathcal{G}$ ) to be such that summing the  $\nu \geq 3$  terms is negligible. This is always the case if we assume the Ramanujan conjecture, condition (4.i).

We must determine the contributions from the  $\nu = 1, 2$  terms. As

$$b_{f \times g}(p^\nu) = b_f(p^\nu) \cdot b_g(p^\nu) \quad (5.5)$$

(Lemma 5.1), we can execute the summations over  $f \in \mathcal{F}_N$  and  $g \in \mathcal{G}_M$ . The main term from  $\nu = 2$  is

$$-2 \cdot \frac{1}{|\mathcal{F}_N| \cdot |\mathcal{G}_M|} \sum_{p=2}^{R_{N,M}^\sigma} \frac{c_{\mathcal{F}} |\mathcal{F}_N| \cdot c_{\mathcal{G}} |\mathcal{G}_M| \log p}{p \log R_{N,M}} \widehat{\phi} \left( 2 \frac{\log p}{\log R_{N,M}} \right). \quad (5.6)$$

If the zeros were normalized by  $\log Q_{f \times g}$  instead of  $\log R_{N,M}$ , it would not be as easy to compute the contributions because the Schwartz functions would be evaluated at points depending on  $Q_{f \times g}$ . This is the main reason we choose to normalize all zeros in a family by the same quantity.

By the Prime Number Theorem (Theorem 2.3), the main term of the sum in (5.6) equals

$$-\frac{c_{\mathcal{F}} \cdot c_{\mathcal{G}}}{2} \phi(0), \quad (5.7)$$

and the error term is negligible. There are three other terms which contribute in the  $\nu = 2$  case:

$$|\mathcal{F}_N|^{1-\delta_2(\mathcal{F})} \cdot |\mathcal{G}_M| \cdot p^{\mu_2(\mathcal{F})}, \quad |\mathcal{F}_N| \cdot |\mathcal{G}_M|^{1-\delta_2(\mathcal{G})} \cdot p^{\mu_2(\mathcal{G})}, \quad |\mathcal{F}_N|^{1-\delta_2(\mathcal{F})} \cdot |\mathcal{G}_M|^{1-\delta_2(\mathcal{G})} \cdot p^{\mu_2(\mathcal{F})+\mu_2(\mathcal{G})}.$$

As we divide by  $|\mathcal{F}_N| \cdot |\mathcal{G}_M|$ , each of the three terms leads to a negligible contribution for test functions with suitably small support.

We are left with handling the  $\nu = 1$  terms. If  $r_{\mathcal{F}}$  or  $r_{\mathcal{G}} = 0$  then we immediately see this term does not contribute for suitably small support. For notational convenience we assume (1.3) and not (1.2) holds, as the argument in each case is similar. We have

$$\begin{aligned} \sum_{f \times g \in \mathcal{F}_N \times \mathcal{G}_M} b_{f \times g}(p) &= \left[ \sum_{f \in \mathcal{F}_N} b_f(p) \right] \cdot \left[ \sum_{g \in \mathcal{G}_M} b_g(p) \right] \\ &= \frac{r_{\mathcal{F}} \cdot r_{\mathcal{G}}}{p} |\mathcal{F}_N| \cdot |\mathcal{G}_M| + O \left( |\mathcal{F}_N|^{1-\delta_{1,\mathcal{F}}} \cdot |\mathcal{G}_M|^{1-\delta_{1,\mathcal{G}}} p^{\mu_{1,\mathcal{F}}+\mu_{1,\mathcal{G}}} \right) \\ &\quad + O \left( |\mathcal{F}_N|^{1-\delta_{1,\mathcal{F}}} \cdot |\mathcal{G}_M| \cdot p^{\mu_{1,\mathcal{F}}-\frac{1}{2}} \right) \\ &\quad + O \left( |\mathcal{F}_N| \cdot |\mathcal{G}_M|^{1-\delta_{1,\mathcal{G}}} \cdot p^{\mu_{1,\mathcal{G}}-\frac{1}{2}} \right). \end{aligned} \quad (5.8)$$

Summing over  $p$ , for test functions with small support the three error terms do not contribute. The main term leads to

$$-2 \sum_p \frac{r_{\mathcal{F}} \cdot r_{\mathcal{G}}}{p} \frac{\log p}{\sqrt{p} \log R_{N,M}} \widehat{\phi} \left( \frac{\log p}{\log R_{N,M}} \right). \quad (5.9)$$

If  $\mathcal{F} \times \mathcal{G}$  were to have rank, this sum would have to contribute. Comparing to equation (1.2), the difference is that in the sum above we have  $\frac{r_{\mathcal{F} \cdot r_{\mathcal{G}}}}{p}$  instead of something like  $\frac{r}{\sqrt{p}}$  times  $\frac{\log p}{\sqrt{p} \log R} \widehat{\phi} \left( \frac{\log p}{\log R} \right)$ . The presence of  $p$  rather than  $\sqrt{p}$  in the denominator means this sum is of size  $(\log R_{N,M})^{-1}$  rather than of size 1. This leads to a lower order correction term to the 1-level density of size  $\frac{r_{\mathcal{F} \cdot r_{\mathcal{G}}}}{\log R_{N,M}}$ .

We now remove the assumption that all the convolutions are primitive; i.e., we now allow a contragredient of an  $f \in \mathcal{F}$  to be in  $\mathcal{G}$ . This can only happen if  $m = n$ . All we require is some control on the number such pairs  $(f, \tilde{f})$  and their contribution. As the families are NT-good, the multiplicities are bounded:  $\mu_f \leq \mu_{\mathcal{F}}$  and  $\mu_g \leq \mu_{\mathcal{G}}$ . Thus the number of such pairs is trivially bounded by  $\min(\mu_{\mathcal{G}}|\mathcal{F}_N|, \mu_{\mathcal{F}}|\mathcal{G}_M|) = O(\min(|\mathcal{F}_N|, |\mathcal{G}_M|))$ .

For any such  $(f, \tilde{f})$  the convolution  $L(s, f \times \tilde{f})$  is not primitive and has a simple pole at  $s = 1$ , contributing two additional terms,  $\phi \left( \pm \frac{\log R}{4\pi} i \right)$ , to the explicit formula (see, for example, [RS]). If  $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ , then

$$\phi(t + iy) = \int_{-\infty}^{\infty} \widehat{\phi}(\xi) e^{2\pi i(t+iy)\xi} d\xi \ll e^{2\pi|y|\sigma}. \quad (5.10)$$

Thus

$$\phi \left( \pm \frac{\log R}{4\pi} i \right) \ll R^{\sigma/2}; \quad (5.11)$$

as we divide by  $|\mathcal{F}_N| \cdot |\mathcal{G}_M|$  and there are only  $O(\min(|\mathcal{F}_N|, |\mathcal{G}_M|))$  such pairs, for  $\sigma$  sufficiently small these two terms have a negligible contribution.

We now show the contribution to the prime sums from these pairs is also negligible if  $\sigma$  is sufficiently small. Any improvement of the exponent  $1/2$  in the Jacquet-Shalika [JS] bound for the Satake parameters suffices; we use the Rudnick-Sarnak<sup>6</sup> [RS] bound: if  $\pi$  is an automorphic representation of  $\text{GL}_r(\mathbb{A}_{\mathbb{Q}})$ , then  $|\alpha_{\pi,j}(p)| \leq p^{\frac{1}{2} - \frac{1}{r^2+1}}$ . Thus, each pair contributes at most

$$\sum_{\nu} \sum_{p \leq R^{\sigma/\nu}} \frac{p^{\frac{\nu}{2} - \frac{\nu}{n^4+1}}}{p^{\nu/2}} \ll_n R^{\sigma}. \quad (5.12)$$

to the prime sums in the explicit formula. As there are only  $O(\min(|\mathcal{F}_N|, |\mathcal{G}_M|))$  pairs, for  $\sigma$  sufficiently small these lead to negligible contributions upon dividing by the family's cardinality,  $|\mathcal{F}_N| \cdot |\mathcal{G}_M|$ .  $\square$

**Remark 5.3.** *The universality in Theorem 1.3 can be surprising at first. In determining the underlying classical compact group of the convolution of two families, all that matters are the distribution of signs of functional equations, the rank of the family, and the family averages of the  $b_f(p)$ 's and  $b_g(p)$ 's (i.e., the family averages of the second moments of the Satake parameters at each unramified prime). Upon convolving two such nice families, the*

<sup>6</sup>The Rudnick-Sarnak bound is stated only for Satake parameters of cuspidal representations, and it trivially extends to isobaric sums of cuspidal representations, hence to arbitrary (not necessarily cuspidal) automorphic representations.

main term is independent of the family ranks; however, there is a lower order correction term which can often be isolated and which does depend on the ranks. Unfortunately the bounds for the errors from the  $\nu \geq 3$  terms, even assuming Ramanujan, will be of the same size, as could the other error bounds from the  $\nu = 1$  and  $\nu = 2$  terms. Conjecturally, however, it is reasonable to expect there to be cancellation in these errors upon summing over the families, and hence that there could be corrections to the 1-level density. For other examples of lower order corrections, see [FI, Mil3, Mil5, Mil6, Mil7, St, Yo1].

