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***Research paper***

# The effect of numerical integration in mixed finite element approximation in the simulation of miscible displacement

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**Abstract**

We consider the effect of numerical integration in finite element procedures applied to a nonlinear system of two coupled partial differential equations describing the miscible displacement of one incompressible fluid by another in a porous meduim. We consider the use of the numerical quadrature scheme for approximating the pressure and velocity by a mixed method using Raviart - Thomas space of index $k$ and the concentration by a standard Galerkin method. We also give some sufficient conditions on the quadrature scheme to ensure that the order of convergence is unaltered in the presence of numerical integration. Optimal order estimates are derived when the imposed external flows are smoothly distributed.

***Keywords:*** *Mixed finite element, Raviart-Thomas spaces, quadrature scheme, molecular dispersion.*

## 1. Introduction

The miscible displacement of one incompressible fluid by another in a reservoir $\Omega \subset \mathbb{R}^2$ of unit thickness and local elevation $z(x), x \in \Omega$, with the Darcy velocity of the fluid mixture given by

$$u = \frac{-k(x)}{\mu(c)}(\nabla p - \gamma_0(c)\nabla z), \tag{1}$$

can be described by differential system that can be put in the slightly more general form [6]

$$\begin{cases} \nabla \cdot u = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a_i(x,c)\left(\frac{\partial p}{\partial x_i} - \gamma_i(x,c)\right)\right] = q, x \in \Omega, t \in [0,T], (a) \\ \phi \frac{\partial c}{\partial t} + u \cdot \nabla c - \nabla \cdot (D\nabla c) = (\hat{c} - c)q = g(x,t,c), x \in \Omega, t \in [0,T], (b) \\ u \cdot \vartheta = 0, \quad x \in \partial\Omega, t \in [0,T], (c) \\ \sum_{i,j} D_{ij}(\phi,u)\frac{\partial c}{\partial x_j}\vartheta_i = 0, \quad x \in \partial\Omega, \quad t \in [0,T], (d) \\ c(x,0) = c_0(x), \quad x \in \Omega. \quad (e) \end{cases} \tag{2}$$

In the above, $p$ is the pressure and the initial pressure, modulo an additive constant, can be determined from (2a) and (2c); $c$ is the concentration, $c_0$ the initial concentration such that $0 \leqslant c_0(x) \leqslant 1$, and the term $\hat{c}$ must be specified where $q > 0$ and $\hat{c} = c$ where $q < 0$; $q = q(x,t)$ is the imposed external flow, positive for injection and negative for projection, $\vartheta$ is the exterior normal to $\partial\Omega$; for compatibility $(q,1) = \int_\Omega q(x,t)dx = 0, \quad t \in [0,T]$; $a = a(c) = a(x,c) = \frac{k(x)}{\mu(c)}$, where $k(x)$ is the permeability of the medium, $\mu(c)$ the viscosity of the fluid; $\gamma_0$ is the density of the fluid and $\gamma(x,c) = \gamma_0(c)\nabla z(x)$. The diffusion coefficient $D = D(\phi,u)$ is a $2 \times 2$ matrix given by

$$D = \phi(x)[d_m I + \mid u \mid \left(d_l E(u) + d_t E^{\perp}(u)\right)], \tag{3}$$

where $\phi$ is the porosity of the meduim and the matrix $E$ is the projection along the direction of flow given by $E(u) = (u_i u_j \diagup |u|^2), E^{\perp} = I - E$, $d_m$ is the molecular diffusion coefficient, and $d_l$ and $d_t$ are respectively, the longitudinal and tranverse dispersion coefficients. The tensor dispersion is more important physically than the molecular diffusion; also, $d_l$ is usually considerably larger than $d_t$.
The effect of numerical integration in finite element method for solving elliptic equations, parabolic equations and hyperbolic equations has been analyzed by Raviart [2], Ciarlet and Ravaiart [3], So-Hsiang Chou and Li Qian [4], Li Qian and Wang Daoyu [5] and others. When numerical integration is not used the problem (2) has been studied by Jim Douglas Jr., Richard E. Ewing and Mary Fanett Wheeler [6] where optimal order estimates are derived. In this paper we consider the use of the numerical quadrature scheme and analyze a continuous-time finite element method based on the use of a mixed finite element procedure to approximate the pressure and the velocity simultaneously, and a standard Galerkin method to approximate the concentration. We shall also give some sufficient conditions on the quadrature scheme which ensure that the order of convergence is unaltered when numerical integration is used.

## 2. Notation and formulation of the finite element procedures

The inner product on $L^2(\Omega)$ or $L^2(\Omega)^2$ is denoted by $(\varphi,\psi) = \int_\Omega \varphi\psi dx$.

