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The effect of feedback on the intensity of the illusion of linearity in high-school students' solving of geometry problems

JELENA PAIĆ-ANTUNOVIĆ and VESNA VLAHOVIĆ-ŠTETIĆ

This study examines the illusion of linearity in high-school students' solving of geometrical problems related to the perimeter and the area of similarly reduced or enlarged figures. Previous studies have shown that students strongly believe that when one dimension of a geometrical figure is increased or decreased by factor *k*, the perimeter and area of that figure increase or decrease by the same factor. Such reasoning is correct when it comes to perimeters, but the area is actually modified by a factor k^2. Participants in our study were third grade high-school students. They have taken an exam consisting of 6 linear and 6 non-linear items on two occasions. The results on the first exam have shown that students were very successful in solving linear problems, but were markedly unsuccessful in solving non-linear problems. Before applying the second exam, half of the students were given feedback regarding their performance on the first exam and had an opportunity to solve some of the items again. On the second exam these students solved more non-linear problems correctly than the students of the control group, but in parallel, they achieved weaker results on the linear problems when compared to the students who were not involved in our intervention.

Key words: illusion of linearity, proportionality, teaching of mathematics, geometry

If renting an apartment costs 1200 kuna (Croatian currency) per month, how much money will I need for a whole year's rent? If I want to make a jam in which the ratio of fruit and sugar is 2:1, how much sugar do I have to buy for 5 kg of fruit? We ask ourselves questions like these daily and to answer them we use proportional reasoning. Because of its wide applicability, both in everyday life and in scientific contexts, proportional reasoning plays an important role in mathematical education in schools. However, since not all of the relations are proportional, deepening knowledge in the field of proportionality can sometimes have a serious drawback. Freudenthal (1983, p. 267) has pointed out that "linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear". This phenomenon can be found in literature under different names such as the *illusion of linearity*, the *linearity trap* or the *linear obstacle* and it is defined as "the tendency to apply properties of linear relations anywhere, thus also in situations where this is inadequate" (De Bock, Van Dooren, Janssens, & Verschaffel, 2007, p. 2).

Children's tendency to use linear reasoning in non-linear problems has been observed in various fields of mathematics, such as elementary arithmetics, algebra (e.g., Hadjidemetriou & Williams, 2010; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010), and probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). The best-known case of children's over-reliance on proportionality comes from the field of geometry (De Bock et al., 2007), specifically from problems that deal with areas and volumes of enlarged or reduced geometrical figures. Many studies have shown that students are very successful in solving problems in which the task is to calculate the effect of an enlargement (or reduction) of one side of a figure on another side or perimeter of that figure, in other words, problems in which enlargements are linear; but are highly unsuccessful in solving problems in which the task is to calculate the effect of an enlargement (or reduction) of one side of a figure on the area or volume of that figure, in other words, problems in which enlargements are not linear, but quadratic or cubic. The principle underlying this type of problems is as follows: Linear enlargements/reductions by factor *k* multiply lengths by factor *k*, areas by factor k^2 and volumes by factor k^3. A crucial aspect of this principle is that these factors depend only on the dimensionality of the magnitudes

Jelena Paić-Antunović, Department of Psychology, Faculty of Humanities and Social Sciences, Ivana Lučića 3, 10000 Zagreb, Croatia. E-mail: jelenapaicantunovic@yahoo.com (the address for correspondence);

Vesna Vlahović-Štetić, Department of Psychology, Faculty of Humanities and Social Sciences, Ivana Lučića 3, 10000 Zagreb, Croatia. E-mail: vvlahovi@ffzg.hr.

involved and are not affected by particularities (e.g. shape) of the figures involved (De Bock et al., 2007). Previous research has shown that most elementary-school students and even high-school students do not know this principle and (incorrectly) believe that if the sides of a figure are doubled then the perimeter and the area of that figure are doubled too (De Bock, Verschaffel, & Janssens, 1998; De Bock, Verschaffel, & Janssens, 2002; Van Dooren, De Bock, Weyers, & Verschaffel, 2004).

