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The effect of random isotropic inhomogeneities on the phase velocity of seismic waves

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SUMMARY

In this paper we study, theoretically and numerically, the influence of 2-D and 3-D random isotropic stationary inhomogeneities on the phase velocities of the transmitted compressional wavefield of an initially plane (or spherical) wave. Due to scattering by the inhomogeneities the wavefield becomes distorted as the wave propagates through the medium. The traveltimes fluctuate when considering different wavefield registrations acquired at the points of surfaces that are parallel to the wavefront of the initial wave. It is usually observed that the slowness obtained from the averaged traveltime differs from the averaged slowness of the medium. In the geophysical literature this effect has been termed the 'velocity shift'.

Using the Rytov approximation we establish formulas for the *frequency- and travel-distance-dependent* phase velocity of the transmitted wavefield in 2-D and 3-D randomly inhomogeneous media. We also compare our analytical results with finite-difference simulations. Good agreement between numerical simulations and theory is observed. The low-frequency limit of our analytical results coincides with the known effective-medium limit of the phase velocity in statistically isotropic inhomogeneous fluids with constant densities. In the high-frequency limit our results coincide with the results previously obtained by the ray-perturbation theory. However, in contrast to the ray theory, our description is not restricted to media with differentiable correlation functions of fluctuations. Moreover, our results quantify the frequency dependence of the velocity shift in the intermediate-frequency range. This frequency dependence is of major importance for estimating this effect in realistic situations.

Key words: finite-difference methods, fractals, ray theory, scattering, seismic velocities wave propagation.

INTRODUCTION

Heterogeneities of geologic media affect traveltimes and velocities of seismic waves. In a randomly inhomogeneous medium characterized by homogeneous statistics (such a medium is said to be stationary), the phase velocity of the direct arrivals differs from the averaged velocity or from the inverse averaged slowness. Recently this phenomenon, termed 'velocity shift', has been considered by Müller, Roth & Korn (1992), Roth, Müller & Snieder (1993), van Avendonk & Snieder (1994) and Ikelle & Yung (1994).

Numerical studies of this effect have involved different approaches to the wavefield simulation. These include finite differences (e.g. Müller *et al.* 1992), the Huygens method of the numerical solution of the eikonal equation (see Roth *et al.* 1993), and numerical procedures based on the ray-perturbation theory [Roth *et al.* (1993) used their eqs (8) and (9) for such numerical simulations].

Analytical studies of the velocity shift, however, have usually been restricted to the ray-perturbation theory, describing the velocity shift in the geometrical optic approximation. The resulting estimation of the velocity shift is independent of the frequency. However, it is physically obvious that the phase velocity of the transmitted wavefield is frequency-dependent, because inhomogeneities interact differently with wavefields of different wavelengths. This was also observed by Yin *et al.* (1994) and Mukerji *et al.* (1995) in numerical and laboratory experiments.

In the case of random media with non-differentiable correlation functions (for example the exponential correlation function: if the Fourier-domain wavenumber is large enough then this correlation function will have a power-law Fourier spectrum describing fractal inhomogeneities), the ray-perturbation theory does not provide any analytical prediction.

In this paper we consider the problem of the velocity shift as part of a theoretical description of the phase velocity of a transmitted wavefield in a random structure. In the case of 1-D randomly inhomogeneous media the frequency- and angle-dependent velocities of the transmitted seismic waves can be described by the generalized O'Doherty–Anstey formulae [which are a variant of the second-order Rytov approximation for 1-D structures; see e.g. Shapiro, Hubral & Zien (1994); Shapiro & Hubral (1995)]. It is observed that in 1-D stationary media the phase velocity is independent of the travel distance (at least in an intermediate asymptotical range of the travel distances). The same is true for the attenuation coefficients (in this paper we consider media without anelastic loss, that is, the attenuation coefficients describe scattering effects only). In contrast, in 2-D and 3-D media the phase velocity, as well as the attenuation coefficient, essentially depends on the travel distance.

Here we present a theory that establishes formulas for the frequency-dependent phase velocity of the transmitted wavefield in 2-D and 3-D randomly inhomogeneous media. This theory is a kinematic extension of the description of the scattering attenuation proposed by Shapiro & Kneib (1993). It is based on the Rytov approximation for the variances and covariances of the phase and of the logarithm of amplitude of the transmitted wavefield. The limitations of the theory are acoustic media with a constant density, isotropic stationary inhomogeneities with spatial sizes (correlation lengths) of the order of or larger than the wavelength, and small fluctuations of the wavefield.