We shall consider $W^{m,s}(\Omega), H^m(\Omega) = W^{m,2}(\Omega), L^2(\Omega) = H^0(\Omega) = W^{0,2}(\Omega)$ and $L^s(\Omega) = W^{0,s}(\Omega)$ for any integer $m \geqslant 0$ and any number $s$ such that $1 \leqslant s \leqslant \infty$
, as the usual Sobolev and Lebesgue spaces on $\Omega$ respectively. The associated norms are denoted as follows: $\|\cdot\|_{m,s} = \|\cdot\|_{W^{m,s}(\Omega)}$, $\|\cdot\|_m = \|\cdot\|_{H^m(\Omega)}$ or $\|\cdot\|_{H^m(\Omega)^2}$ as appropriate, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{L^2(\Omega)^2}$ as appropriate, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$.
Let $X$ be any of $L^s$ or Sobolev spaces; for a function $f(x,t)$ defined on $\Omega \times [0,T]$ we set $\|f\|^2_{L^2(X)} = \int_0^T \|f(\cdot,t)\|^2_X dt$, $\|f\|_{L^\infty(X)} = ess \sup\limits_{0\leqslant t\leqslant T} \|f(\cdot,t)\|_X$.
Set $H(div;\Omega) = \{v, v \in L^2(\Omega)^2, \nabla\cdot v \in L^2(\Omega)\}$, provided with the norm

$$\|v\|_{H(div;\Omega)} = \left\{ \sum_{i=1}^{2} \|v_i\|^2_{0,\Omega} + \|\nabla\cdot v\|^2_{0,\Omega} \right\}^{\frac{1}{2}};$$

and let

$$V = \{v; v \in H(div;\Omega), v\cdot\vartheta = 0 \quad \text{on} \quad \partial\Omega\}$$

$$\text{and} \quad W = L^2(\Omega) \diagup \{\varphi \equiv \text{constant on } \Omega\}.$$

## Assumptions (A)

(i) The external flow is smoothly distributed, and the coefficients and domain are sufficiently regular as to allow a smooth solution of the differential problem.
(ii) The functions $a_i(x,c), (c \in [0,1])$ are bounded above and below by positive constants and there exist a uniform positive constant $M$ such that $|q(x,t)| + |\nabla z(x)| + |\gamma_0(x,c)| \leqslant M$; and the matrix $D$ should be uniformly positive-definite:

$$\sum_{i,j=1}^{2} D_{ij}(\phi,u)\xi_i\xi_j \geqslant D_0 |\xi|^2, \xi \in \mathbb{R}^2,$$

with $D_0$ being independent of $x$ and $u$.

(iii) $g$ is Lipschitz continuous and the various bounds that are used for the coefficients and their derivatives need hold only in a neighborhood of the solution of the differential problem.

Let $h = (h_c, h_p)$, where $h_c$ and $h_p$ are positive, and different in general. Let $M_h = M_{h_c} \subset H^1(\Omega)$ be a standard finite element space of index at least $l$ associated with a quasi-regular polygonalization $T_{h_c}$ of $\Omega$ and having the following approximation and inverse hypotheses:

$$\inf_{z_h \in M_h} \| z - z_h \|_l \leqslant M h_c^l \| z \|_{l+1}, z \in H^{l+1}(\Omega), \qquad (a)$$
$$\| z_h \|_m \leqslant M h_c^{-m} \| z_h \|, 1 \leqslant m \leqslant l+1, z_h \in M_h \qquad (b) \qquad (4)$$

Suppose that $\Omega$ is a polygonal domain. Let $\tilde{V}_h \times \tilde{W}_h$ be one of the Raviart-Thomas spaces of index at least $k$ associated with a quasi-regular triangulation or quadrilateralization $T_{h_p}$ of $\Omega$ such that the elements have diameters bounded by $h_p$.

Set $V_h = \{v \in \tilde{V}_h : v\cdot\vartheta = 0 \quad \text{on} \quad \partial\Omega\}$
and $W_h = \tilde{W}_h \diagup \{\varphi \equiv \text{constant on} \quad \Omega\}$.

The approximation properties of $V_h \times W_h$ are given by the following relations:

$$\inf_{v_h \in V_h} \| v - v_h \|_{L^2(\Omega)^2} \leqslant M \| v \|_{H^{k+1}(\Omega)^2} h_p^{k+1}, \qquad (a)$$
$$\inf_{v_h \in V_h} \| v - v_h \|_V \leqslant M\{\| v \|_{H^{k+1}(\Omega)^2} + \| \nabla\cdot v \|_{H^{k+1}(\Omega)}\} h_p^{k+1}, (b) \qquad (5)$$
$$\inf_{w_h \in W_h} \| w - w_h \|_W \leqslant M \| w \|_{H^{k+1}(\Omega)} h_p^{k+1}, \qquad (c)$$

whenever the norms on the right-hand side are finite.