Possible explanations for the occurrence of the illusion of linearity

Even though our knowledge base about the illusion of linearity is slowly increasing, a complete theoretical framework that conclusively explains this phenomenon does not exist to date. De Bock et al. (2007) analyzed various results of previous research and tried to derive possible explanations for this widespread phenomenon. They identified three groups of explanatory factors that supposedly underlie the occurrence and persistence of students' over-reliance on proportionality.

The first and most important explanation for the tendency to over-rely on linearity seems to lie in the omnipresence of linear relations in our everyday life and the observation that human cognition often tends to rely on intuitive instead of analytical modes of thinking (De Bock et al., 2007). Starting from the early childhood the concept of linearity develops from exposure to linear relations and learning and it gradually becomes very conspicuous. Concurrent with the development of the concept, first cases of its inappropriate usage start to appear. To explain why this happens De Bock et al. (2007) rely on Fischbein's theory of intuition in mathematical reasoning. Fischbein (1987, 1999) described intuitive knowledge as a type of immediate and self-evident cognition, based on salient problem characteristics, leading to generalizations, generating great confidence and often persisting despite formal learning. He assumed that the idea of proportionality belongs to that category of knowledge and that it is shaped through early and repeated experiences with situations that reflect the concept of linearity. The idea that students use intuitive knowledge in mathematical reasoning got its empirical verification in a study by De Bock, Van Dooren, Janssens, and Verschaffel (2002). These authors have conducted interviews with 12-13 and 15-16-year old students while solving a non-proportional word problem. After confrontation with the problem, students almost immediately gave a proportional solution. When they were asked afterwards why they chose that particular solution, they could not explain, but they were quite certain that both their computational method and solution were correct.

Besides work done by Fischbein, the distinction between intuitive and analytical modes of thinking has been studied by many cognitive psychologists, especially by advocates of dual process theories of thinking (e.g., Evans, 2003; Stanovich & West, 2000). These researchers make a distinction between two types of reasoning systems. The first system, often called S1 or heuristic system, is characterized as automatic, associative, unconscious and requiring less effort and cognitive resources. It is responsible for quick reactions and is often based on salient features of the problem situation and stored "prototype" situations (Sloman, 1996; according to De Bock et al., 2007). The second system, S2 or analytical system, is characterized as controlled, deliberate and requiring much more resources from our working memory. Considering the fact that S1 system results in valuable responses, people develop a tendency to use it even in situations that require usage of the analytical (S2) system. Applied to the illusion of linearity, it means that once students begin to realize the applicability of the linear model in a wide spectrum of situations, a kind of "linear heuristic" is developed inside the S1 system, which in consequence results in quick and often correct responses. Looking from that perspective, students' mistakes on non-proportional problems do not stem from their deficient mathematical knowledge (which is part of the S2 system), but from a tendency to give quick and impulsive answers.

After the tendency to reason proportionally is established, some experiences that children have in school seem to affect its further development and firming. The study by Van Dooren et al. (2005) has shown that Belgian elementary school pupils have a strong tendency to apply the linear model while solving non-linear arithmetic word problems. This tendency was first observed among second-graders, and in the period from third to fifth grade, when proportionality is taught in school, it was constantly increasing. De Bock et al. (2007) claim that the quick and technically correct execution of known procedures is often the focus in mathematical education, while the understanding of the purpose and area of applicability of a certain procedure is not explicitly taught. Hatano (2003; according to Van Dooren et al., 2010) claims that children acquire a kind of "routine expertise" – the ability to complete school mathematic tasks quickly and accurately without much understanding; instead of "adaptive expertise" – the ability to apply meaningfully learned procedures flexibly and creatively. This kind of teaching in schools restricts gained mathematical knowledge to a specific domain of applicability that students have encountered in school, and reduces the process of choosing the appropriate procedure to identification of distinctive features of a problem situation. Hence, when students encounter a problem of similar structure to the ones they solved while learning some mathematical procedure, they will be inclined to use that same procedure, regardless of its applicability in that situation. Unfortunately, this kind of approach is often reinforced in schools. Reusser (1988) concluded that only a few school problems encourage students to use profound semantic analysis of the problem. Problems that are used in teaching are often stereotypical, so using routine strategies enables students to reach a correct solution to many of them without much thinking. If we take into consideration the fact that school exams have a time limit and that the end results are what is scored and graded, then it should not come as a surprise that students adopt this approach. When tasks are separated from school context and given in an "authentic" form, as real problems that need to be solved, students' success rates significantly increase (e.g., De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003; Reusser & Stebler, 1997; Van Dooren, De Bock, Janssens, & Verschaffel, 2007). Unfortunately, this positive effect of authentic context is lost once the same problems are applied again within the standard school context.