THEORY

Dynamic equations and formulation of the problem

We consider a plane scalar wave propagating in a 2-D or 3-D randomly inhomogeneous medium with constant density. In this medium we define the local squared slowness $p^2(\mathbf{r}) \equiv (1/c_0^2)(1+2n(\mathbf{r}))$, where the function $n(\mathbf{r})$ is assumed to be a realization of a stationary statistically isotropic random field with zero average $\langle n(\mathbf{r})\rangle$ (the angular brackets denote statistical averaging). We also assume that this random field is ergodic. In these notations the local propagation velocity c in each point $\mathbf{r}$ of the medium is defined by $c(\mathbf{r}) \equiv c_0(1+2n(\mathbf{r}))^{-1/2}$. The constant c_0 gives the propagation velocity in a homogeneous *reference medium.* The function $n(\mathbf{r})$ can be considered as a function describing velocity fluctuations in the case of $|n| \ll 1$: $c(\mathbf{r}) \approx c_0(1-n(\mathbf{r}))$. Note, however, that the quantity c_0 is not the average velocity, but rather the velocity obtained from the averaged squared slowness.

In these notations the wave equation reads:

$$\Delta u - p^2(\mathbf{r})\frac{\partial^2 u}{\partial t^2} = 0, \tag{1}$$

where u describes a scalar wavefield (for example the pressure).

In the theoretical treatment we consider propagation of time-harmonic waves, that is, the time dependence of u is given by the term $\exp(-i\omega t)$, which we will omit below. At an arbitrary point of the random medium we can express the wavefield in the following form:

$$u(\mathbf{r}) = u_0(\mathbf{r})\exp(\chi(\mathbf{r}) + i\phi(\mathbf{r})), \tag{2}$$

where $u_0(\mathbf{r})$ is the wavefield in the homogeneous reference medium, the function $\chi(\mathbf{r})$ denotes fluctuations of the logarithm of the wavefield's amplitude, and the function $\phi(\mathbf{r})$ denotes fluctuations of its unwrapped phase (that is, the phase that changes continuously from zero to infinity, rather than from $-\pi$ to π).

The plane wavefield $u_0(\mathbf{r})$ can, in turn, be written in a similar form:

$$u_0(\mathbf{r}) = A_0\exp(i\phi_0(\mathbf{r})), \tag{3}$$

where A_0 is its amplitude and $\phi_0(\mathbf{r})$ is its unwrapped phase.

Let us now consider the following geometry of the problem (see also Fig. 6). An initial plane wave propagates along the x-axis and impinges normally from a layer of a homogeneous medium on a layer of a 3-D (or 2-D) randomly inhomogeneous medium. The wavefront of this wave is parallel to the plane (y, z). The inhomogeneous medium starts at $x = 0$. The positive direction of the x-axis points into the inhomogeneous medium.

In order to estimate the phase velocity of the wavefield in a random medium we must find *traveltimes* at points located along a plane defined by the equation $x = L$, where L is a constant travel distance. The expression

$$v = \frac{L}{\langle \text{traveltime} \rangle} \tag{4}$$

will give the phase velocity if arrivals of a given phase are picked. However, in the case of time-harmonic waves the traveltime is given by

$$\text{traveltime} = \frac{\text{phase}}{\omega}, \tag{5}$$

where the phase is assumed to be unwrapped. Taking into account eqs (2)–(4), we now obtain

$$v = \frac{\omega L}{\phi_0(L) + \langle\phi(L)\rangle}. \tag{6}$$

Therefore, we have reduced the problem to the consideration of the quantity $\langle\phi(L)\rangle$.

Weak- and strong-fluctuation regions

In order to describe the quantity $\langle\phi(L)\rangle$ we consider scattering in two different regimes of wave propagation: regions of weak and strong fluctuations of the wavefield. At an arbitrary point $\mathbf{r}$ of a random medium the wavefield can be written as

$$u(\mathbf{r}) = \langle u(\mathbf{r})\rangle + u_f(\mathbf{r}), \tag{7}$$

where $\langle u\rangle$ is *the coherent field* (the coherent field is the wavefield averaged over the statistical ensemble of medium's realizations; it is also called *the mean field*), and u_f is the fluctuation of u, subsequently called the incoherent field. Its mean is $\langle u_f\rangle = 0$.