**Example 5.4.** We give an interesting example of Theorem 1.3. Consider families  $\mathcal{F}_i$  of weight- $k_i$  holomorphic cuspidal newforms of prime level  $N$  ( $k_i$  fixed,  $N \rightarrow \infty$ ); perhaps we might want to take the sub-families of even or odd sign. These families  $\mathcal{F}_i$  are NT-good, as is  $\mathcal{F}_1 \times \mathcal{F}_2$  when  $k_1 \neq k_2$ . By [ILS] each  $\mathcal{F}_i$  has orthogonal symmetry. As these are  $GL_2$  holomorphic cuspidal newforms, we know the Ramanujan conjectures and thus condition (4.i) holds. From the Petersson formula, (1.3) holds with  $r_{\mathcal{F}_i} = 0$ . Thus these families have orthogonal symmetries and hence their symmetry constants are  $c_{\mathcal{F}_i} = -1$ . Therefore  $c_{\mathcal{F}_1 \times \mathcal{F}_2} = c_{\mathcal{F}_1} \cdot c_{\mathcal{F}_2} = 1$ , implying that  $\mathcal{F}_1 \times \mathcal{F}_2$  has symplectic symmetry. In particular, all elements should have even sign (which we do get from Rankin-Selberg). Note this is a  $GL_4$  family of  $L$ -functions. Is there a larger natural  $GL_4$  family containing it (analogous to the quadratic Dirichlet characters sitting inside all Dirichlet characters)? Further, if  $k_1, k_2$  and  $k_3$  are distinct then  $c_{\mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3} = c_{\mathcal{F}_1} \cdot c_{\mathcal{F}_2} \cdot c_{\mathcal{F}_3} = -1$ , implying  $\mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3$  has orthogonal symmetry.<sup>7</sup>

### 5.3. Convolution by a Fixed Form.

*Proof of Theorem 1.8.* We may assume all the convolutions are primitive. As  $\mathcal{G}$  is NT-good, from our cardinality assumption there are at most  $\mu_{\mathcal{G}} = O(1)$  imprimitive convolutions. Arguing as in the proof of Theorem 1.3 we can show they have a negligible contribution for sufficiently small support.

Using the explicit formula to calculate the 1-level density, we have the expansion

$$D_{1, f \times \mathcal{G}_M}(\phi) = \widehat{\phi}(0) - \frac{2}{|\mathcal{G}_M|} \sum_{g \in \mathcal{G}_M} \sum_{\nu=1}^{\infty} \sum_{p=2}^{R_M^\sigma} \frac{b_f(p^\nu) b_g(p^\nu) \log p}{p^{\nu/2} \log R_M} \widehat{\phi} \left( \nu \frac{\log p}{\log R_M} \right) + o(1). \quad (5.13)$$

There will be no contribution from the  $\nu \geq 3$  terms if we have sufficiently good bounds for  $\frac{b_f(p^\nu) b_g(p^\nu)}{p^{\nu/2}}$ , or if we have some power savings (relative to  $|\mathcal{G}_M|$ ) in  $\sum_{g \in \mathcal{G}_M} b_g(p^\nu)$ . For example, if we take  $f$  to be any nice  $L$ -function on  $GL_2$  (say a holomorphic cuspidal newform of weight  $k$  and level  $N$  or an even Maass form), then we have good bounds on  $b_f(p^\nu)$ . In the holomorphic

<sup>7</sup>It is known that  $f_1 \times f_2$  is automorphic. While it is not known that  $f_1 \times f_2 \times f_3$  is automorphic, we do know that  $L(s, f_1 \times f_2 \times f_3)$  is entire; the automorphy follows from standard functoriality conjectures, and would imply the  $L$ -function is primitive. See [Bu, Ga, Ram] for details.

case, we know Ramanujan and  $b_f(p^\nu) \ll 1$ ; in the Maass case we have  $b_f(p^\nu) \ll p^{7/64}$  (see the appendix by Kim and Sarnak in [K]). We can quantify exactly what bounds we need on  $b_g(p^\nu)$  for each  $g \in \mathcal{G}_M$  ( $\nu \geq 3$ ), and these bounds are available in many cases of interest. We then execute the summation over  $p$ , which gives  $(\log R_M)^{-1}$ , and then we trivially handle the sum over  $\mathcal{G}_M$ .

As  $\mathcal{G}$  satisfies (1.3), for  $\nu = 1$  a simple calculation shows there is no contribution in the limit as  $M \rightarrow \infty$  for sufficiently small support. We are left with the crucial case of  $\nu = 2$ ; note that this is the term that determines the symmetry type: If it is 0 (resp., 1 or  $-1$ ), we have unitary (resp., symplectic or orthogonal). From our assumption that  $\mathcal{G}$  satisfies (1.5), we can execute the summation  $\sum_{g \in \mathcal{G}_M} b_g(p^2)$ , and we find that the main contribution from  $\nu = 2$  is just

$$-\frac{2}{|\mathcal{G}_M|} \sum_{p=2}^{R_M^\sigma} \frac{[c_{\mathcal{G}} \cdot |\mathcal{G}_M| + |\mathcal{G}_M|^{1-\delta_2} p^{\mu_2}] b_f(p^2) \log p}{p \log R_M} \widehat{\phi} \left( 2 \frac{\log p}{\log R_M} \right). \quad (5.14)$$

For sufficiently small support, as  $b_f(p^2)$  is bounded by some power of  $p$ , the second term doesn't contribute. We are left with

$$-2c_{\mathcal{G}} \sum_{p=2}^{R_M^\sigma} \frac{b_f(p^2) \log p}{p \log R_M} \widehat{\phi} \left( 2 \frac{\log p}{\log R_M} \right). \quad (5.15)$$

Thus the symmetry will be the product of  $c_{\mathcal{G}}$  and the above sum. If  $f$  is a Dirichlet character  $\chi$ , then  $b_f(p^2) = \chi(p)^2$ . If  $\chi$  is quadratic then  $\chi(p)^2 = 1$  and the symmetry constant will be  $c_{\mathcal{G}}$  again; if  $\chi$  is not quadratic then the sum of  $\chi(p)^2$  (times the other factors) over the primes is  $o(1)$ , yielding unitary symmetry. If  $f$  is a nice  $\mathrm{GL}_2$   $L$ -function (say holomorphic cuspidal Hecke newform or Hecke-Maass), then the prime sum is  $-\frac{1}{4}\phi(0)$  because

$$\begin{aligned} b_f(p^2) &= \alpha_{f,1}^2(p) + \alpha_{f,2}^2(p) \\ &= (\alpha_{f,1}(p) + \alpha_{f,2}(p))^2 - 2 \\ &= a_f(p)^2 - 2 \\ &= [a_f(p^2) + 1] - 2 \\ &= a_f(p^2) - 1, \end{aligned} \quad (5.16)$$

where we have used the fact that  $f$  is a Hecke eigenform to say  $a_f(p)a_f(p) = a_f(p^2) + 1$  (at least for  $p$  relatively prime to the conductor). The  $a_f(p^2)$  will be related to the symmetric square  $L$ -function associated to  $f$ , and by GRH for that  $L$ -function, its sum over primes is negligible (see [ILS] for details). Thus the  $\nu = 2$  terms contribute  $(-2c_{\mathcal{G}}) \cdot (-\frac{1}{4}\phi(0)) = c_{\mathcal{G}} \cdot \frac{1}{2}\phi(0)$ .

Setting  $c_f = 1$  if  $f$  is a quadratic Dirichlet character, 0 if  $f$  is a non-quadratic Dirichlet character, and  $c_f = -1$  if  $f$  is a Hecke holomorphic or Maass form, we find that the 1-level density of  $f \times \mathcal{G}$  is

$$\widehat{\phi}(0) - c_f \cdot c_{\mathcal{G}} \cdot \frac{1}{2}\phi(0). \quad (5.17)$$

□

**Remark 5.5.** *These results are similar to those obtained by Rubinstein in his thesis, where he considered the convolution of the family of quadratic Dirichlet  $L$ -functions with a fixed  $\mathrm{GL}_n$  form; see [Rub]. In our notation, if  $f$  is self-dual, then  $c_f = +1$  (resp.,  $c_f = -1$ ) if  $L(s, \mathrm{sym}^2 f)$  (resp.,  $L(s, \wedge^2 f)$ ) has a pole at  $s = 1$ . If  $f$  is not self-dual then  $c_f = 0$ .*

## 6. CONVOLVING FAMILIES OF SYMMETRIC POWERS OF MODULAR FORMS

Families of  $L$ -functions attached to holomorphic modular forms and their functorial liftings are often NT-good, at least under the assumption of standard conjectures. The main purpose of this section is to provide further examples illustrating Theorem 1.3. Additional examples (independent of this section) involving elliptic curves are given in §7.

Let  $\mathcal{H}_k$  be a Hecke eigenbasis of the space of modular cusp forms of weight  $k$  for the full modular group  $\mathrm{SL}_2(\mathbb{Z})$ . Then  $|\mathcal{H}_k| = \frac{k}{12} + O(1)$ . We denote the average over  $\mathcal{H}_k$  by

$$\langle A_f \rangle_{\mathcal{H}_k} := \frac{1}{|\mathcal{H}_k|} \sum_{f \in \mathcal{H}_k} A_f. \quad (6.1)$$

We normalize  $f \in \mathcal{H}_k$  so its leading Fourier coefficient is one, viz.,

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} \exp(2\pi i n z) \quad (6.2)$$

$$L(s, f) = \sum_{n=1}^{\infty} a_f(n) n^{-s} = \prod_p (1 - a_f(p) p^{-s} + p^{-2s})^{-1} \quad (6.3)$$

$$= \prod_p (1 - \alpha_f(p) p^{-s})^{-1} (1 - \alpha_f(p)^{-1} p^{-s})^{-1}, \quad \Re s > 1, \quad (6.4)$$

$$\Lambda(s, f) = 2(2\pi)^{-(s+\frac{k-1}{2})} \Gamma\left(s + \frac{k-1}{2}\right) L(s, f) = (-1)^{k/2} \Lambda(1-s, f). \quad (6.5)$$

Here  $\alpha_f(p), \alpha_f(p)^{-1}$  are the Satake parameters at  $p$ . Since we never need to look at Satake parameters simultaneously for two different primes, we usually omit  $p$  and write simply  $\alpha_f$ . It is well known that  $f$  uniquely determines an automorphic cuspidal unitary *self-dual* representation  $\pi$  of  $\mathrm{GL}_2$  with trivial central character [Gel]. Moreover  $\pi_{\infty}$  is the discrete series representation of weight  $k$ .<sup>8</sup> In what follows we will implicitly use this identification and rarely bother to talk about the representation  $\pi$  *per se*. From the completed  $L$ -function in equation (6.5) (in particular from its gamma factor) it follows that the analytic conductor of  $\mathcal{F}_k$  is  $R_k \asymp k^2$ .

Because  $\mathcal{H}_k$  consists of forms of full level,  $\pi_p$  is unramified for all  $p$ . The following orthogonality relations for the Fourier coefficients  $\{a_f(n)\}$  are crucial:

<sup>8</sup>Some authors prefer to say that this  $\pi_{\infty}$  has weight  $k-1$ . We follow the convention in [CM].