Students think about school as of a context that is different from everyday life and their approach to school tasks can be seen in studies in which they solved those tasks during individual interviews (De Bock, Van Dooren, et al., 2002; Hadjidemetriou & Williams, 2010). Statements like: "In your calculations you can only involve numbers that are given", "I thought that the width was relevant too, but because in the text no reference was made to the width, I decided to work with the height only", "You never should ground a solution in mathematics on a drawing. You have to ground it on formulas. Drawings are less accurate", or "No [I wouldn't do it this way], if it was a real realistic graph, but this is the way I would do it in test" illustrate students' understanding of school problems and a way they have to be solved. De Bock et al. (2007) describe this phenomenon using the term "game of school word problems". Like in every other game, in this one as well, there are rules. Some of them have already been mentioned – the goal is to get a correct final solution in a limited amount of time – but there are others. For example, the rule that every problem is solvable, that a correct solution can be obtained through recognition and execution of familiar mathematical operations on given data and while doing this, not much attention should be paid on a linguistic context or graphical representation of the problem at hand. These rules are part of "didactical contract" between students and teachers. They are acquired through interaction and, even though they are rarely explicitly stated, play an important role in the educational process (De Bock et al., 2007).

The first two groups of explanatory factors we discussed have a general nature and explain why the illusion of linearity is so widespread across various fields. De Bock et al. (2007) also mention a third group of factors in which they incorporate more specific elements of the situations in which this phenomenon occurs. They can be further divided into two subgroups. The first subgroup relates to students' deficiencies in knowledge of a certain mathematical field. Those are, for example, some shortcomings in the general geometrical knowledge, difficulties with the notions of area and volume, misconceptions about probability laws and so on.

The second subgroup consists of specific features of the mathematical problems that are used and which contribute to the over-use of the linear model. Until now several factors from this subgroup have been identified. Some of them are: different ways of formulating and presenting a task (De Bock, Verschaffel, et al., 2002), the type of problems (Van Dooren et al., 2005), the size of a scaling factor (Lapaine, 2010), presence of graphical representations of the problem (De Bock, Verschaffel, et al., 2002; De Bock et al., 2003) and presence of a linear solution in multiple-choice tasks (Rajter, 2006).

What stands behind students' solutions?

The literature on the illusion of linearity is almost completely concentrated on non-proportional relations and difficulties that children experience while dealing with them. Proportional relations, on the other hand, are given much less attention. A possible reason is that most students are very successful in solving proportional problems. This can leave an impression that they have thoroughly mastered those relations, and the difficulties they have with non-proportional problems stem from an unjustified generalization of those well-known proportional strategies in situations where they are not appropriate. However, there are some indications that this is not the case. The mere fact that students have difficulties distinguishing between situations that are properly modeled using linearity from those that are not, tells us something about students' understanding of proportionality. Modestou and Gagatsis (2010) argue that the traditional conception that defines proportional reasoning simply as the ability to solve proportions is incomplete. Hence they propose a new model of proportional reasoning in which two new dimensions are incorporated – meta-analogical awareness and analogical reasoning – besides the old conception of proportional thinking. A similar opinion has been stated by Cramer, Post, and Currier (1993, according to De Bock et al., 2007, p. 8) who argue that "a proportional reasoner cannot be defined simply as one who knows how to set up and solve a proportion" and think that mathematical textbooks do not sufficiently emphasize the ability to discriminate between linear and non-linear situation. Further argument supporting a notion that students have trouble understanding proportional relations comes from the studies that were attempting to reduce the negative impact of the illusion of linearity (e.g., De Bock, Verschaffel, et al., 2002; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004; Van Dooren, De Bock, Weyers, et al., 2004). When students have been made aware that not all problems can be solved by using the proportional model, the proportion of correct answers on items where proportionality is appropriate significantly decreased. Greer (2010) thinks that this finding implies that the previous high level of correct responses on such items included some false positives.