The weak form of (2) is defined by finding the map $\{c,u,p\} : [0,T] \to H^1 \times V \times W$ such that

$$\begin{cases} (\phi c_t, z) + (u\cdot\nabla c, z) + (D(u)\nabla c, \nabla z) = (g(c), z), \quad z \in H^1(\Omega), 0 < t \leqslant T, (a) \\ A(c; u, v) + B(v,p) = (\gamma(c), v), \quad v \in V, \quad 0 < t \leqslant T, \qquad (b.1) \\ B(u,\varphi) = -(q,\varphi), \quad \varphi \in W, \quad 0 \leqslant t \leqslant T, \qquad (b.2) \\ c(0) = c_0, \qquad (c) \end{cases} \qquad (6)$$

where $c(0) = c(x,0); D(u) = D(\phi,u); \quad c_t = \dfrac{\partial c}{\partial t}; \quad u\cdot\nabla c \in L^2(\Omega)$,

$$A(\theta;\alpha,\beta) = (\frac{1}{a(\theta)}\alpha, \beta) = \sum_{i=1}^{2} (\frac{1}{a_i(\theta)}\alpha_i, \beta_i) = \int_\Omega \sum_{i=1}^{2} \frac{1}{a_i(\theta)} \alpha_i\beta_i dx,$$
$$\alpha, \beta \in V, \quad \theta \in L^\infty(\Omega),$$
$$B(\alpha,\varphi) = -(\nabla\cdot\alpha, \varphi) = -\int_\Omega \nabla\cdot\alpha\varphi dx, \quad \varphi \in W.$$

Following [1], we now give a general description of the corresponding formulation of (6) when numerical integration is present.
In what follows let $f$ be $c$ or $p$ as appropriate, and $s$ be $l$ or $k$ as appropriate.

Let $T_{h_f}$ be a quasi-regular polygonalisation of the set $\bar{\Omega}$ with elements $(K_f, P_{K_f}, \sum_{K_f})$ with diameters $\leqslant h_f$.

The following assumptions shall be made

(i) The family $(K_f, P_f, \sum_f)$, $K_f \in T_{h_f}$ for all $h_f$ is a regular affine family with a single reference finite element $(\hat{K_f}, \hat{P_f}, \hat{\sum_f})$.
(ii) $\hat{P_f} = P_s(\hat{K_f})$, the set of polynomials of degree less than or equal to $s$.
(iii) The family of triangulations or quadrilateralizations $\bigcup\limits_{h_f} T_{h_f}$ satisfies an inverse hypothesis.
(iv) Each polygonalization $T_{h_f}$ is associated with a finite-dimensional subspace $M_h$ or $V_h$ or $W_h$ of trial functions which is contained in $H^1(\Omega) \cap C^0(\bar{\Omega})$.

We now introduce a quadrature scheme over the reference set $\hat{K_f}$. A typical integral $\int_{\hat{K_f}} \hat{\phi}(\hat{x}) d\hat{x}$ is approximated by $\sum\limits_{l_f=1}^{L_f} \hat{\omega}_{l_f} \hat{\phi}(\hat{b}_{l_f})$, where the points $\hat{b}_{l_f} \in \hat{K_f}$ and the numbers $\hat{\omega}_{l_f} > 0, \quad 1 \leqslant l_f \leqslant L_f$ are respectively the nodes and the weights of the quadrature.
Let $F_{K_f} : \hat{x} \in \hat{K_f} \to x \equiv F_{K_f}(\hat{x}) \equiv B_{K_f}\hat{x} + b_{K_f}$ be the inversible affine mapping from $\hat{K_f}$ onto $K_f$ with the Jacobian of $F_{K_f}$, $\det(B_{K_f}) > 0$. Any two functions $\phi$ and $\hat{\phi}$ on $K_f$ and $\hat{K_f}$ ar related as $\phi(x) = \hat{\phi}(\hat{x})$ for all $x = F_{K_f}(\hat{x})$, $\hat{x} \in \hat{K_f}$.
The induced quadrature scheme over $K_f$ is

$$\int_{K_f} \phi(x) dx = \det(B_{K_f}) \int_{\hat{K_f}} \hat{\phi}(\hat{x}) d\hat{x} \approx \sum_{l_f=1}^{L_f} \omega_{l_f,K_f} \phi(b_{l_f,K_f}),$$

with $\omega_{l_f,K_f} \equiv \det(B_{K_f})\hat{\omega}_{l_f}$, and $b_{l_f,K_f} \equiv F_{K_f}(\hat{b}_{l_f})$, $1 \leqslant l_f \leqslant L_f$.