**Lemma 6.1.** *We have*

$$\frac{1}{|\mathcal{H}_k| + O(1)} \sum_{f \in \mathcal{H}_k} \frac{\zeta(2)}{L(1, \text{sym}^2 f)} a_f(m) a_f(n) = \delta(m, n) + \mathcal{E}, \quad (6.6)$$

where

$$\delta(m, n) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad (6.7)$$

and

$$\mathcal{E} = \begin{cases} O_\ell \left( \frac{(mn)^{1/4} \log mn}{k^{5/6}} \right) & \text{if } m, n \text{ have no more than } \ell \text{ factors} \\ O \left( \frac{\sqrt{mn}}{2^k} \right) & \text{if } 12\pi\sqrt{mn} \leq k. \end{cases} \quad (6.8)$$

Formula (6.6) is a consequence of the Petersson formula (see equation 2.12 of [ILS]). Note that the left-hand side of (6.6) is just the average  $\langle a_f(m) a_f(n) \rangle_{\mathcal{H}_k}$ , except for the presence of the weights  $\zeta(2)/L(1, \text{sym}^2 f)$ . This is called the *harmonic averaging* of  $a_f(m) a_f(n)$  and often makes the analysis more tractable (see [DM, ILS, Mil7, Ro]). If we were interested in bounding the order of vanishing at the central point in the family then the harmonic weights would cause difficulty (see Remarks 2.11 and 6.1 in [HM]).

Following [ILS], by additional work we can remove the harmonic weights in the 1-level density. The cost is a slight worsening of the constants  $\delta_1, \delta_2, \delta_3$  in the definition of NT-good. Alternatively, we can simply redefine the average  $\langle a_f(m) a_f(n) \rangle_{\mathcal{H}_k}$  to be given by the left-hand side of (6.6).

Note that

$$a_f(p^n) = \alpha_f^n + \alpha_f^{n-2} + \dots + \alpha_f^{-n+2} + \alpha_f^{-n}, \quad (6.9)$$

so from Definition 2.1 it follows immediately that

$$\begin{aligned} b_f(p) &= a_f(p) \\ b_f(p^2) &= a_f(p^2) - 1. \end{aligned} \quad (6.10)$$

These formulas, together with the orthogonality relations (6.6), already suffice to prove conditions (3.i) and (3.ii) of Definition 1.1 with  $\delta_1 = \delta_2 = 1/6$ , any  $\mu_1 > 1/4$ ,  $\mu_2 > 1/2$ , rank zero and, most importantly, with symmetry constant  $-1$  (note that  $a_f(p^n) = a_f(p^n) a_f(1)$  and  $\delta(1, p^n) = 0$  for  $n = 1, 2$ , whereas  $-1 = -a_f(1) a_f(1)$ ). Conditions (1) and (2) are obvious, and the Ramanujan conjecture (condition (4)) is known for these  $f$  by Deligne. We therefore recover the result from [ILS, Ro] that the family  $\{\mathcal{H}_k\}$  as  $k \rightarrow \infty$  has orthogonal 1-level density (at least for small support).

For small support of test functions, one cannot in general pinpoint the exact underlying symmetry in the orthogonal case. However, with the help of the root number (sign of the functional equation), the symmetry should be  $\text{SO}(\text{even})$  if all the functional equations have positive sign and  $\text{SO}(\text{odd})$  if all have negative sign. Determining the sign of the functional equation is most easily done through the local Langlands correspondence. Since we will be building automorphic representations starting from modular forms of full level, all finite places ( $p$  prime) contribute local root numbers equal to  $+1$ , and we only need the archimedean

local correspondence. Moreover, since the only archimedean place of  $\mathbb{Q}$  is  $\mathbb{Q}_\infty = \mathbb{R}$  we can simplify the notation a bit. The reader who wants an authoritative survey of the archimedean Langlands correspondence should read Knapp's article [Kn].

The archimedean local correspondence for  $GL_n(\mathbb{A}_\mathbb{Q})$  is a bijection  $\rho \leftrightarrow \pi_\infty$  between admissible representations  $\rho : \mathcal{W}_\mathbb{R} \rightarrow GL_n(\mathbb{C})$  and *irreducible* admissible representations  $\pi_\infty$  of  $GL_n(\mathbb{R})$ . Here  $\mathcal{W}_\mathbb{R} := \mathbb{C}^\times \cup j\mathbb{C}^\times$  (disjoint union) is a multiplicative group with  $j^2 = -1$ , and  $j$  acts on  $\mathbb{C}^\times$  by  $jzj = \bar{z}$ .  $\mathcal{W}_\mathbb{R}$  is the *Weil group* of  $\mathbb{R}$ ; it can also be identified with an obvious multiplicative subgroup of the quaternions. We will not discuss the meaning of admissibility here.

Irreducible admissible representations of  $\mathcal{W}_\mathbb{R}$  are one or two dimensional. There are two families of inequivalent one-dimensional representations, each parametrized by a complex number  $t \in \mathbb{C}$ . They are denoted  $\{[+, t]\}$  and  $\{[-, t]\}$ . Additionally, there are two-dimensional representations; they are parametrized by an integer  $k \geq 2$  and a complex number  $t \in \mathbb{C}$ . They are denoted  $[k, t]$ . There are no irreducible admissible representations of dimension greater than two, and any (finite-dimensional) admissible representation of  $\mathcal{W}_\mathbb{R}$  is fully reducible (decomposes as a direct sum of irreducible ones).

The correspondence assigns  $[+, 0]$  to the trivial representation and  $[-, 0]$  to the ‘‘sign’’ representation  $x \mapsto \text{sgn}(x) = x|x|^{-1}$  of  $GL(1, \mathbb{R})$ . The discrete-series representation of weight  $k \geq 2$  corresponds to  $[k, 0]$ . The parameter  $t \in \mathbb{C}$  parametrizes twists: either by the character  $|x|^t$  of  $GL(1, \mathbb{R})$  or by  $|\det(x)|^t$  of  $GL(2, \mathbb{R})$ .

In order to characterize the archimedean components of functorial liftings of automorphic representations, we need to understand the effect of certain operations on representations of  $\mathcal{W}_\mathbb{R}$ .

**Lemma 6.2.** *Let  $(-)^{\kappa}$  be ‘+’ for  $\kappa$  even, ‘-’ for  $\kappa$  odd. Then for all  $m \geq 1$ ,  $k > k' \geq 2$ ,  $t, t' \in \mathbb{C}$ :*

$$\wedge^2[k, t] \simeq [(-)^k, 2t] \tag{6.11}$$

$$\text{sym}^m[+, t] \simeq [+, mt] \tag{6.12}$$

$$\text{sym}^m[-, t] \simeq [(-)^m, mt] \tag{6.13}$$

$$\text{sym}^{2m+1}[k, t] \simeq \bigoplus_{\ell=0}^m [(2\ell+1)(k-1)+1, (2m+1)t] \tag{6.14}$$

$$\text{sym}^{2m}[k, t] \simeq [(-)^{m(k-1)}, 2mt] \oplus \bigoplus_{\ell=1}^m [2\ell(k-1)+1, 2mt] \tag{6.15}$$

$$[+, t] \otimes [+, t'] \simeq [-, t] \otimes [-, t'] \simeq [+, t+t'] \tag{6.16}$$

$$[+, t] \otimes [-, t'] \simeq [-, t'] \otimes [+, t] \simeq [-, t+t'] \tag{6.17}$$

$$[+, t] \otimes [k, t'] \simeq [-, t] \otimes [k, t'] \simeq [k, t+t'] \tag{6.18}$$

$$[k, t] \otimes [k', u] \simeq [k', u] \otimes [k, t] \simeq [k+k'-1, t+t'] \oplus [k-k'+1, t+t'] \tag{6.19}$$

$$[k, t] \otimes [k, t'] \simeq [2k-1, t+t'] \oplus [+, t+t'] \oplus [-, t+t'] \tag{6.20}$$

The proof is easy and we omit it. Cogdell and Michel prove (6.14) and (6.15) in [CM]. The archimedean  $\varepsilon$ - and  $L$ - (gamma) factors are as follows<sup>9</sup> (see [Kn]):

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2) \quad (6.21)$$

$$\Gamma_{\mathbb{C}}(s) := \Gamma_{\mathbb{R}}(s) \Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s} \Gamma(s) \quad (6.22)$$

$$L(s, [+ , t]) = \Gamma_{\mathbb{R}}(s+t) \quad \varepsilon([+ , t]) = 1 \quad (6.23)$$

$$L(s, [- , t]) = \Gamma_{\mathbb{R}}(s+t+1) \quad \varepsilon([- , t]) = i \quad (6.24)$$

$$L(s, [k, t]) = \Gamma_{\mathbb{C}}\left(s+t+\frac{k-1}{2}\right) \quad \varepsilon([k, t]) = i^k. \quad (6.25)$$

Finally,  $\varepsilon$ - and  $L$ -factors are multiplicative with respect to direct sums of representations of  $\mathcal{W}_{\mathbb{R}}$  (which, via the archimedean Langlands correspondence, are associated to isobaric sums of irreducible admissible representations of  $\mathrm{GL}_{n_i}(\mathbb{R})$ ), and if  $\rho \leftrightarrow \pi_{\infty}$  then  $L(s, \pi_{\infty}) = L(s, \rho)$ , and similarly for  $\varepsilon$ -factors.

With these results in hand we can easily determine the underlying symmetry type of various families obtained by functorial operations starting from  $\{\mathcal{H}_k\}$ . However, we introduce one last bit of notation: since, for  $f \in \mathcal{H}_k$ , the automorphic representation  $\pi_f$  is unitary and has trivial central character, the parameter  $t \in \mathbb{C}$  is always zero in our applications and we will adopt the following:

**Convention:** We write  $[k]$  for  $[k, 0]$ ,  $[+]$  for  $[+, 0]$ , and  $[-]$  for  $[-, 0]$ . Also define  $[1, t] := [+ , t] \oplus [- , t]$  and  $[1] := [1, 0]$ . Then equation (6.20) is the special case  $k = k'$  of (6.19). Note that  $[1, t]$  is a *reducible* two-dimensional representation of  $\mathcal{W}_{\mathbb{R}}$ .