If different thinking processes can lie beneath correct answers on proportional problems, can the same be true for (mostly) incorrect answers on non-proportional problems?

Greer (2010) tried to answer this question by explaining possible ways of reasoning that can stand in the background of selecting a proportional solution to a non-proportional problem:

1. The student has carefully thought about the situation and decided that a linear interpretation is appropriate.
2. The student has carefully thought about the situation and decided that a linear interpretation is not appropriate, but does not know how to formulate an appropriate alternative, so he gives the simplest, linear response.
3. The student is reacting to more or less superficial cues, such as characteristic linguistic phrases which suggest that the solution can be found routinely; therefore a common linear answer is appropriate.
4. Regardless of correctness of his/her interpretation of problem situation, the student just wants to provide the researcher with an answer and get out of the testing situation. It seems naïve to assume complete cognitive engagement in the context of school, and the same surely applies to the experimental context.

What we are trying to say is that equating a correct or an incorrect response with a correct or an incorrect reasoning is problematic, because more than one interpretation is possible. Unless we have observed students' thinking process during the problem solving, we cannot know which interpretation is applicable.

There are also studies suggesting that some interpretations are, if nothing else, more probable than others. Those studies explored the process that underlies solving of this type of mathematical problems. According to Nesher (1992) this process consists of several phases. During the reading of the text a student has to understand its meaning, construct the situational model of the problem and create a formal representation. Based on the developed problem representation, appropriate computational procedures are chosen. The final phase in problem solving is the interpretation and verification of the result. Studies that have examined this process have shown that students tend to skip some of these phases. For example, in a study by De Bock, Van Dooren, et al. (2002) students have often skipped all of the steps apart from choosing and conducting calculations. The decision about which calculations should be performed was based on routine recognition of a certain type of problem, and after that, most of the time was dedicated to calculating and after checking for possible calculation errors, the final solution was given. Similar conclusion was reached by Vlahović-Štetić and Zekić (2005). It is worth mentioning that this finding was characteristic for both types of problems, in other words, students have used this routine procedure while solving both proportional and non-proportional problems. Based on these results, we can conclude that correct responses on proportional problems and incorrect responses on non-proportional problems are a product of the same superficial problem-solving procedure. Consequently, this would imply that results of studies about the illusion of linearity speak more of a typical approach that students have towards solving this type of problems than of their actual knowledge or understanding of proportionality. If we want to explore whether students are indeed capable or incapable of discriminating proportional and non-proportional situations, we should somehow intervene into their problem-solving habits. If we could manage to stimulate students to analyze the problems more thoroughly before giving their final response, we might get a different estimation of their success rate in proportional and non-proportional problem situations.

The aim of this study was to design an intervention whose primary goal was to increase students' interest in problems and persuade them to approach them more cautiously and try to analyze represented relations before giving a final solution. We wanted to explore if these conditions would have a beneficial effect on students' ability to discriminate between proportional and non-proportional relations; and if they did, would that ability translate to their later encounter with similar problems in ordinary conditions.

METHOD

Participants

Participants were 121 third grade high-school students from Zagreb. They were divided into two groups, an experimental and a control group, in a way that two intact classes were selected for one and two for the other group. To make sure that selected groups are equivalent based on students' mathematical abilities, we chose classes following the same mathematical program in school. The experimental group consisted of 59 (38 girls and 21 boys) and the control group of 62 (40 girls and 22 boys) students.

Materials and procedure

In the study two comparable versions of an exam with mathematical problems (Form A for the first examination and Form B for the second one), answer sheets and feedback sheets were used.