Accordingly, we introduce the quadrature error functionals

$$E_{K_f}(\phi) \equiv \int_{K_f} \phi(x) dx - \sum_{l_f=1}^{L_f} \omega_{l_f} \phi(b_{l_f}), \qquad (7)$$

$$\hat{E}(\hat{\phi}) \equiv \int_{\hat{K}_f} \hat{\phi}(\hat{x})d\hat{x} - \sum_{l_f=1}^{L_f} \hat{\omega}_{l_f}\hat{\phi}(\hat{b}_{l_f}), \tag{8}$$

which are related by

$$E_{K_f}(\phi) = \det(B_{K_f})\hat{E}(\hat{\phi}). \tag{9}$$

The quadrature scheme is exact for the space of functions $\hat{\phi}$, if $\hat{E}(\hat{\phi}) = 0, \forall \hat{\phi}$.
If the approximations for the concentration, the velocity and the pressure are denoted by $C, U$ and $P$, respectively, then using these quadrature formulas, the continuous-time approximation procedure of (6) is given by finding the map $\{C,U,P\} : [0,T] \to M_h \times V_h \times W_h$ such that

$$\begin{cases} C(0) = c_0 \quad \text{small: } L^2(\Omega)\text{-or } H^1(\Omega)\text{-projection of } c_0 \\ \text{ into } M_h \text{ or some interpolation of } c_0 \text{ into } M_h \\ (\phi C_t, z)_h + (U\cdot\nabla C, z)_h + (D(U)\nabla C, \nabla z)_h = (g(C), z)_h, \quad (a) \\ z \in M_h, \quad t\in[0,T] \\ A_h(C;U,v) + B_h(v,P) = (\gamma(C), v)_h, \quad v\in V_h, \quad t\in[0,T], (b.1) \\ B_h(U,\varphi) = -(q,\varphi)_h, \quad \varphi\in W_h, \quad t\in[0,T], \quad (b.2) \end{cases} \tag{10}$$

where

$$(\alpha,\beta)_h = \sum_{K_f\in T_{h_f}} \sum_{l_f=1}^{L_f} \omega_{l_f,K_f}(\alpha\beta)(b_{l_f,K_f})$$

$$A_h(\theta;\alpha,\beta) = (\frac{1}{a(\theta)}\alpha,\beta)_h = \sum_{i=1}^{2}(\frac{1}{a_i(\theta)}\alpha_i,\beta_i)_h$$

$$= \sum_{K_f\in T_{h_f}} \sum_{l_f=1}^{L_f} \omega_{l_f,K_f}(\sum_{i=1}^{2}(\frac{1}{a_i(\theta)}\alpha_i\beta_i)(b_{l_f,K_f})$$

$$B_h(\alpha,\varphi) = -(\nabla\cdot\alpha,\varphi)_h = -\sum_{K_f\in T_{h_f}} \sum_{l_f=1}^{L_f} \omega_{l_f,K_f}(\nabla\cdot\alpha\varphi)(b_{l_f,K_f}).$$

The analysis of the convergence of finite element methods will make use of two useful projections.
Let the map $\{\tilde{u},\tilde{p}\} : [0,T] \to V_h \times W_h$ be the projection of the pressure solution $\{u,p\}$ given by

$$\begin{cases} A(c;\tilde{u},v) + B(v,\tilde{p}) = (\gamma(c),v), \quad v\in V_h, \quad (a) \\ B(\tilde{u},\varphi) = -(q,\varphi), \quad \varphi\in W_h. \quad (b) \end{cases} \tag{11}$$

Then, by [6] , the map exists and (5) implies that

$$\|u-\tilde{u}\|_V + \|p-\tilde{p}\|_W \leqslant M\{\inf_{v\in V_h}\|u-v\|_V + \inf_{\varphi\in W_h}\|p-\varphi\|_W\}$$

$$\leqslant M\|p\|_{L^\infty(H^{k+3}(\Omega))}h_p^{k+1} \tag{12}$$

where $M$ depends only on uniform bounds for $a_i(c)$, but not on $c$ itself.
Next, let $\tilde{c} : [0,T] \to M_h$ be the projection of $c$ given by

$$(D(u)\nabla(\tilde{c}-c),\nabla z) + (u\cdot\nabla(\tilde{c}-c),z) + (\lambda(\tilde{c}-c),z) = 0, z\in M_h, \tag{13}$$

where $\lambda = 1+q^+$.