As a measure of wavefield fluctuations we introduce the parameter ε defined as the ratio of the incoherent field to the coherent field:

$$\varepsilon \equiv \frac{|u_f(L)|}{|\langle u(L)\rangle|}. \tag{8}$$

In a random medium without energy dissipation, the intensity of the coherent field $I_c = |\langle u\rangle|^2$ attenuates due to energy transfer from $\langle u\rangle$ to u_f. The expression of energy conservation in terms of the intensities follows from eq. (7) after multiplication with the corresponding complex-conjugate equation and averaging:

$$I_t = I_c + I_n, \tag{9}$$

where $I_t = \langle |u|^2 \rangle$ is the total intensity and $I_n = \langle |u_f|^2 \rangle$ is the intensity of the incoherent wavefield. Taking into account definition (8) we obtain

$$\langle \varepsilon^2 \rangle = I_t / I_c - 1. \tag{10}$$

The region of weak wavefield fluctuations is limited to small propagation distances, where $\langle \varepsilon^2 \rangle \ll 1$, i.e. the total intensity is dominated by the coherent intensity. For large L the coherent intensity is small, i.e. we have $\langle \varepsilon^2 \rangle \gg 1$ and this part of the medium is the region of strong wavefield fluctuations. The transition from the weak-fluctuation region to the strong-fluctuation region occurs where $\langle \varepsilon^2 \rangle = \mathcal{O}(1)$.

For small travel distances L in the weak-fluctuation region $\langle \varepsilon^2 \rangle \ll 1$ and we can roughly estimate this quantity from the exponential attenuation of the coherent intensity. Assuming that the total intensity is constant (i.e. neglecting backscattering and inelasticity) we obtain from (10) $\langle \varepsilon^2 \rangle \approx 2\alpha L$, where α is *the scattering coefficient of the coherent (i.e. mean) field.* If the correlation distance a of the inhomogeneities is of the same order as or larger than the wavelength λ then the coefficient α can in turn be roughly estimated as $\alpha = \mathcal{O}(\sigma^2 k^2 a)$, where $k = 2\pi/\lambda$ and σ^2 is the variance of the quantity n described above (i.e. velocity fluctuations). This estimation is valid for 2-D as well as 3-D media and can be directly obtained from eqs (A2–16) and (A2–17) of Shapiro & Kneib (1993), for example.

The wavefield behaves differently in the regions of weak and strong wavefield fluctuations. We solve the problem in the region of small wavefield fluctuations. Note that the ray-perturbation theory is also valid in the weak-fluctuation region only. Later in this section we consider the applicability of our results to the strong-fluctuation region using the estimation of the quantities ε and α given above.

The averaged phase of the wavefield

In the following we assume that the quantity ε is a small parameter. Using eqs (2) and (3), we obtain:

$$\frac{u(\mathbf{r})}{u_0(\mathbf{r})} = e^{\chi + i\phi} \tag{11}$$

With χ as defined in eq. (2), we obtain:

$$\phi = \frac{1}{i} \ln\left[\frac{u}{u_0}\left|\frac{u_0}{u}\right|\right] = \frac{1}{i} \ln\left[\frac{u}{u_0}\sqrt{\frac{u_0}{u}\frac{u_0^*}{u^*}}\right]. \tag{12}$$

After inserting eq. (7), omitting terms of order higher than $\mathcal{O}(\varepsilon^2)$ and averaging, we obtain the following result:

$$\langle \phi \rangle = \frac{1}{i}\left[\frac{1}{2}\ln\frac{\langle u \rangle}{u_0} - \frac{1}{2}\ln\frac{\langle u \rangle^*}{u_0^*} + \frac{1}{4}\langle \varepsilon_u^{*2} - \varepsilon_u^2 \rangle\right], \tag{13}$$

where $\varepsilon_u = u_f / \langle u \rangle$ and $|\varepsilon_u| = \varepsilon$.