Exams consisted of 12 items. Half of the items were non-proportional problems (problems for which it is not appropriate to reason linearly), and the other half were proportional problems (problems for which it is appropriate to reason linearly). Within each of these categories there were two items involving one of three types of geometrical figures – squares, circles, and irregular figures. In the proportional problems students were asked to calculate the effect of increasing one dimension (height, length, or diameter) of a figure on the perimeter of that figure, and in the non-proportional problems, on the area of the figure. In most items it was not explicitly stated that it was necessary to calculate

Table 1
Examples of experimental items

	Proportional problems	Non-proportional problems
Squares	Miško needs 7 hours to cut the hedge around the square-shaped garden with a side of 40 m. How much time would he need to cut the hedge around the garden of a same shape if its side was 6 times longer?	The restaurant “Galion” has 2 wedding venues. They are both square-shaped, but one of them is 3 times longer than the other one. The optimal number of guests in the smaller venue is 100. What is the optimal number of guests in the larger venue?
Circles	The specialty of pizzeria “Ljubica” is stuffed crust pizza. If there are 26 pieces of cheese needed for the crust of pizza with a diameter of 30 cm, how many pieces of cheese are needed for a smaller pizza with a diameter of 15 cm?	Maja’s mother is baking a birthday cake for her daughter. Everything is ready except for the icing that goes on the top of the cake. If 1 chocolate was needed for the last year’s cake with a diameter of 14 cm, how many chocolates will be needed for this year’s cake that has 3 times longer diameter?
Irregular figures	From the oval table that is 5 m long 18 people can simultaneously take food. How many people could simultaneously take food from the 3 times longer table of a same shape?	Petra wants to draw a picture of a butterfly on the wall of her room. To see how much paint she would need, she first drew it on a paper and found out that for the butterfly that is 12 cm wide 2 bottles of paint are needed. If the width of the same butterfly on the wall is 60 cm, how many bottles of paint would she need?

the perimeter or the area, but those measures were replaced by other, indirect measures, which are directly proportional to them, e.g. the time needed to walk around a lake or the amount of paint needed to cover a surface. During the item construction we tried to keep all other possibly relevant task variables, such as mathematical and language complexity, as similar as possible. A number of items were taken from tests used by De Bock et al. (1998) and Vlahović-Štetić, Pavlin-Bernardić and Rajter (2010), but most of them were constructed for the purpose of this study. Examples of items are listed in Table 1.

Examination was conducted with every class of students separately in the regular class time and it was performed in two phases. In the first phase the procedure was the same for both groups. Prior to solving the exam participants were given the following instruction: “You are looking at a booklet with 12 mathematical word problems. Please solve these problems in the same order as they are presented, paying attention to the ordinal number of a problem. Once you have solved the problem, write your end result into the attached answer sheet under the corresponding number. You will have enough time to solve these 12 problems, so please be as careful as possible. Every solution should be accompanied by a procedure you used to come to the final result. The procedure does not have to be written in the form of mathematical operations, but may be a graphical representation or an explanation in your own words. You can use a calculator for your computations.” After reading the instructions all students started solving the items from Form A and entered their final solutions into their answer sheets. They were given 30 minutes to complete this task, but most of them solved the entire exam within 15-20 minutes.

The second phase of the examination took place one or two days after the first one, but now the procedure for the two groups of students was different. In the control group, the same procedure as the first time was followed. After the participants were given the instructions, they took the second test (Form B). Students in the experimental group first received feedback sheets and their own answer sheets from the previous test. The feedback sheet consisted of the first three items from Form A and their correct final solutions. The following operating instructions were read: “In the volume in front of you are the first three problems from the test you took yesterday together with their correct final solutions. Your task is to re-read these problems and to compare the solutions from your answer sheets with the provided correct ones. If your solution differs from the correct one, try to determine where you made a mistake and then try to solve the problem again correctly.” The purpose of the feedback was to allow participants to recognize that not all the problems in the test were equal, i.e., to identify the problems that cannot be solved using a linear model. This intervention was designed to encourage students who solved non-proportional problems in the first test as if they were proportional to review their original strategy. After that, the participants in the experimental group also took the second test (Form B).

RESULTS

In the first phase of the experiment 121 subjects participated and in the second one, there were 114. Students’ responses on the proportional and non-proportional items were considered correct and given 1 point when they resulted from a mathematically appropriate reasoning process; therefore, final solutions that differed from the correct ones due to minor technical mistakes were also considered correct, as long as most of the solution procedure was mathe-

Table 2
Number of participants with a specified number of correctly solved non-proportional and proportional items (N = 121)

Number of correct responses	Non-proportional items	Proportional items
0	89	0
1	17	1
2	10	1
3	3	2
4	1	10
5	0	21
6	1	86
Mdn	0	6
Q	0.5	0.5

Note. Mdn = medians, *Q* = semi-interquartile ranges.

matically appropriate. All other kinds of erroneous answers and cases in which no solution at all was given were scored with 0 points.