Then, at any point $x\in\Omega$, decomposing $\nabla\xi$ into orthogonal components $\alpha$ and $\beta$, respectively parallel to $u$ and orthogonal to $u$, and using the assumption that $d_l \geqslant d_t$, by [6],

$$(D(u)\nabla\xi,\nabla\xi) + (u\cdot\nabla\xi,\xi) + (\lambda\xi,\xi) \geqslant (\phi(d_m + d_t\,|\,u\,|)\nabla\xi,\nabla\xi) + (\xi,\xi), \tag{14}$$

$$< d_t E(u)\nabla\xi + d_t E^\perp\nabla\xi, \nabla\xi >_{R^2} = d_l\,|\,\alpha\,|^2 + d_t\,|\,\beta\,|^2 \geqslant d_t\,|\,\nabla\xi\,|^2, \tag{15}$$

$$\|c-\tilde{c}\| + h_c\|c-\tilde{c}\|_l \leqslant M\|c\|_{l+1}h_c^{l+1}; \tag{16}$$

$$\|\frac{\partial}{\partial t}(c-\tilde{c})\| \leqslant M\{\|c\|_{l+1} + \|c_t\|_{l+1}\}h_c^{l+1}, \tag{17}$$

where $M$ depends on the $L^\infty$-norm of $u$ and $u_t$ and the ellipticity constant associated with $d_m\phi(x)$. There exists a constant $M$[6, 9, 12] such that

$$\|\nabla\tilde{p}\|_{L^\infty(L^\infty(\Omega))} + \|\nabla\tilde{c}\|_{L^\infty(L^\infty(\Omega))} + \|\tilde{c}_t\|_{L^\infty(H^{l+1})} \leqslant M \tag{18}$$

## 3. Lemmas

We point out that the general point of view in Ciarlet [1] for elliptic problems has provided a guide line for our development here. In what follows, let $S_h$ denote $M_h$ or $V_h$ or $W_h$ as appropriate.

**Lemma 3.1.** [1] *Assume that, for some integer $s \geqslant 1$,*

*(i) $\hat{P}_f = P_s(\hat{K}_f)$,*

*(ii) the union $\bigcup_{l_f=1}^{L_f}\left\{\hat{b}_{l_f}\right\}$ contains a $P_s(\hat{K}_f)$-unisolvent subset and/or the quadrature scheme is exact for the space $P_{2s}(\hat{K}_f)$. Then*

$$M_1\|w\|_h \leqslant \|w\| \leqslant M_2\|w\|_h, \quad w\in S_h,$$

$$|(w_1,w_2)_h| \leqslant M\|w_1\|_h\|w_2\|_h, \quad w_1,w_2\in S_h, \quad \textit{where} \quad \|w\|^2 \equiv (w,w)_h$$

**Lemma 3.2.** [1] *Assume $\tilde{g}\in C^0(K_f)$. Then for all $w_1,w_2\in S_h$,*

$$|E_{K_f}(\tilde{g}w_1w_2)| \leqslant M\|\tilde{g}\|_{L^\infty(K_f)}\|w_1\|_{L^2(K_f)}\|w_2\|_{L^2(K_f)},$$

*where $E_{K_f}(\cdot)$ is the quadrature error functional in (7).*

**Lemma 3.3.** [4] *Assume that, for some integer $s\geqslant 1, \hat{P} = P_s(\hat{K}_f)$ and that $\hat{E}(\hat{\phi}) = 0, \quad \forall\hat{\phi}\in P_{2s-1}(\hat{K}_f)$.*
*Then there exists a constant $M$ independent of $K_f\in T_{h_f}$ and $h_f$ such that for any*

$$\tilde{g}\in W^{s+1,\infty}(K_f), \quad \tilde{q}\in P_s(K_f), \tilde{q}'\in P_s(K_f),$$

$$|E_{K_f}(\tilde{g}\tilde{q}_{x_i}\tilde{q}'_{x_j})| \leqslant Mh_{f,K_f}^{s+1}\|\tilde{g}\|_{W^{s+1,\infty}(K_f)}\|\tilde{q}\|_{H^s(K_f)}|\tilde{q}'|_{H^1(K_f)},$$

*where $\quad h_{f,K_f} = diam(K_f)$.*

**Lemma 3.4.** [4] *Under the same hypotheses as in Lemma 3.3. Furthermore assume that there exists a number $q_0$ satisfying $s+1 \geqslant \dfrac{2}{q_0}$. Then there exists a constant $M$ independent of $K_f\in T_{h_f}$ and $h_f$ such that for any $\tilde{g}\in W^{s+1,q_0}(K_f)$ and any $w\in P_s(K_f)$,*

$$|E_{K_f}(\tilde{g}w)| \leqslant Mh_{f,K_f}^{s+1}(meas(K_f))^{\frac{1}{2}-\frac{1}{q_0}}\|\tilde{g}\|_{W^{s+1,q_0}(K_f)}\|w\|_{H^1(K_f)}.$$