Substituting now in eq. (13) the following representation of the coherent field: $\langle u \rangle = \sqrt{I_c}\exp(i\phi_c)$, where ϕ_c represents the phase of the coherent wavefield, we obtain:

$$\langle \phi \rangle = \phi_c - \phi_0 - \frac{1}{4i}\langle (\varepsilon_u^2 - \varepsilon_u^{*2}) \rangle. \tag{14}$$

An analogous treatment yields:

$$\langle \chi \rangle = \ln\left(\left|\frac{\langle u \rangle}{u_0}\right|\right) - \frac{1}{4}\langle (\varepsilon_u^2 + \varepsilon_u^{*2}) \rangle, \tag{15}$$

$$\langle \chi\phi \rangle = -\frac{1}{4i}\langle (\varepsilon_u^{*2} - \varepsilon_u^2) \rangle \tag{16}$$

[note that $\ln|\langle u \rangle / u_0|$ is a quantity of order $\mathcal{O}(\varepsilon^2)$].

Taking into account that the variance of a property X is defined by

$$\sigma_x^2 \equiv \langle (X - \langle X \rangle)^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2, \tag{17}$$

and the cross-variance $\sigma_{\chi\phi}^2$ reads

$$\sigma_{\chi\phi}^2 \equiv \langle (\chi - \langle \chi \rangle)(\phi - \langle \phi \rangle) \rangle = \langle x\phi \rangle - \langle \chi \rangle \langle \phi \rangle, \tag{18}$$

using eqs (14)–(16) and neglecting terms of order higher than $\mathcal{O}(\varepsilon^2)$, we obtain the following:

$$\sigma_{\chi\phi}^2 = -\frac{1}{4i}\langle (\varepsilon_u^{*2} - \varepsilon_u^2) \rangle. \tag{19}$$

Eq. (19) shows that the last term in eq. (14) is the same as $-\sigma_{\chi\phi}^2$. Therefore, eq. (14) becomes

$$\langle \phi \rangle = \phi_c - \phi_0 - \sigma_{\chi\phi}^2. \tag{20}$$

We wish to emphasize that this equation can also be obtained without any assumption of weak fluctuation of the wavefield. For this, however, the normal distribution of the quantities χ and ϕ in eq. (11) must be assumed. To show this let us statistically average the left- and right-hand sides of eq. (11). The left-hand side then provides

$$|\langle u \rangle / u_0|\, e^{i(\phi_c - \phi_0)}. \tag{21}$$

Averaging of the right-hand side of eq. (11) requires averaging of the exponential function. Assuming that the quantities χ and ϕ are normally distributed, and taking into account definitions (17) and (18), we obtain:

$$\langle e^{\chi + i\phi} \rangle = e^{\langle \chi \rangle + i\langle \phi \rangle}\, e^{(\sigma_x^2 - \sigma_\phi^2 + 2i\sigma_{\chi\phi}^2)/2}. \tag{22}$$

Equating the imaginary parts of the logarithms of relations (21) and (22) we again obtain eq. (20), now, however, without any assumption of weak fluctuation of the wavefield.

Substituting eq. (20) into eq. (6) we obtain

$$v = \frac{\omega L}{\phi_c(L) - \sigma_{\chi\phi}^2(L)}. \tag{23}$$

In eq. (23) there are two unknown quantities. These are ϕ_c and $\sigma_{\chi\phi}^2$. The first one is found from a consideration of the coherent wavefield in a random medium. For this we applied the Bourret approximation (see Appendix A), which is well known in the theory of multiple scattering (Rytov *et al.* 1987). The second quantity, $\sigma_{\chi\phi}^2$, is found using the Rytov approximation (Ishimaru 1978; for the derivation in 2-D media see our Appendix B). Both these approximations are valid under our assumptions.

The phase velocity in random media

Eqs (6), (20), (A16), (A17), (B13) and (B14) provide the following results for plane waves. The phase velocity in 2-D random media is given by

$$v_{2D} \approx c_0\left[1 - 4k^2\pi\int_0^\infty \frac{\sin^2(\xi^2 L/(2k))}{\xi^2 L}\Phi_n^{2D}(\xi)\,d\xi + 4k^2\pi\int_{2k}^\infty \frac{\Phi_n^{2D}(\xi)}{\sqrt{\xi^2-4k^2}}\,d\xi\right]^{-1}, \quad (24)$$

where $k \equiv \omega/c_0$ is the wavenumber in the homogeneous reference medium, $\Phi_n^{2D}(\xi)$ is the 2-D power spectrum of velocity fluctuations (i.e. the 2-D Fourier transform of the autocorrelation function of the quantity n) and ξ is the absolute value of the Fourier-domain wavenumber.