Kolmogorov-Smirnoff tests showed that all distributions significantly deviated from a normal distribution. Score distributions on the proportional items were negatively asymmetrical and distributions of the non-proportional items were positively asymmetrical. Therefore we decided to analyze our data using non-parametrical statistics.

Students' performance on both the non-proportional and the proportional items in the first test (Form A) is shown in Table 2. On the non-proportional items participants achieved extremely weak results (*Mdn* = 0; *Q* = 0.5), but at the same time, they were highly successful in solving the proportional items (*Mdn* = 6; *Q* = 0.5).

To check whether students' poor performance on the non-proportional items was a result of the illusion of linearity, we qualitatively analyzed their incorrect responses on those items. Out of all incorrect answers, 96% of them were solutions that could be obtained through a proportional reasoning process.

Table 3 shows the percentages of correct responses of the students in the experimental group on the first three items in the first test and after their confrontation with the feedback sheets. The proportional item (item 2) was already correctly solved by most of the students (96.6%) in the first test, so those students did not have to solve it again within the feedback presentation. The few students who made an error on this item in the first test managed to recognize their error, so all students answered this item correctly with the aid of the given feedback. The difference in students' performance on this item in the first test and after confrontation with the feedback sheets was not statistically significant (Z = -1.000; $p > .05$).

Compared to the first examination students' performance on the non-proportional items improved with the aid

Table 3
Comparison of the performance of the experimental group on the first three items in Form A (N = 59) and after their confrontation with the feedback sheets (N = 55)

	Form A			Feedback			Wilcoxon
Item	*Mdn*	*Q*	%	*Mdn*	*Q*	%	Z
1	0	0.5	27.1	1	0.5	70.9	-4.796**
2	1	0.0	96.6	1	0.0	100.0	-1.000
3	0	0.0	0.0	0	0.5	34.5	-4.359**

Note. % = percentage of students who have solved the item correctly.
** $p < .01$

Table 4
Comparison of the results of the experimental and the control group on both types of items in the first and the second examination

Group	Form A				Form B			
	Mdn	*Q*	%	Mann-Whitney *Z*	*Mdn*	*Q*	%	Mann-Whitney *Z*
				Non-proportional items				
Experimental	0	0.5	7.9	-0.224	1	1.0	25,5	-5.299**
Control	0	0.5	7.5		0	0.0	4,8	
				Proportional items				
Experimental	6	0.5	91.2	-0.982	4	1.0	75,2	-6.496**
Control	6	0.5	93.3		6	0.0	97,5	

Note. ** $p < .01$

of the given feedback. The percentage of correct responses increased from 27.1% to 70.9% (Z = -4.796; p < .01) for the item involving squares (item 1) and from 0.0% to 34.5% (Z = -4.359; p < .01) for the item involving irregular figures (item 3).

To examine if the feedback also influenced students' scores on the items in the second test, we analyzed differences in average number of correct responses of participants from two the groups on both test forms. Medians, semi-interquartile ranges and results of Mann-Whitney U test are shown in Table 4.

In the first test (Form A) both groups of students successfully solved almost all proportional items (Mdn = 6; Q = 0.5) and almost none of the non-proportional items (Mdn = 0; Q = 0.5). The differences in results of the two groups of participants were not statistically significant on neither of the two types of items. In the examination after the introduction of the feedback (Form B) the situation changed and differences appeared for both types of items. On the non-proportional items the experimental group performed significantly better (Z = -5.99; p < .01) and on the proportional items significantly worse (Z = -5.229; p < .01) than the control group of students.