**6.1. Families of Symmetric Powers.** We begin with the family  $\mathcal{G}_k^{(M)} = \mathrm{sym}^M \mathcal{H}_k$  for a fixed  $M \geq 1$ , and study the limit as  $k \rightarrow \infty$ . It is conjectured, and we assume this as a hypothesis, that every  $f \in \mathcal{H}_k$  has a self-dual automorphic cuspidal functorial lift  $g = \mathrm{sym}^M f \simeq \otimes' (\mathrm{sym}^M f_v)$  whose local factors  $\mathrm{sym}^M f_v$  are defined through the local Langlands correspondence (by composition with the  $M$ -th symmetric-power of the defining representation of  $\mathrm{GL}_2(\mathbb{C})$ ). This is known for  $M = 1, 2, 4$  by work of Hecke, Gelbart-Jaquet, Kim-Shahidi, and Kim [GJ, KiSh1, K]. Under this hypothesis, the family  $\mathcal{G}_k^{(M)} = \mathrm{sym}^M \mathcal{H}_k$  consists of primitive  $L$ -functions for  $\mathrm{GL}_{M+1}$ .

**Theorem 6.3.** *With the assumptions above, the family  $\mathcal{G}^{(M)}$  is NT-good with symmetry constant  $c_{\mathcal{G}} = (-1)^M$ .*

*Proof.* Firstly,  $|\mathcal{G}_k^{(M)}| = |\mathcal{H}_k| = k + O(1) \rightarrow \infty$  as  $k \rightarrow \infty$ , so the cardinality condition holds.

Now,  $f_{\infty} \simeq [k]$  for all  $f \in \mathcal{H}_k$ , so  $g_{\infty} \simeq \mathrm{sym}^M f_{\infty}$  are all *isomorphic* admissible representations of  $\mathrm{GL}_{M+1}(\mathbb{R})$  as  $f$  varies over  $\mathcal{H}_k$ . In addition, all the non-archimedean places are unramified, so the analytic conductors  $Q_g$  are completely determined by  $g_{\infty}$ , and hence are *constant* in  $\mathcal{G}_k$ .

<sup>9</sup>With respect to standard Lebesgue measure on  $\mathbb{R}$  and provided the additive character used to define the Fourier transform is  $x \mapsto e^{2\pi i x}$ .

To compute the  $\varepsilon$ - and  $\Gamma$ -factor  $L_\infty(s, \text{sym}^M f)$  we use the Langlands correspondence, Lemma 6.2, and equations (6.23)–(6.25). Recall that  $f_\infty \simeq [k]$ .

We split into two cases:  $M = 2m$  and  $M = 2m + 1$ .

- $M = 2m + 1$ .

$$\Gamma(s, \text{sym}^{2m+1} f) = \prod_{\ell=0}^m \Gamma_{\mathbb{C}} \left( s + \frac{(2\ell+1)(k-1)}{2} \right) \quad (6.26)$$

$$\varepsilon(s, \text{sym}^{2m+1} f) = \begin{cases} i^k & m \equiv 0 \pmod{4} \\ -1 & m \equiv 1 \pmod{4} \\ -i^k & m \equiv 2 \pmod{4} \\ 1 & m \equiv 3 \pmod{4}. \end{cases} \quad (6.27)$$

- $M = 2m$ .

$$\Gamma(s, \text{sym}^{2m} f) = \Gamma_{\mathbb{R}} \left( s + \frac{1-(-1)^m(k-1)}{2} \right) \prod_{\ell=1}^m \Gamma_{\mathbb{C}} \left( s + \frac{2\ell(k-1)}{2} \right) \quad (6.28)$$

$$\varepsilon(s, \text{sym}^{2m} f) = 1. \quad (6.29)$$

As explained in [DM], the contribution of the archimedean places to the analytic conductor  $Q_g$  can be read off from the gamma factors (cf., equations (2.10) and (2.11) at the end of §2): each factor  $\Gamma_{\mathbb{R}}(s + T)$  contributes a factor of  $T/2$  to the analytic conductor, and each factor  $\Gamma_{\mathbb{C}}(s + T)$  contributes  $T(T+1)/4 \asymp T^2/4$ . Equations (6.26) and (6.28) reveal that the analytic conductor is  $Q_g \asymp (k/2)^{M+1}$  if  $M$  is odd,  $Q_g \asymp (k/2)^M$  if  $M$  is even. This verifies the conductors condition. The error terms are handled using the Ramanujan bounds of Deligne.

To analyze the crucial conditions on prime sums, let us write  $\alpha_p, \alpha_p^{-1}$  for the Satake parameters of  $f_p$ . Then those of  $(\text{sym}^M f)_p$  are

$$\alpha_p^M, \alpha_p^{M-2}, \dots, \alpha_p^{-M+2}, \alpha_p^{-M}. \quad (6.30)$$

Writing  $a(p^n), b(p^n)$  for  $a_f(p^n), b_f(p^n)$  and  $B(p^n)$  for  $b_{\text{sym}^M f}(p^n)$ , we have:

$$\begin{aligned} B(p) &= \alpha_p^M + \alpha_p^{M-2} + \dots + \alpha_p^{-M+2} + \alpha_p^{-M} \\ &= a(p^M) \end{aligned} \quad (6.31)$$

$$\begin{aligned} B(p^2) &= \alpha_p^{2M} + \alpha_p^{2M-4} + \dots + \alpha_p^{-2M+4} + \alpha_p^{-2M} \\ &= a(p^{2M}) - a(p^{2M-2}) + a(p^{2M-4}) - a(p^{2M-6}) + \dots + (-1)^M a(1). \end{aligned} \quad (6.32)$$

Once again the orthogonality relations of (6.6) prove the condition on the prime sum (with rank zero) and the prime square sum, with symmetry constant  $c_{\text{sym}^M \mathcal{H}_k} = (-1)^M$ . This reveals underlying symplectic symmetry when  $M = 2m$  is even and orthogonal when  $M = 2m + 1$  is odd. In the latter case, by looking at the  $\varepsilon$ -factor (root number), we expect that the symmetry is  $\text{SO}(\text{even})$  for  $m \equiv 3 \pmod{4}$ ,  $\text{SO}(\text{odd})$  for  $m \equiv 1 \pmod{4}$ , and full orthogonal when  $m$  is even. Furthermore, in this last case the symmetry is  $\text{SO}(\text{even})$  (resp.,  $\text{SO}(\text{odd})$ ) when  $k/2$  is even (resp.,  $k/2$  is odd).  $\square$

**Remark 6.4.** *Güloğlu has obtained results for larger support for symmetric-power families [Gü].*

## 6.2. Convolutions of Symmetric Powers.

**Theorem 6.5.** *Fix  $M, N \geq 1$  and consider the families  $\mathcal{F}_k^{(M)} = \text{sym}^M \mathcal{H}_k$  and  $\mathcal{G}_{k'}^{(N)} = \text{sym}^N \mathcal{H}_{k'}$ . Assume that the convolutions  $f \times g$ ,  $f \in \mathcal{F}_k^{(M)}$ ,  $g \in \mathcal{G}_{k'}^{(N)}$  are automorphic. Let  $\mathcal{H}_{k,k'}^{(M,N)} = \mathcal{F}_k^{(M)} \times \mathcal{G}_{k'}^{(N)}$  (where, as usual, we discard the non-cuspidal  $f \times f$  when  $M = N$  and  $k = k'$ ). Then, as  $k, k' \rightarrow \infty$  in such a way that  $\log k' / \log k \rightarrow 1$ , the family  $\mathcal{H}^{(M,N)} = \mathcal{F}^{(M)} \times \mathcal{G}^{(N)}$  is NT-good with symmetry constant  $c_{\mathcal{H}^{(M,N)}} = (-1)^{M+N} = c_{\mathcal{F}^{(M)}} \cdot c_{\mathcal{G}^{(N)}}$ .*

**Remark 6.6.** *The automorphicity of the convolutions  $f \times g$  is known when  $M + N \leq 3$  [Ram, KiSh1].*

*Proof.* For simplicity we will only consider the case when  $k = k' \rightarrow \infty$ ; the proof of the general case differs from this case only in trivial details.

As in the previous section, all non-archimedean places of  $f \in \mathcal{F}_k^{(M)}$ ,  $g \in \mathcal{G}_k^{(N)}$  are unramified. We will once more split into cases when  $M, N$  are even or odd.

Using Lemma 6.2 we obtain<sup>10</sup>:

$$\text{sym}^{2m+1}[k] \otimes \text{sym}^{2n+1}[k] \simeq \bigoplus_{\substack{0 \leq \ell \leq m \\ 0 \leq \lambda \leq n}} ([2(\ell + \lambda + 1)(k - 1) + 1] \oplus [2|\ell - \lambda|(k - 1) + 1]) \quad (6.33)$$

$$\begin{aligned} \text{sym}^{2m}[k] \otimes \text{sym}^{2n}[k] &\simeq [(-)^{m+n}] \oplus \bigoplus_{1 \leq \ell \leq m} [2\ell(k - 1) + 1] \oplus \bigoplus_{1 \leq \lambda \leq n} [2\lambda(k - 1) + 1] \\ &\oplus \bigoplus_{\substack{1 \leq \ell \leq m \\ 1 \leq \lambda \leq n}} ([2(\ell + \lambda)(k - 1) + 1] \oplus [2|\ell - \lambda|(k - 1) + 1]) \end{aligned} \quad (6.34)$$

$$\begin{aligned} \text{sym}^{2m+1}[k] \otimes \text{sym}^{2n}[k] &\simeq \bigoplus_{0 \leq \ell \leq m} [(2\ell + 1)(k - 1) + 1] \\ &\oplus \bigoplus_{\substack{0 \leq \ell \leq m \\ 1 \leq \lambda \leq n}} ([ (2\ell + 2\lambda + 1)(k - 1) + 1 ] \oplus [ |2\ell - 2\lambda + 1|(k - 1) + 1 ]). \end{aligned} \quad (6.35)$$

The  $\varepsilon$ -factors are as follows:

$$\varepsilon(\text{sym}^{2m+1}[k] \otimes \text{sym}^{2n+1}[k]) = \varepsilon(\text{sym}^{2m}[k] \otimes \text{sym}^{2n}[k]) = +1 \quad (6.36)$$

$$\varepsilon(\text{sym}^{2m+1}[k] \otimes \text{sym}^{2n}[k]) = \begin{cases} (-1)^{(m+1)(n-m)+(m+1)^2 \frac{k}{2}} & m < n \\ (-1)^{\frac{(m-n)(m+n+1)}{2} + (m+1)^2 \frac{k}{2}} & m \geq n. \end{cases} \quad (6.37)$$

We omit explicitly writing down the  $\Gamma$ -factors, but observe that every term  $[a(k-1)+1]$  with  $a > 0$  contributes a factor  $\asymp \frac{a^2}{4} \cdot k^2$  to the analytic conductor  $Q_{f \times g}$ . Hence, up to an additive constant, the analytic log-conductors are  $\log Q_{f \times g} \sim 2(2m+1)(n+1) \log k$ , resp.  $2m(2n+$

<sup>10</sup>Recall  $[1] := [+]\oplus[-]$ , and observe that  $[+]\otimes[1] \simeq [-]\otimes[1] \simeq [1]$ .