## 4. Error Estimates

**Theorem.** *Let* $\{c,u,p\},\{C,U,P\},\{\tilde{u},\tilde{p}\},\tilde{c}$ *satisfy* (6), (10), (11) *and* (13), *respectively.*
*Let f denote c or p as appropriate and s denote l or k as appropriate. Assume that*

(i) $\hat{P}_f = P_s(\hat{K}_f)$,

(ii) *the quadrature scheme* $\int_{\hat{K}_f}\hat{\phi}(\hat{x})d\hat{x} \approx \sum_{l_f=1}^{L_f}\hat{\omega}_{l_f}\hat{\phi}(\hat{b}_{l_f}), \hat{\omega}_{l_f}$ *is exact for the space* $P_{2s}(\hat{K}_f)$ *and/or exact for the space* $P_{2(s-1)}(\hat{K}_f)$, *and the union* $\bigcup_{l_f=1}^{L_f}\left\{\hat{b}_{l_f}\right\}$ *contains a* $P_s(\hat{K}_f)$*-unisolvent subset.*
*Then, if* $C(0)$ *is determined in such a way that*

$\| C(0)-\tilde{c}(0) \| \leqslant M \| c_0 \|_{l+1} h_c^{l+1}$, *then for* $l\geqslant 1, \quad k\geqslant 0$ *and* $h$ *sufficiently small,*

$$\|c-C\|_{L^\infty(L^2(\Omega))}+\|u-U\|_{L^\infty(V)}+\|p-P\|_{L^\infty(W)}$$
$$\leqslant M\Big[\big\{1+\|c\|_{L^\infty(H^{l+1})(\Omega)}+\|c_t\|_{L^2\left(H^{l+1}(\Omega)\right)}\big\}h_c^{l+1}+$$
$$+\|p\|_{L^\infty\left(H^{k+3}(\Omega)\right)}h_p^{k+1}\Big].$$

*Proof.* With (12) and (16) known, the convergence analysis will have only to bound

$$U-\tilde{u}, \quad P-\tilde{p}, \quad \text{and} \quad C-\tilde{c}.$$

Let $E(w_1w_2)=(w_1,w_2)-(w_1,w_2)_h$.
We first consider the estimate of $U-\tilde{u}$ and $P-\tilde{p}$. Manipulation of (2.3b), (2.7b) and (11) leads to

(a)

$$A_h(C;U-\tilde{u},v)+B_h(v,P-\tilde{p})=A(c;\tilde{u},v)-A(C;\tilde{u},v)+$$
$$+(\gamma(C)-\gamma(c),v)+E(\frac{1}{a(C)}\tilde{u}v)+E(-\nabla\cdot v\tilde{p})-E(\gamma(C)v), \quad v\in V_h \tag{19}$$

(b) $\quad B_h(U-\tilde{u},\varphi)=E(q\varphi)+E(-\nabla\cdot\tilde{u}\varphi), \quad \varphi\in W_h.$

Existence and uniqueness of $U$ and $P$ can be proved based on ideas of [13, 14]. Hence, as in [6], it follows from assumptions (A), the quasi-regularity of the grid combined with the bound (12) and Lemmas 3.1, 3.3 and 3.4 that

$$\|U-\tilde{u}\|_V+\|P-\tilde{p}\|_W \leqslant M\big[(1+\|\tilde{u}\|_\infty)\|c-C\|+h_c^{l+1}\big(\|\tilde{u}\|_{W^{l+1,q_0}}+$$
$$+\|\tilde{p}\|_{l+1}+\|\gamma(C)\|_{W^{l+1,q_0}}+\|q\|_{W^{l+1,q_0}}+\|\nabla\tilde{u}\|_l\big)$$
$$\leqslant M\big[\| c-C \| +h_c^{l+1}\big] \tag{20}$$

where the constant $M$ depends only on constants in (A).