We used the Wilcoxon signed-rank test to analyze the changes in performance of both groups of participants separately. The performance of the experimental group improved on the non-proportional items (Z = -4.632, p < .01) and deteriorated on the proportional items (Z = -4.108, p < .01). Interpretation of the changes in performance of the control group is somewhat more complicated. Score medians did not change, but semi-interquartile ranges decreased which means that in the second examination more students achieved a score equal to the median. Since the median score on the non-proportional items is the minimum possible score on the test (Mdn = 0), we can conclude that students' performance weakened on this type of items. For the proportional items it is exactly the opposite. This time more students achieved the maximum score on the test (Mdn = 6), which means that they improved their performance on these items. Both differences are significant at the 5% percent level of significance.

In the previous analysis we have treated the two groups of participants in their entirety and thereby ignored the fact that not all participants in the experimental group benefited equally from the feedback. Although all participants in this group had the opportunity to find out whether their solutions in the first test were correct or wrong, only some of them managed to discover their mistakes and to find a way to reach the right solutions.

Therefore, we decided to split up the experimental group into two subgroups, depending on the number of correct responses on the two non-proportional items after the feedback was given. The first subgroup consisted of the students who solved at least one non-proportional item successfully (the successful subgroup) and the second one of the students who solved none of these items successfully (the unsuccessful subgroup). We compared the results achieved in the second test by the students from these two subgroups of the experimental group with the results of the students from control group. Due to great differences in size of these three samples, we decided not to test the differences in results, but only to compare them. On the non-proportional items, the control group and the unsuccessful subgroup achieved similar results (4.8% and 4.2% of correct responses), while the successful subgroup was distinctively better (32.5% of correct responses). On the proportional items, the successful subgroup had the least correct responses (70.7%), the unsuccessful subgroup a little bit more (77.1%) and control group the most (97.5%). In other words, the effect of feedback on scores in the second test was different depending on students' success in solving the non-proportional items within the feedback presentation.

DISCUSSION

The aim of the present study was to examine the effect of feedback on students' success rate in solving non-proportional and proportional problems. Several analyses were performed on the test results, before and after the introduction of feedback. In the initial test, no differences in the success rate of the two groups of participants were found – almost all participants achieved very high scores on the proportional and very low scores on the non-proportional items.

Between the first and the second test the experimental group received feedback on the accuracy of their solutions for the first three items from the first test. Most participants learned that their solution for the proportional problem was correct, while their solutions for the two non-proportional problems were incorrect. It should be noted that participants were not told about a difference between these two types of problems. Hence, they could only find out that one of their answers was correct and two were incorrect. In the time they had at their disposal to discover their mistakes and to reach the correct solution, not all of the participants were equally successful. Due to the given feedback, the percentage of participants who correctly solved the non-proportional problem about a square increased from 27.1% to 70.9% and for the non-proportional problem about an irregular figure from 0% to 34.5%. Changes in the performance on both non-proportional problems proved to be statistically significant.

We hypothesize that this improvement is based on a dual effect of the feedback. On the one hand, participants were confronted with the information that they had made a mistake on problems that they considered being simple and of which they believed they solved them properly. We have not asked our participants to express their level of certainty about their solutions, but several previous studies (e.g. Lapaine, 2010; Molnar, 2010; Vlahovic-Štetić & Zekić, 2005) have shown that for this type of mathematical problems students generally report high levels of certainty about their solutions, even when those solutions are incorrect. The fact that some of our subjects commented that the problems in the first test were very easy, that those are the problems for first graders and so on, suggests that the same holds for this study. We think that the information about the number of errors that students made in these "simple" problems caused cognitive dissonance in them; hence they were motivated to try to discover the reason why they made those mistakes. On the other hand, we assume that the feedback could have served as an incentive for students to change their initial approach for solving these problems. Studies have shown that many students, while solving mathematical problems, skip all phases in the solution process except choosing and performing calculations (De Bock, Van Dooren, et al., 2002). Since in this case the final solution was already given and the instruction was to reveal the process by which it was obtained, the participants were forced to redirect their efforts towards the remaining phases of the solution process, i.e. understanding the meaning, constructing the situational model, creating a formal representation and verifying the solution of the problem. In such circumstances, as it turned out, much more participants managed to discover a correct way of solving these problems.

In the second examination, after the introduction of the feedback, the results of both groups of participants changed and the differences in success of the two groups proved to be statistically significant.