1)  $\log k$ , resp.  $2(m+1)(2n+1)\log k$  corresponding to the cases (6.33), resp. (6.34), resp. (6.35) above (we assumed  $m \geq n$  in the first two cases).

By Theorem 1.3, it only remains to show that  $\mathcal{F}^{(M)} \times \mathcal{G}^{(N)}$  is NT-good. The argument above shows that the conductor condition is satisfied. When  $M = N$  and  $k = k'$  the representations  $f \times f$  are not cuspidal; hence we must discard  $O(k)$  of them. Note that  $|\mathcal{F}_k^{(M)} \times \mathcal{G}_k^{(N)}| = k^2 + O(k)$  (so the cardinality condition holds) and that possibly shrinking the family introduces error terms of size  $O(1/k)$ , which are quite admissible. Properties of cardinality and the handling of error terms (by Ramanujan) are thus valid. We need not verify the conditions on prime sums explicitly: the reason is that there are no ramified primes and conductors are essentially constant. Hence Lemma 5.1 and the argument in the proof of Theorem 1.3 suffice to prove that  $\mathcal{F}^{(M)} \times \mathcal{G}^{(N)}$  is NT-good with symmetry constant  $c_{\mathcal{F}^{(M)} \times \mathcal{G}^{(N)}} = c_{\mathcal{F}^{(M)}} \cdot c_{\mathcal{G}^{(N)}} = (-1)^{M+N}$ .

Therefore, for small support, the 1-level density of the family  $\mathcal{F}^{(M)} \times \mathcal{G}^{(N)}$  agrees with symplectic for  $M + N$  even, whereas for  $M = 2m + 1$ ,  $N = 2n$  the symmetry is orthogonal and the root number, as read off from equations (6.36) and (6.37), determines whether the underlying symmetry is SO(even) or SO(odd). Equation (6.36) holds even when  $k \neq k'$ , but the form of equation (6.37) is specific to the case  $k = k'$ .  $\square$

## 7. CONVOLVING FAMILIES OF ELLIPTIC CURVES

We now consider the interesting case of convolving two families of elliptic curves. Specifically, consider the one-parameter families

$$\begin{aligned} \mathcal{F}_N : \quad y^2 &= x^3 + A_1(T)x + B_1(T), \quad T \in [N, 2N - 1] \\ \mathcal{G}_M : \quad y^2 &= x^3 + A_2(S)x + B_2(S), \quad S \in [M, 2M - 1], \end{aligned} \quad (7.1)$$

where the polynomials  $A_1(T)$  through  $B_2(S)$  have integer coefficients. If we specialize  $T$  to  $t$  we obtain an elliptic curve  $E_{\mathcal{F}}(t)$  with discriminant  $\Delta_{\mathcal{F}}(t)$  and conductor  $C_{\mathcal{F}}(t)$ ; similarly if we specialize  $S$  to  $s$  we obtain an elliptic curve  $E_{\mathcal{G}}(s)$  with discriminant  $\Delta_{\mathcal{G}}(s)$  and conductor  $C_{\mathcal{G}}(s)$ . The conductors are products of powers of primes dividing the discriminants. It is known (see [BCDT, TW, Wi]) that the  $L$ -function of an elliptic curve of conductor  $C$  agrees with a weight-2 cuspidal newform of level  $C$ . Thus if  $E_i$  are elliptic curves with conductors  $C_i$  and associated newforms  $f_i$ , by  $L(s, E_1 \times E_2)$  we mean the Rankin-Selberg convolution  $L(s, f_1 \times f_2)$ , which is a  $\mathrm{GL}_4$   $L$ -function. The arithmetic conductor  $Q(f_1 \times f_2)$  of such  $L(s, f_1 \times f_2)$  is an integer satisfying

$$(C_1 C_2)^2 / (C_1, C_2)^4 \leq Q(f_1 \times f_2) \leq (C_1 C_2)^2 / (C_1, C_2), \quad (7.2)$$

where  $(C_1, C_2)$  is the greatest common divisor of  $C_1$  and  $C_2$ ; see for example [HaMi]. We often write  $Q(C_1, C_2)$  for  $Q(f_1 \times f_2)$ .

We are interested in the behavior of  $\mathcal{F}_N \times \mathcal{G}_M$  as  $N$  and  $M$  tend to infinity. The gamma factors for these  $\mathrm{GL}_4$   $L$ -functions depend neither on the specific curve nor on the family. As

such, since we need only identify the analytic conductor up to a constant, we may use the integer  $Q(C_1, C_2)$  as the analytic conductor.

We normalize the low lying zeros for the convolution  $L$ -function by the average of the logarithms of the analytic conductors. Thus, we set

$$\log R_{N,M} := \frac{1}{NM} \sum_{t=N}^{2N-1} \sum_{s=M}^{2M-1} \log Q(C_{\mathcal{F}}(t), C_{\mathcal{G}}(s)). \quad (7.3)$$

We need  $R_{N,M}$  to tend to infinity with  $N$  and  $M$ . A weak estimate on the size of  $Q(C_{\mathcal{F}}(t), C_{\mathcal{G}}(s))$ , namely that the average log-conductor in (7.3) tends to infinity with  $N$  and  $M$ , suffices for our purposes.

To show this requires a few basic facts about elliptic curves. An elliptic curve  $E : y^2 = x^3 + a_4x + a_6$  has discriminant  $\Delta = -16(4a_4^3 + 27a_6^2)$  and  $j$ -invariant  $j = 3a_4^3/(4a_4^3 + 27a_6^2)$ ; it is also convenient to set  $c_4 = -48a_4$  and  $c_6 = -864a_6$ . Let  $R$  be the ring of integers for some local field  $K$ ;  $K$  is a local field which is complete with respect to a discrete valuation  $v$ . Let  $\mathcal{M} = \{x \in K : v(x) > 0\}$  be the maximal ideal of  $R$ , and let  $k = R/\mathcal{M}$  be the residue field. If  $a_i \in R$  and  $v(c_4) < 4$  or  $v(c_6) < 6$ , then the equation for the elliptic curve is minimal with respect to the valuation  $v$ .

**Theorem 7.1.** *Notation as above, assume that there are non-constant monic integral polynomials  $f_1(x)$  and  $g_1(x)$  such that  $f_1(x)$  divides  $\Delta_{\mathcal{F}}(x)$  and  $g_1(x)$  divides  $\Delta_{\mathcal{G}}(x)$ . To simplify the analysis, assume  $f_1(x)$  does not divide either  $c_{\mathcal{F},4}(x)$  or  $c_{\mathcal{F},6}(x)$  (and similarly for  $g_1(x)$ ). Define the average log-conductor by (7.3). If  $j_{\mathcal{F}}(T)$  and  $j_{\mathcal{G}}(S)$  are both non-constant, then for some  $a > 0$*

$$\frac{\log NM}{(\log \log \min(N, M))^a} \ll_{\mathcal{F}, \mathcal{G}} \log R_{N,M} \ll_{\mathcal{F}, \mathcal{G}} \log NM. \quad (7.4)$$

The proof follows from basic facts on solutions to Diophantine equations and properties of elliptic curves, and is given in Appendix A.

The following observation ensures that, except for a negligible fraction of the time, the  $L$ -functions in the convolved family are good (i.e., primitive).

**Lemma 7.2.** *Assume  $j_{\mathcal{F}}(T)$  and  $j_{\mathcal{G}}(S)$  are non-constant. The Rankin-Selberg convolution of  $E_{\mathcal{F}}(t)$  and  $E_{\mathcal{G}}(s)$  is imprimitive for at most  $O(\min(N, M))$  of the  $NM$  pairs  $(s, t)$ .*

*Proof.* Without loss of generality assume  $N \leq M$ . If for some pair  $(s, t)$  we have  $E_{\mathcal{F}}(t)$  and  $E_{\mathcal{G}}(s)$  are associated to the same weight-2 cuspidal newform, then the Rankin-Selberg convolution will be imprimitive (and divisible by  $\zeta(s)$ ); call such a pair *bad*. If two elliptic curves are isomorphic, then they have the same  $j$ -invariant. Thus for a bad pair,

$$j_{\mathcal{F}}(t) = \frac{3A_1(t)^3}{4A_1(t)^3 + 27B_1(t)^2} = \frac{3A_2(s)^3}{4A_2(s)^3 + 27B_2(s)^2} = j_{\mathcal{G}}(s). \quad (7.5)$$

As we are assuming  $j_{\mathcal{F}}(t)$  and  $j_{\mathcal{G}}(s)$  are non-constant, for each fixed  $t$  there are only finitely many solutions to  $j_{\mathcal{G}}(s) = j_{\mathcal{F}}(t)$  (the number is bounded by the degrees of  $A_2(s)^3$  and

$4A_2(s)^3 + 27B_2(s)^2$ ). Thus of the  $NM$  pairs  $(s, t)$ , at most  $O(N)$  of the pairs have a Rankin-Selberg convolution divisible by the Riemann zeta function. As the only non-primitive  $L$ -functions  $L(s, f \times g)$  for  $f, g$  primitive weight-2 cuspidal newforms of levels  $N_1$  and  $N_2$  arise when  $f = g$ , the remaining pairs yield primitive  $L$ -functions.  $\square$

We now prove our main result about convolving two families of elliptic curves.