We now turn to the examination of the concentration equation.
Let $\eta=c-\tilde{c}, \quad \xi=C-\tilde{c}$ and $E(w_1w_2)=(w_1,w_2)-(w_1,w_2)_h$.
Subtract (2.3a) from (2.7a), apply (13), set $z=\xi$ and use the following relation

$$(\phi\xi_t,\xi)_h=\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)_h \quad \text{to obtain}$$

$$\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)_h+(U\cdot\nabla\xi,\xi)_h+(D(U)\nabla\xi,\nabla\xi)_h=(\phi\eta_t,\xi)-(\lambda\eta,\xi)+$$
$$-((U-u)\cdot\nabla\tilde{c},\xi)-((D(U)-D(u))\nabla\tilde{c},\nabla\xi)+E(\phi\tilde{c}_t\xi)+E(U\cdot\nabla\tilde{c}\xi)+$$
$$+E(D(U)\nabla\tilde{c}\nabla\xi)+E(g(C)\xi)=\sum_{i=1}^{9}R_i. \tag{21}$$

First, we shall bound the left-hand side of (19). As in [6], it follows from Lemma 3.1 and (15) that

$$\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)_h+(D(U)\nabla\xi,\nabla\xi)_h\geqslant\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)_h+\big(\phi(d_m+d_t\,|\,U\,|)\nabla\xi,\nabla\xi\big)_h \tag{22}$$

Using the argument of [6], it follows from Lemma 3.1 that

$$(U\cdot\nabla\xi,\xi)_h=-\frac{1}{2}(q\xi,\xi)_h-\frac{1}{2}B_h(u-U,\xi^2-\varphi), \quad \varphi\in W_h \quad \text{and}$$

$$\inf_{\varphi\in W_h}|(\nabla\cdot(u-U),\xi^2-\varphi)_h|\leqslant Mh_p\,\|\nabla\cdot(u-U)\|_\infty\|\nabla(\xi^2)\|_{L^1(\Omega)}$$
$$\leqslant Mh_p\{\|\nabla\cdot(u-\tilde{u})\|_\infty+\|\nabla\cdot(\tilde{u}-U)\|_\infty\}\|\xi\|\,\|\nabla(\xi)\|$$
$$\leqslant M\{\|p\|_{k+3}h_p^k+\|c-C\|\}\|\xi\|\|\nabla\xi\|\leqslant M\{1+\|c-C\|^2\}\|\xi\|^2+\varepsilon\|\nabla\xi\|^2$$
$$\leqslant M\{1+\|\xi\|^2\}\|\xi\|^2+\varepsilon\|\nabla\xi\|^2.$$

Thus

$$|(U\cdot\nabla\xi,\xi)_h|\leqslant M\{1+\|\xi\|^2\}\|\xi\|^2+\varepsilon\|\nabla\xi\|^2. \tag{23}$$

Hence

$$\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)_h+(U\cdot\nabla\xi,\nabla\xi)_h+(D(U)\nabla\xi,\nabla\xi)_h\geqslant\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)_h+$$
$$+(\{\phi(d_m+d_t|U|)-\varepsilon\}\nabla\xi,\nabla\xi)-M\{1+\|\xi\|^2\}\|\xi\|^2. \tag{24}$$

Now, we need to bound the right-hand side of (21). By using (18), we have

$$|R_1|+|R_2|+|R_3|+|R_5|\leqslant M\big[\|\eta_t\|+\|\eta\|+$$
$$+\|\nabla\tilde{c}\|_\infty\|u-U\|_{L^2(\Omega)^2}+\|c-C\|\big]\|\xi\|$$
$$\leqslant M\big[\|\eta_t\|^2+\|\eta\|^2+\|u-U\|^2_{L^2(\Omega)^2}+\|\xi\|^2\big].$$

Use [6] and (18) to see that

$$|R_4|=|-((D(U)-D(u))\nabla\tilde{c},\nabla\xi)|\leqslant M\|\nabla\tilde{c}\|_\infty\|u-U\|_{L^2(\Omega)^2}\|\nabla\xi\|$$
$$\leqslant M\|u-U\|^2_{L^2(\Omega)^2}+\varepsilon\|\nabla\xi\|^2.$$

By Lemma 3.4 and (18)

$$|R_6|=|E(\phi\tilde{c}_t\xi)|\leqslant Mh_c^{l+1}\|\tilde{c}_t\|_{l+1}\|\xi\|_1\leqslant Mh_c^{2(l+1)}+\varepsilon\|\xi\|_1^2$$

Observe that $R_7=E(U\cdot\nabla\tilde{c}\xi)=E((U-u)\cdot\nabla\tilde{c}\xi)+E(u\cdot\nabla\tilde{c}\xi)$
Thus, using Lemma 3.2 and (18), we see that

$$|R_7|\leqslant M\big[\|\nabla\tilde{c}\|_\infty+\|U-u\|_{L^2(\Omega)^2}\|\xi\|+h_c^{l+1}\|\tilde{c}\|_l\|\xi\|_1\big]$$
$$\leqslant M\big[\|U-u\|^2_{L^2(\Omega)^2}+\|\xi\|^2+h_c^{2(l+1)}\big]+\varepsilon\|\xi\|_1^2$$