The obtained pattern of changes in the results suggests that there are several explanatory factors. For clarity, we will first focus on the results on the non-proportional items. In the second test the experimental group achieved slightly, but significantly better results than the control group, which can probably be attributed to the positive influence of the given feedback. However, the feedback was not of equal benefit to all participants. To most of the students in the experimental group, it proved to be a sufficient incentive for reviewing their original solving strategy and detecting the non-proportional nature of the problems. When we have singled out this subset of participants, we have observed that their results were significantly better than the results of the other participants. To a smaller part of the students in the experimental group our intervention has given the opportunity to verify the accuracy of their solutions, but that information was not sufficiently helpful for them to find the right solution for the problems. In the second test, the results of the students from this subset of the experimental group were similar to those of the control group. These findings suggest that mere presentation of the feedback was not sufficient for causing a positive shift in the success rate of the experimental group in solving non-proportional problems; it was students' own success in solving this type of problems during the presentation of the feedback that facilitated their improvement in the second test.

In the control group of participants, the results on the non-proportional problems deteriorated in the second test. This finding was not expected. Namely, if the illusion of linearity is a phenomenon that is, in the absence of serious interventions, highly resistant to change, there is no reason to expect that in two days, which was the length of the period between the two measurements, something would alter it significantly. Even if something happened in this period (e.g. if the participants have been thinking about the problems after the examination, talking to each other, etc), then, by all accounts students' results in the second test could get better, but not worse. How can we then explain the obtained findings? What we could assume is that the participants in the control group have, in the absence of any intervention, approached the second test in the same way as the first one, and then, after gaining the impression that it is the same (or a very similar) test, put less effort into profound analysis of the problems and gave way to automatism. The fact that the results of this group improved on proportional items, also fits into this explanation. Namely, if the automatic response is the reaction that could be defined as "set up and solve a proportion", then that response must be the wrong solution to non-proportional problems, but at the same time it will necessarily be the right solution to all proportional ones. If we accept this explanation, we have another argument in favor of the feedback. With its introduction, we were able to prevent the experimental group from falling into this trap of automaticity.

For the proportional items changes in the results were observed as well, and the direction of these changes was opposite to the ones observed for the non-proportional items. On this type of items the control group achieved better and the experimental group worse results than in the initial test. We have already suggested a possible explanation for the results of the control group, so we will now focus on the results of the experimental group. Some previous studies (e.g., De Bock et al., 1998, 2002; Van Dooren, De Bock, Hessels, et al., 2004) reported a similar effect – when an intervention has helped to reduce the number of errors on non-proportional problems, the price of that success was paid by the proportional problems. De Bock et al. (2007) concluded that once students begin to doubt the applicability of linear model in problem situations for which it is not appropriate, they sometimes begin to doubt its applicability in situations for which it is. The results that we have obtained are consistent with this explanation. Of the total number of errors that participants in the experimental group have made on the proportional problems, the largest share were the solutions that would have been obtained if these problems were solved as if they were non-proportional.

In other words, it seems that our feedback has suppressed students' tendency to solve non-proportional problems as if they were proportional, but has at the same time attracted a new trend, contrary to the previous one, to solve proportional problems as if they were non-proportional. We could say that the feedback helped decreasing the impact of the illusion of linearity in the experimental group, but it did not improve students' ability to meaningfully discriminate between these two types of problems and to better understand the mathematical principles that stand in their ground. When we compared the total score of the control and experimental group in the second test, the difference in their results was not statistically significant.

The aim of this study was to examine the impact of feedback on the accuracy of solving proportional and non-proportional problems. Initial examination has shown that students are very successful in solving proportional problems, while they are markedly unsuccessful in solving non-proportional ones, mostly because they solve them as if they were proportional. Feedback on the accuracy of their solutions has helped many participants to discover their mistake and to reach the correct solutions to problems that were presented during feedback. This intervention had a significant effect on students' performance in the second test with similar items. Participants in the experimental group performed better on non-proportional items compared to the control group, but at the same time their results on proportional problems got worse. It remains an open question what kind of educational intervention could help students discriminate between proportional and non-proportional problems more proficiently and solve both types of problems successfully, but without the negative effects that were observed in the present study as well as in previous ones.

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