In 3-D random media we obtain:

$$v_{3D} \approx c_0\left[1 - 4k^2\pi^2\int_0^\infty \frac{\sin^2(\xi^2 L/(2k))}{\xi L}\Phi_n^{3D}(\xi)\,d\xi + k\pi\int_0^\infty \ln\left(\frac{2k+\xi}{2k-\xi}\right)^2 \Phi_n^{3D}(\xi)\xi d\xi\right]^{-1}, \quad (25)$$

where $\Phi_n^{3D}(\xi)$ denotes a 3-D power spectrum of velocity fluctuations.

Sometimes, for analytical as well as for numerical computations, another form of the last integral terms of the above formulas is useful. For the 2-D case the last term in the brackets of eq. (24) can be replaced by

$$-\pi k^2\int_0^\infty N_0(kr)J_0(kr)B_n(r)r dr, \quad (26)$$

where N_0 and J_0 are corresponding Neumann and Bessel functions and $B_n(r)$ is the autocorrelation function of the velocity fluctuations. (Due to our assumption of statistical isotropy of the media under consideration this function depends on the absolute value r of the correlation lag only.) Finally, the last term in the brackets of eq. (25) can be replaced by

$$k\int_0^\infty \sin(2kr)B_n(r)\,dr. \quad (27)$$

In addition, we analyse these results for two types of statistically isotropic random media: with exponential and with Gaussian autocorrelation functions. In the following we call these media exponential and Gaussian, respectively. In a Gaussian medium the correlation function $B_n(r)$ is

$$B_n(r) = \sigma^2 e^{-r^2/a^2}, \quad (28)$$

where a is the correlation length of the velocity fluctuations and σ is their standard deviation.

The Fourier transforms of the correlation function are

$$\Phi_n^{2D}(\xi) = \frac{\sigma^2 a^2}{4\pi} e^{-\xi^2 a^2/4}, \quad (29)$$

and

$$\Phi_n^{3D}(\xi) = \frac{\sigma^2 a^3}{8\pi\sqrt{\pi}} e^{-\xi^2 a^2/4}. \quad (30)$$

The corresponding quantities of an exponential medium are

$$B_n(r) = \sigma^2 e^{-r/a}, \quad (31)$$

$$\Phi_n^{2D}(\xi) = \frac{\sigma^2 a^2}{2\pi(1+\xi^2 a^2)^{3/2}}, \quad (32)$$

$$\Phi_n^{3D}(\xi) = \frac{\sigma^2 a^3}{\pi^2(1+\xi^2 a^2)^2}. \quad (33)$$

Substituting the analytical expressions of the power spectra of the velocity fluctuations in the above formulas for the phase sometimes provides explicit analytical results.

Let us consider a 2-D Gaussian medium. For this, eq. (29) must be substituted in (24). The first integral term in (24) is $\sigma_{\chi\phi}^2/(kL)$. The quantity $\sigma_{\chi\phi}^2$ for a Gaussian medium is calculated in Appendix C. Therefore, for the first integral term we obtain

$$\sigma_{\chi\phi}^2/(kL) = \frac{\sigma^2 a^2 k^2}{2}\sqrt{\frac{\pi}{kL}}\left[\left(1+\frac{a^4k^2}{16L^2}\right)^{1/4} \times \cos\left(\frac{1}{2}\arcsin\sqrt{1+\frac{a^4k^2}{16L^2}}^{\,-1}\right) - \left(\frac{a^2k}{4L}\right)^{1/2}\right] \quad (34)$$

After inserting the fluctuation spectrum and substituting $x \equiv \xi^2$, we obtain the second integral term in (24),

$$-\frac{1}{2}k^2\sigma^2a^2\int_{4k^2}^\infty \frac{e^{-xa^2/4}}{\sqrt{x^2-4k^2x}}\,dx \quad (35)$$

$$= -\frac{1}{2}k^2\sigma^2a^2\left(\int_0^\infty - \int_0^{4k^2}\right)\frac{e^{-xa^2/4}}{\sqrt{x^2-4k^2x}}\,dx \quad (36)$$

$$= -\frac{1}{2}k^2\sigma^2a^2 e^{-k^2a^2/4}\left(i\pi I_0\left(\frac{k^2a^2}{2}\right) + K_0\left(-\frac{k^2a^2}{2}\right)\right) \quad (37)$$

$$= -\frac{1}{2}k^2\sigma^2a^2 e^{-k^2a^2/4}K_0\left(\frac{k^2a^2}{2}\right). \quad (38)$$

Here I_0 and K_0 are modified Bessel functions. To perform the integration we used tables of definite integrals [Gradshteyn & Ryzhik (1983), integrals (1) and (2) from p. 322]. Therefore, all terms in brackets in eq. (24) in the case of Gaussian media now have closed analytical forms.