**Theorem 7.3.** *Consider two one-parameter families of elliptic curves (elliptic surfaces over  $\mathbb{Q}$ ):*

$$\begin{aligned} \mathcal{E}_{\mathcal{F}} : y^2 &= x^3 + A_1(T)x + B_1(T) \\ \mathcal{E}_{\mathcal{G}} : y^2 &= x^3 + A_2(S)x + B_2(S). \end{aligned} \tag{7.6}$$

Let  $\mathcal{F}_N$  be the specialization of  $\mathcal{E}_{\mathcal{F}}$  with  $t \in [N, 2N - 1]$ ,  $\mathcal{G}_M$  be the specialization of  $\mathcal{E}_{\mathcal{G}}$  with  $s \in [M, 2M - 1]$ , and set  $\mathcal{F} = \cup \mathcal{F}_N$  and  $\mathcal{G} = \cup \mathcal{G}_M$ . Assume  $\log N \ll \log M \ll \log N$  and

- (1) the first family is an elliptic curve over  $\mathbb{Q}(T)$  of rank  $r_{\mathcal{F}}$  and non-constant  $j_{\mathcal{F}}(T)$ ;
- (2) the second family is an elliptic curve over  $\mathbb{Q}(S)$  of rank  $r_{\mathcal{G}}$  and non-constant  $j_{\mathcal{G}}(S)$ ;
- (3) the average log-conductor of  $\mathcal{F}_N \times \mathcal{G}_M$  satisfies (7.4);
- (4) the Fourier coefficients of each family satisfy either (1.2) or (1.3).

Then Theorem 1.3 holds for the family  $\mathcal{F} \times \mathcal{G}$ ; the symmetry is symplectic and the rank is 0.

**Remark 7.4.** *Rosen and Silverman [RoSi] show that (1.2) is a consequence of Tate's conjecture [Ta]: Let  $\mathcal{E}/\mathbb{Q}$  be an elliptic surface and  $L_2(\mathcal{E}, s)$  be the  $L$ -series attached to  $H_{\text{ét}}^2(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$ .  $L_2(\mathcal{E}, s)$  has a meromorphic continuation to  $\mathbb{C}$  and  $-\text{ord}_{s=1} L_2(\mathcal{E}, s) = \text{rank } NS(\mathcal{E}/\mathbb{Q})$ , where  $NS(\mathcal{E}/\mathbb{Q})$  is the  $\mathbb{Q}$ -rational part of the Néron-Severi group of  $\mathcal{E}$ . Further,  $L_2(\mathcal{E}, s)$  does not vanish on the line  $\text{Re}(s) = 1$ .*

*Tate's conjecture is known for rational surfaces<sup>11</sup>. Theorem 7.3 should be true for families with constant  $j$ -invariants; however, for such families Michel's result on the average second moments of the Fourier coefficients is not available, and one must show by direct calculation that (1.4) holds.*

*Proof.* For the  $L$ -function attached to  $E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s)$ , the explicit formula (2.10) becomes

$$\begin{aligned} \sum_{\ell} \phi \left( \gamma_{E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s), \ell} \frac{\log R_{N,M}}{2\pi} \right) &= \frac{A_{E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s)}}{\log R_{N,M}} \widehat{\phi}(0) \\ &\quad - 2 \sum_p \sum_{\nu=1}^{\infty} \widehat{\phi} \left( \frac{\nu \log p}{\log R_{N,M}} \right) \frac{b_{E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s)}(p^{\nu}) \log p}{p^{\nu/2} \log R_{N,M}}, \end{aligned} \tag{7.7}$$

where

$$A_{E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s)} = \log Q(E_{\mathcal{F}}(t), E_{\mathcal{G}}(s)) + o(1). \tag{7.8}$$

The  $o(1)$  error follows from Theorem 7.1, where we showed  $R_{N,M}$  cannot be too small. Note (7.7) may be slightly off in that, if  $E_{\mathcal{F}}(t) = E_{\mathcal{G}}(s)$ , then the  $L$ -function associated to  $E_{\mathcal{F}}(t) \times$

<sup>11</sup>An elliptic surface  $y^2 = x^3 + A(T)x + B(T)$  is rational if and only if one of the following is true: either  $0 < \max\{3 \deg A, 2 \deg B\} < 12$  or  $3 \deg A = 2 \deg B = 12$  and  $\text{ord}_{T=0} T^{12} \Delta(T^{-1}) = 0$ .



$E_G(s)$  is imprimitive. We would have a superposition of zeros of two primitive  $L$ -functions, one of which has a pole. Fortunately, by Lemma 7.2, this occurs for at most  $O(\min(N, M))$  of the  $NM$  pairs; as we divide by  $NM$  this contribution is negligible.<sup>12</sup>

Thus summing (7.7) over  $t \in [N, 2N - 1]$  and  $s \in [M, 2M - 1]$ , and recalling the definition of the 1-level density and the average log-conductor, we find

$$D_{1, \mathcal{F}_N \times \mathcal{G}_M}(\phi) = \widehat{\phi}(0) + o(1) - \frac{2}{NM} \sum_{t=N}^{2N-1} \sum_{s=M}^{2M-1} \sum_p \sum_{\nu=1}^{\infty} \widehat{\phi} \left( \frac{\nu \log p}{\log R_{N,M}} \right) \frac{b_{E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s)}(p^\nu) \log p}{p^{\nu/2} \log R_{N,M}}. \quad (7.9)$$

By Lemma 5.1,  $b_{E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s)}(p^\nu) = b_{E_{\mathcal{F}}(t)}(p^\nu) \cdot b_{E_{\mathcal{G}}(s)}(p^\nu)$  when  $L(s, E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s))$  is primitive. We use this for all  $E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s)$ , as the  $O(\min(N, M))$  instances where this is false lead to a difference that is  $o(1)$ .

There is trivially no contribution in (7.9) for  $\nu \geq 3$ . As for each  $E_{\mathcal{F}}(t) \times E_{\mathcal{G}}(s)$  the conductor is at most  $(NM)^b$  for some  $b$ , at primes dividing the conductor if necessary we may adjust the coefficients at  $p$  and  $p^2$  and introduce an error at most  $o(1)$ . This is because the worst case is if  $(NM)^b$  is the product of the first  $\ell$  primes, where  $p_\ell \ll \log(NM)^b$ . This would lead to a sum bounded by

$$\frac{1}{\log R_{N,M}} \sum_{p \leq \log(NM)^b} \frac{1}{\sqrt{p}} \ll \frac{\sqrt{\log((NM)^b)}}{\log R_{N,M}} = o(1), \quad (7.10)$$

where the last inequality follows from the lower bound for the average log-conductor.

The proof is completed by showing our family is NT-good. We must check the four conditions of Definition 1.1. The first is straightforward (with Lemma 7.2 a key ingredient), the second (on the size of the log-conductors) follows from our assumption that the average log-conductor satisfies (7.4). The fourth is an easy consequence of the Hasse bound. We are left with the third condition, which concerns the sums over primes and squares of primes. We handle the prime sums first.

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<sup>12</sup>Here we are using Corollary 1.7, which says it suffices to show there is a power savings in the number of bad pairs. Alternatively, instead of using Lemma 7.2 we could show that the multiplicity of any elliptic curve in our parametrizations is  $O(1)$ .

The needed result for the sum of the Fourier coefficients at the primes is true because (1.3) is satisfied with  $r = 0$ . To see this, note

$$\begin{aligned}
& \frac{1}{NM} \sum_{t=N}^{2N-1} \sum_{s=M}^{2M-1} \sum_p \widehat{\phi} \left( \frac{\log p}{\log R_{N,M}} \right) \frac{b_{E_{\mathcal{F}(t)} \times E_{\mathcal{G}(s)}}(p) \log p}{\sqrt{p} \log R_{N,M}} \\
&= \frac{1}{NM} \sum_{t=N}^{2N-1} \sum_{s=M}^{2M-1} \sum_p \widehat{\phi} \left( \frac{\log p}{\log R_{N,M}} \right) \frac{b_{E_{\mathcal{F}(t)}}(p) b_{E_{\mathcal{G}(s)}}(p) \log p}{\sqrt{p} \log R_{N,M}} \\
&= \sum_p \widehat{\phi} \left( \frac{\log p}{\log R_{N,M}} \right) \left[ \frac{1}{N} \sum_{t=N}^{2N-1} b_{E_{\mathcal{F}(t)}}(p) \right] \left[ \frac{1}{M} \sum_{s=M}^{2M-1} b_{E_{\mathcal{G}(s)}}(p) \right] \frac{\log p}{\sqrt{p} \log R_{N,M}}.
\end{aligned} \tag{7.11}$$

We analyze the  $t$ -sum; the  $s$ -sum follows similarly. Let  $a_{E_{\mathcal{F}(t)}}(p) = b_{E_{\mathcal{F}(t)}}\sqrt{p}$ ; by Hasse's bound we have  $|a_{E_{\mathcal{F}(t)}}(p)| \leq 2\sqrt{p}$ , and these correspond to the associated  $L$ -function having functional equation  $u \rightarrow 2 - u$ . Let  $A_p(\mathcal{E}_{\mathcal{F}}) = \frac{1}{p} \sum_{t \bmod p} a_{E_{\mathcal{F}(t)}}(p)$ . We have

$$\frac{1}{N} \sum_{t=N}^{2N-1} b_{E_{\mathcal{F}(t)}}(p) = \frac{1}{N} \left( \frac{N}{p} \sum_{t \bmod p} \frac{a_{E_{\mathcal{F}(t)}}(p)}{\sqrt{p}} + O(p) \right) = \frac{A_p(\mathcal{E}_{\mathcal{F}})}{\sqrt{p}} + O\left(\frac{p}{N}\right). \tag{7.12}$$

The  $O(p/N)$  term (and the corresponding  $O(p/M)$  term from the  $s$ -sum) lead to  $o(1)$  contributions if  $\widehat{\phi}$  has suitably restricted support. We are left with the  $A_p(\mathcal{E}_{\mathcal{F}})A_p(\mathcal{E}_{\mathcal{G}})/p$  term. Thus (7.11) becomes

$$\sum_p \widehat{\phi} \left( \frac{\log p}{\log R_{N,M}} \right) \frac{A_p(\mathcal{E}_{\mathcal{F}})A_p(\mathcal{E}_{\mathcal{G}}) \log p}{p^{3/2} \log R_{N,M}} + o(1). \tag{7.13}$$

As  $A_p(\mathcal{E}_{\mathcal{F}})$  and  $A_p(\mathcal{E}_{\mathcal{G}})$  are bounded independent of  $p$  (see [De], or [Mic] for an explicit bound in terms of the curves), the above sum is  $O(1)$  and hence negligible upon division by  $NM$ .