Similar as in estimation of $R_7$, we have

$$R_8 = E(D(U)\nabla\tilde{c}\nabla\xi) = E((D(U)-D(u))\nabla\tilde{c}\nabla\xi) + E(D(u)\nabla\tilde{c}\nabla\xi)$$
$$\leqslant M\big[\|\nabla\tilde{c}\|_\infty \|U-u\|_{L^2(\Omega)^2}\|\nabla\xi\| + h_c^{l+1}\|\tilde{c}\|_l\|\nabla\xi\|\big]$$
$$\leqslant M\big[\|U-u\|^2_{L^2(\Omega)^2} + h_c^{2(l+1)}\big] + \varepsilon\|\nabla\xi\|^2.$$

Note that

$$R_9 = E(g(C)\xi) = E((g(C)-g(\tilde{c}))\xi) + E(g(\tilde{c})\xi).$$

Thus, using Lemma 3.4 and (2.1b) to see that

$$|R_9| \leqslant Mh_c^{l+1}\big[\|\xi\|_{l+1} + \|g(\tilde{c})\|_{l+1}\big]\|\xi\|_1$$
$$\leqslant Mh_c^{l+1}\big[h_c^{-(l+1)}\|\xi\| + \|g(\tilde{c})\|_{L^\infty(H^{l+1})}\big]\|\xi\|_1$$
$$\leqslant M\big[h_c^{2(l+1)} + \|\xi\|^2\big] + \varepsilon\|\xi\|_1^2.$$

Then, combine the above estimates $|R_i|, 1\leqslant i\leqslant 9$, and use (12), (16), (17) and (20) to obtain

$$\sum_{i=1}^{9}|R_i| \leqslant M\big[\{1+\|c\|^2_{l+1}+\|c_t\|^2_{l+1}\}h_c^{2(l+1)} + \|p\|^2_{k+3}h_p^{2(k+1)} + \|\xi\|^2\big] +$$
$$+\varepsilon\|\nabla\xi\|^2 \qquad (25)$$

Then, (24) and (25) imply that

$$\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)_h + \big(\phi\{d_m+d_t|U|\}-\varepsilon\}\nabla\xi,\nabla\xi\big)$$
$$\leqslant M\big[\{1+\|\xi\|^2\}\|\xi\|^2 + \{1+\|c\|^2_{l+1}+\|c_t\|^2_{l+1}\}h_c^{2(l+1)} +$$
$$+\|p\|^2_{k+3}h_p^{2(k+1)} + \|\xi\|^2\big] \qquad (26)$$

where $M$ depends on certain lower norms of the solution of the differential problem but not on the solution of the approximation problem.
Make the induction hypothesis that

$$\|\xi\|_{L^\infty(L^2(\Omega))} \leqslant 1; \qquad (27)$$

certainly, for any reasonable choice of the initial condition (27) holds for $t=0$. Thus (27) will hold for $t\leqslant T_h$ for some $T_h>0$; we shall show for $h=(h_c,h_p)$ sufficiently small that $T_h=T$ and that convergence will take place asymptotically at an optimal rate.
Integrate (26) in time and assume that

$$\|\xi(0)\| \leqslant M\|c_0\|_{l+1}h_c^{l+1}. \qquad (28)$$

Then, it follows from (26), (27) and Gronwall's Lemma that

$$\|\xi\|^2 \leqslant M\big[\{1+\int_0^t\|c\|^2_{l+1}d\tau + \int_0^t\|c_t\|^2_{l+1}d\tau\}h_c^{2(l+1)} + \int_0^t\|p\|^2_{k+3}d\tau h_p^{2(k+1)} + \|\xi(0)\|^2\big], \qquad (29)$$

thus, use (28) to obtain

$$\|\xi\|_{L^\infty(L^2(\Omega))} \leqslant M\big[\{1+\|c_0\|^2_{l+1}+\|c\|_{L^2(H^{l+1}(\Omega))}+\|c_t\|_{L^2(H^{l+1}(\Omega))}\}h_c^{l+1} + \|p\|_{L^2(H^{k+3}(\Omega))}h_p^{k+1}\big]. \qquad (30)$$

To complete the argument, note that (30) implies that the induction hypothesis (27) holds for small $h$.
Therefore use (30) with the inequalities (12),(16), (20) and the triangle inequality to obtain

$$\|c-C\|_{L^\infty(L^2(\Omega))} + \|u-U\|_{L^\infty(V)} + \|p-P\|_{L^\infty(W)}$$
$$\leqslant M\big[\{1+\|c\|_{L^\infty(H^{l+1}(\Omega))}+\|c_t\|_{L^2(H^{l+1}(\Omega))}\}h_c^{l+1} + \|p\|_{L^\infty(H^{k+3}(\Omega))}h_p^{k+1}\big]. \qquad (31)$$

□

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