We are left with showing that (1.5) (the second part of the third condition of Definition 1.1) holds, i.e., analyzing the prime square sums (the sums of  $b_{E_{\mathcal{F}(t)}}(p^2)$  over  $t$  and  $b_{E_{\mathcal{G}(s)}}(p^2)$  over  $s$ ). As we have assumed  $j_{\mathcal{F}}(T)$  and  $j_{\mathcal{G}}(S)$  are non-constant, this follows immediately from work of Michel [Mic], who showed that for a one-parameter family  $\mathcal{F}$  over  $\mathbb{Q}(T)$  with non-constant  $j_{\mathcal{F}}(T)$  that

$$\sum_{t \bmod p} a_{E_{\mathcal{F}(t)}}(p)^2 = p^2 + O(p^{3/2}). \tag{7.14}$$

The exponent in the error term cannot be improved in general, and may be related to family specific lower order correction terms to the 1-level density; see [Mil3]. From (5.16) and our

normalizations<sup>13</sup> we have  $b_{E_{\mathcal{F}(t)}}(p^2) = p^{-1}a_{E_{\mathcal{F}(t)}}(p)^2 - 2$ , which implies

$$\begin{aligned} \sum_{t=N}^{2N-1} b_{E_{\mathcal{F}(t)}}(p^2) &= \frac{N}{p} \sum_{t \bmod p} b_{E_{\mathcal{F}(t)}}(p^2) + O(p) \\ &= \frac{N}{p} \sum_{t \bmod p} \frac{a_{E_{\mathcal{F}(t)}}(p)^2}{p} - 2N + O(p) \\ &= -N + O(p^2) = -|\mathcal{F}_N| + O(p). \end{aligned} \tag{7.15}$$

Thus (1.5) holds with  $c_{\mathcal{F}} = -1$ ; an analogous result holds for sums of  $b_{E_{\mathcal{G}(s)}}(p^2)$ .

Therefore the two families have orthogonal symmetry (as was already known), but the convolution family has symplectic symmetry ( $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}} = (-1)^2 = 1$ ).  $\square$

**Remark 7.5.** *The conditions of Theorem 7.3 are quite weak, and are easily seen to be satisfied in many cases of interest (for example, by many of the families studied in [ALM, Fe]).*

## APPENDIX A. AVERAGE LOG-CONDUCTORS FOR ELLIPTIC CURVE FAMILIES

We prove Theorem 7.1. The upper bound follows trivially from (7.2) and bounds relating the discriminant of an elliptic curve to its conductor. We prove the lower bound through a series of lemmas. We first introduce some notation. Let  $f_1(x), \dots, f_{k_1}(x)$  be the distinct monic irreducible factors of  $\Delta_{\mathcal{F}}(x)c_{\mathcal{F},4}(x)c_{\mathcal{F},6}(x)$ , and let  $g_1(x), \dots, g_{k_2}(x)$  be the distinct monic irreducible factors of  $\Delta_{\mathcal{G}}(x)c_{\mathcal{G},4}(x)c_{\mathcal{G},6}$ . By relabeling if necessary, we may assume  $f_1(x) | \Delta_{\mathcal{F}}(x)$ , and similarly  $g_1(x) | \Delta_{\mathcal{G}}(x)$ .

Further, we may assume *all* of the  $k_1 + k_2$  polynomials are relatively prime. If some  $f_i(x)$  and  $g_j(x)$  were not relatively prime, then we could find a fixed  $x'$  such that  $g_1(x+x'), \dots, g_{k_2}(x+x')$  are relatively prime (as functions of  $x$ ) to the  $f_i(x)$ 's. Thus instead of considering the interval  $[M, 2M]$  we would consider the interval  $[M-c, 2M-c]$ , which for  $M$  large is still approximately  $[M, 2M]$ . Hence we may assume  $f_1(x) \nmid \Delta_{\mathcal{G}}(x)$  and  $g_1(x) \nmid \Delta_{\mathcal{F}}(x)$ , and without loss of generality we may assume  $N \leq M$ .

The proof is completed by showing that, for some constants  $\epsilon, \delta, a > 0$ , at least  $\epsilon NM / (\log \log N)^a$  of the  $L(u, E_{\mathcal{F}(t)} \times E_{\mathcal{G}(s)})$  have arithmetic conductor at least  $(NM)^\delta$ . We do this by showing for at least  $\epsilon NM / (\log \log N)^a$  of the pairs  $(t, s)$  that we can find a number at least  $N^\delta$  dividing  $f_1(t)$  and the conductor  $C_{\mathcal{F}}(t)$  but not any other  $f_i(t)$ , and a number at least  $M^\delta$  (relatively prime to the number at least  $N^\delta$ ) dividing  $g_1(s)$  and  $C_{\mathcal{G}}(s)$  but not any other  $g_i(s)$ . As the arithmetic conductor is an integer, it will then be divisible by at least  $(NM)^\delta$ , which implies the lower bound in (7.4). We do this through the following series of lemmas.

**Lemma A.1.** *There exists an integer  $c$  (a product of distinct primes) and an integer  $r$  such that, for  $i \neq j$ , for a positive fraction of  $t \in [N, 2N]$  we have  $(f_i(t), f_j(t)) | c^r$  (and similarly for the  $g$ 's).*

<sup>13</sup>Remember  $b_{E_{\mathcal{F}(t)}}(p)\sqrt{p} = a_{E_{\mathcal{F}(t)}}(p)$ . We must be careful in our normalizations, since we wish our elliptic curve  $L$ -functions to have a functional equation as  $u \rightarrow 1-u$  (not as  $u \rightarrow 2-u$ ).

*Proof.* Let  $i \neq j$ . By the Euclidean algorithm, there exists a  $c_{ij}$  (independent of  $x$ ) such that if  $p|(f_i(x), f_j(x))$  then  $p|c_{ij}$ . Let  $c_f$  be the product of 6 and the prime divisors  $p \geq 5$  of the  $c_{ij}$ 's. Choose an  $x_0$  such that  $f_i(x_0) \neq 0$  for all  $i$ . Let  $r_f$  be the largest integer such that if  $p|c_f$  then  $p^{r_f+1}$  divides none of the  $f_i(x_0)$ . Then for all  $i \neq j$  the greatest common divisor of  $f_i(c_f^{r_f+1}x + x_0)$  and  $f_j(c_f^{r_f+1}x + x_0)$  divides  $c_f^{r_f}$ . We similarly construct  $c_g$  and  $r_g$  so that the greatest common divisors of the  $g_i$ 's divides  $c_g^{r_g}$ . Let  $c$  equal the product of the prime divisors of  $c_f c_g$  and  $r = \max(r_f, r_g)$ . We change variables, sending  $t \rightarrow c^{r+1}t + x_0$ . A positive fraction of  $t \in [N, 2N]$  satisfy this condition. We similarly change  $s \rightarrow c^{r+1}s + s_0$ . For ease of exposition we denote these polynomials by  $\tilde{f}_i$  and  $\tilde{g}_j$ ; thus  $\tilde{f}_i(x) = f_i(c^{r+1}x + x_0)$ .  $\square$

It is possible that  $\tilde{f}_i(x)$  is divisible by a fixed square (or higher power) for all  $x$ ; for example,  $x^4 - x^2 + 20$  is always divisible by 4. Further, if  $\Delta_{\mathcal{F}}(x)c_{\mathcal{F},4}(x)c_{\mathcal{F},6}(x) = a_{\mathcal{F}} \prod_i f_i(x)$ , for some  $x$  an  $\tilde{f}_i(x)$  could share a factor with  $a_{\mathcal{F}}$ . The following lemma handles such primes.

**Lemma A.2.** *Notation as in Lemma A.1, let  $C$  be the product of all numbers that divide an  $\tilde{f}_i(x)$  for all  $x$ , a  $\tilde{g}_j(x)$  for all  $x$ , or  $a_{\mathcal{F}}a_{\mathcal{G}}$ . Then there exists an integer  $m$  such that, for a positive fraction of  $t \in [N, 2N]$ , Lemma A.1 holds and if  $p|C$  then  $p^m$  does not divide  $f_i(t)$  for all  $i$  (and similarly for the  $g$ 's).*

*Proof.* Let  $\tilde{f}_i$  and  $\tilde{g}_j$  be as in Lemma A.1. Choose an  $x_1$  such that  $\tilde{f}_i(x_1)$  and  $\tilde{g}_j(x_1)$  are non-zero for all  $i$  and  $j$ . Arguing as before, after a simple linear change of variables we can ensure that at most a fixed power of  $C$  divides our polynomials for any  $x$ . Specifically, consider  $\tilde{f}_i(C^m x + x_1)$ ; for  $m$  sufficiently large, if  $p|C$  then  $p^m \nmid \tilde{f}_i(C^m x + x_1)$  (and the same is true for the  $\tilde{g}_j$ 's). Let  $\hat{f}_i(x) = \tilde{f}_i(C^m x + x_1)$  (and similarly for  $\hat{g}_j$ ).  $\square$

We have shown that for a positive fraction of all  $t \in [N, 2N]$  and  $s \in [M, 2M]$ : (i) the greatest common divisor of the  $\hat{f}_i(t)$ 's is at most  $c^r$  and the greatest common divisor of the  $\hat{g}_j(s)$ 's is at most  $c^r$ ; (ii) the product of all the squares or factors of  $a_{\mathcal{F}}a_{\mathcal{G}}$  that divide a  $\hat{f}_i(t)$  for all  $t$  is at most  $C^m$  (and similarly for  $\hat{g}_j(s)$ ). We would like to say the arithmetic conductor is at least  $\hat{f}_1(t)\hat{g}_1(s)$ , except there are two problems: (i) we must show  $\hat{f}_1(t)$  divides the conductor of  $E_{\mathcal{F}}(t)$  (and similarly for  $\hat{g}_1(s)$  and  $E_{\mathcal{G}}(s)$ ); (ii) we must show  $(\hat{f}_1(t), \hat{g}_1(s))$  is small. We handle (i) first.

**Lemma A.3.** *Let  $d = \max(\sum_i \deg f_i, \sum_j \deg g_j) + 2$ . For a positive fraction of  $t \in [N, 2N]$  and  $s \in [M, 2M]$ , the results of Lemmas A.1 and A.2 hold, the conductor of  $E_{\mathcal{F}}(t)$  is  $\gg N^{1/d}$  and the conductor of  $E_{\mathcal{G}}(s)$  is  $\gg M^{1/d}$*

*Proof.* Notation as in Lemmas A.1 and A.2 and conditions as in Theorem 7.1, we show that for a positive fraction of the time that the conductor of  $E_{\mathcal{F}}(t)$  is  $\gg N^{1/d}$ . Recall the following basic facts (see for example [Nag]) for an integral polynomial  $D(t)$  of degree  $k$  and discriminant  $\delta$ :

- (1) Let  $p$  be a prime not dividing the coefficient of  $x^k$ . Then  $D(t) \equiv 0 \pmod{p}$  has at most  $k$  incongruent solutions.

- (2) Suppose  $p \nmid \delta$ . Then the number of incongruent solutions of  $D(t) \equiv 0 \pmod p$  equals the number of incongruent solutions of  $D(t) \equiv 0 \pmod{p^\alpha}$ .

Note that if the discriminant of  $h(x)$  is  $\delta$ , then the discriminant of  $h(ax+b)$  is  $a^n\delta$  for some  $n$ . Let  $D$  be the product of the prime divisors of the discriminants and leading coefficients of all the  $\widehat{f}_i$ 's and  $\widehat{g}_j$ 's, as well as any missing primes at most  $d$ . We make one last change of variables: for sufficiently large  $n$  consider

$$F_i(x) = \widehat{f}_i(D^n x + x_2), \quad G_j(x) = \widehat{g}_j(D^n x + x_2), \quad (\text{A.1})$$

where  $x_2$  is chosen so that all  $\widehat{f}_i(x_2)$  and  $\widehat{g}_j(x_2)$  are non-zero. The advantage is that the degree of divisibility of  $F_i(x)$  (resp.,  $G_j(x)$ ) by primes dividing the discriminants,  $c_{\mathcal{F},4}(x)$  and  $c_{\mathcal{F},6}(x)$  (resp.,  $c_{\mathcal{G},4}(x)$  and  $c_{\mathcal{G},6}(x)$ ), leading coefficients or at most  $d$  is bounded independent of  $x$ , say by  $k$ . It is now immediate that, for a positive fraction of  $x$ ,  $F_i(x)$  has a  $d^{\text{th}}$  power free factor  $\gg N^{\deg F_i/d}$ . To see this, let  $\nu_{F_i}(p^d)$  denote the number of solutions to  $F_i(x) \equiv 0 \pmod{p^d}$ . For  $p \nmid D$ ,  $p$  does not divide the discriminant of  $F_i$  and thus  $\nu_{F_i}(p^d) = \nu_{F_i}(p) \leq \deg F_i$ . Thus the fraction of  $t$  giving  $F_i(t)$   $d$ -power free (except for divisors of  $D$ ) is at least

$$\prod_{p \nmid D} \left(1 - \frac{\nu_{F_i}(p^d)}{p^d}\right) \geq \prod_{p \nmid D} \left(1 - \frac{\deg f_i}{p^d}\right) \geq \prod_{p \nmid D} \left(1 - \frac{1}{p^{d-1}}\right). \quad (\text{A.2})$$

As  $d$  was chosen to be at least 3, this last factor is larger than  $\prod_p (1 - p^{-2}) = 6/\pi^2$ . By our linear change of variables (how we defined the  $F_i$ ), the number of times a  $p \mid D$  divides  $F_i(x)$  is bounded independent of  $x$  and  $i$ , say by  $k$ . Thus, for a positive fraction of  $t$ ,  $F_i(t)$  has a  $d$  power free part at least  $F_i(t)^{1/d}/D^k$ . As the greatest common divisors of any two of the  $F_i$  is at most  $c^r$ , for a positive fraction of  $t$  we have  $F_1(t)$  has a  $d$  power free factor of size at least  $F_1(t)^{1/d}/c^r D^k$  that is relative prime to the  $F_i(t)$  for  $i \neq 1$ .

We need only show this factor (which is at least  $F_1(t)^{1/d}/c^r D^k$ ) divides the conductor. This follows by showing the conductor of the elliptic curve  $y^2 = x^3 + A_1(t)x + B_1(t)$  is minimal for each  $p \nmid cCD$  that divides  $F_1(t)^{1/d}$ . This follows from our assumption that  $j_{\mathcal{F}}(T)$  is not constant, as this implies that  $c_{\mathcal{F},4}(x)$  and  $c_{\mathcal{F},6}(x)$  are not identically zero. Thus neither are  $C_{\mathcal{F},4}(x)$  or  $C_{\mathcal{F},6}(x)$  (where we have used the obvious notation to represent the linear change of variables). By assumption (see the conditions of Theorem 7.1), as the irreducible polynomial factors of  $c_{\mathcal{F},4}(x)$  and  $c_{\mathcal{F},6}(x)$  were included in our list of the  $f_i$ 's, and we assumed  $f_1(x)$  is relatively prime to either  $c_{\mathcal{F},4}(x)$  or  $c_{\mathcal{F},6}(x)$ , for  $p \nmid cCD$  with  $p \mid F_1(t)$ ,  $p$  cannot divide both  $C_{\mathcal{F},4}(x)$  and  $C_{\mathcal{F},6}(x)$ . Thus the elliptic curve is minimal for such primes  $p$ , implying the conductor is at least  $F_1(t)^{1/d}/c^r C^m D^k$ .

Thus as  $N \rightarrow \infty$ , for a positive fraction of  $t$  the conductor of  $E_{\mathcal{F}}(t)$  is  $\gg N^{\deg F_1/d} \gg N^{1/d}$ ; an analogous statement holds a positive fraction of the time for the conductor of  $E_{\mathcal{G}}(s)$ . We call such  $t$  and  $s$  *good*.  $\square$

**Remark A.4.** We chose  $d = \max(\sum_i \deg f_i, \sum_j \deg g_j) + 2$  and not  $\max(\max_i \deg f_i, \max_j \deg g_j) + 2$  because of Lemma A.5.

The following lemma completes the proof of Theorem 7.1.

**Lemma A.5.** *Notation as in Theorem 7.1, for some  $\epsilon, a > 0$  for at least  $\epsilon NM/(\log \log N)^a$  of the pairs  $(t, s) \in [N, 2N] \times [M, 2M]$  the results of Lemmas A.1 through A.3 hold, and the greatest common divisor of  $\prod_i F_i(t)$  and  $\prod_j G_j(s)$  is bounded independent of  $t$  and  $s$ .*

*Proof.* Consider the positive fraction of  $t$  and  $s$  that are good. We must make sure that each such  $G_j(s)$  is essentially relatively prime to the  $F_i(t)$ . If so, then since the arithmetic conductor is an integer it would have to be  $\gg N^{1/d}M^{1/d}$  (remember the arithmetic conductor comes from the arithmetic conductors of  $E_{\mathcal{F}}(t)$  and  $E_G(s)$ , and these are  $\gg N^{1/d}$  and  $\gg M^{1/d}$ ). For a good  $t$ , the worst case for common factors of  $\prod_i F_i(t)$  and  $\prod_j G_j(s)$  is when  $\prod_i F_i(t)$  is the product of the first  $\ell$  primes. We can easily handle the bounded contributions from  $c$  (Lemma A.1),  $C$  (Lemma A.2) or  $D$  (see the proof of Lemma A.3), and thus we need only investigate primes  $p$  such that  $p > D$  and  $p \nmid cC$ . Letting  $\mu = \deg \Delta_{\mathcal{F}}(x)C_{\mathcal{F},4}(x)C_{\mathcal{F},6}(x)$ , the product of the  $F_i(t)$ 's is  $\ll N^\mu$ . Thus

$$\begin{aligned} \prod_{p \leq p_\ell} p &\ll N^\mu \\ \sum_{p \leq p_\ell} \log p &\ll \mu \log N \Rightarrow p_\ell \ll \mu \log N. \end{aligned} \tag{A.3}$$

We may need to discard some good  $s$  because a  $G_j(s)$  is not relatively prime to  $p_1 \dots p_\ell$ . Thus, even though we are going to use  $F_1(t)$  and  $G_1(s)$ , we must make sure that there are no large common factors of  $\prod_i F_i(t)$  and  $\prod_j G_j(s)$ , as otherwise the conductor of the Rankin-Selberg convolution could be reduced (see the division by the greatest common divisor in the left hand side of (7.2)). As  $d > \sum_j \deg g_j$ , we have  $\sum_j \mu_{G_j}(p) < d$  for  $p \nmid D$ ; this allows us to obtain the needed estimate.

We have already handled  $p \leq d$  and  $p$  dividing a discriminant in our construction of the good  $s$ . For each  $j$ , the product in (A.2) for  $G_1(s)$  is modified by a factor no worse than

$$\begin{aligned} \prod_{\substack{d < p \leq p_\ell \\ p \nmid cCD}} \left( 1 - \frac{\sum_j \nu_{G_j}(p)}{p} \right) &\geq \prod_{d < p \leq p_\ell} \left( 1 - \frac{d}{p} \right) \\ &\gg \exp \left( -2009d \cdot \log \sum_{d < p \leq p_\ell} \frac{1}{p} \right) \\ &\gg (\log p_\ell)^{-2009d} \gg (\log \log N)^{-2009d} \end{aligned} \tag{A.4}$$

(the last bound is Mertens' theorem, see for example [Da]). We therefore obtain a  $d$  power free factor of the conductor of  $E_G(s)$  whose common factors with  $F_1(t) \dots F_{k_1}(t)$  is bounded by  $c^r D^k$ .  $\square$

This completes the proof of Theorem 7.1.  $\square$

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