Generally, however, it is not always possible to perform the integration in (24) and (25) analytically and it is very instructive to analyse the frequency dependence of the phase velocities. It is clear that the frequency-dependent part of the phase velocity is given by the difference $v - c_0$. In the following, instead of this quantity we consider the velocity shift, which we introduce in accordance with its definition by Roth *et al.* (1993):

$$\frac{\delta v}{v_0} = \frac{v - v_0}{v_0}, \quad (39)$$

where $v_0 \equiv \langle 1/c(\mathbf{r})\rangle^{-1} \approx c_0(1+\sigma^2/2)$. We will neglect all terms of order higher than $\mathcal{O}(\sigma^2)$. Taking this into account and considering eqs (24) and (25), along with the above definition (39) for the case of small values of the velocity shift, we obtain

$$\frac{\delta v}{v_0} = \sigma^2 S(ka, L/a), \quad (40)$$

where $S(ka, L/a)$ is a function of two arguments. Before we analyse the form of the function $S(ka, L/a)$, however, let us consider its asymptotical features in the high- and low-frequency limits.

High- and low-frequency asymptotic solutions

Let us first consider a 2-D Gaussian medium. The high-frequency limit of formula (34), where $ak \to \infty$, provides

$$\sigma_{\chi\phi}^2/(kL) \to \sigma^2 L\frac{\sqrt{\pi}}{2a}. \quad (41)$$

This asymptotic result can also be directly obtained from the Taylor expansion of the cosine function in (C2).

In the limit $ka \to \infty$, the second integral term in the brackets of eq. (24), given by eq. (38), tends to zero as

$$\frac{\sqrt{\pi}}{2}\sigma^2 ka\, e^{-k^2a^2}. \tag{42}$$

Therefore, in the high-frequency limit it provides no contribution.

Analogous calculations can be made for plane waves in 3-D Gaussian media and for point sources in 2-D and 3-D Gaussian media [see formulas (B15) and (B16) in Appendix B]. Neglecting all terms of order higher than $\mathcal{O}(\sigma^2)$, we found the following results for the high-frequency (i.e. geometrical-optic) limit.

2-D, plane wave:

$$\frac{\delta v}{v_0} = \frac{\sigma^2}{2}\left(\frac{\sqrt{\pi}L}{a} - 1\right); \tag{43}$$

3-D, plane wave:

$$\frac{\delta v}{v_0} = \sigma^2\left(\frac{\sqrt{\pi}L}{a} - 1\right); \tag{44}$$

2-D, point source:

$$\frac{\delta v}{v_0} = \frac{\sigma^2}{2}\left(\frac{\sqrt{\pi}L}{3a} - 1\right); \tag{45}$$

3-D, point source:

$$\frac{\delta v}{v_0} = \sigma^2\left(\frac{\sqrt{\pi}L}{3a} - 1\right). \tag{46}$$

These are exactly the same results as those given by ray-perturbation theory (van Avendonk & Snieder 1994).

More generally, let us consider the high-frequency limits of the integral terms in the brackets of eqs (24) and (25). Then the first terms must be considered in the geometrical-optic limit $ka \gg L/a$ (Fresnel zone $\sqrt{L\lambda}$ is smaller than the size of inhomogeneities, i.e. the diffraction effects are weak). In this case we can approximate the sine functions by the first terms of their Taylor expansion [the large values of $\xi \gg 1/a$ do not give a significant contribution in the integrals because of vanishing fluctuation spectra; see Ishimaru (1978), p. 361 for an excellent explanation of this argument]. We obtain the following expressions for the first integral terms:

$$-\pi L \int_0^\infty \xi^2 \Phi_n^{2D}(\xi)\, d\xi, \tag{47}$$

$$-\pi^2 L \int_0^\infty \xi^3 \Phi_n^{3D}(\xi)\, d\xi. \tag{48}$$

The second integral terms are independent of L/a and therefore can be considered just for $ka \gg 1$. Using again the argumentation with vanishing fluctuation spectra for large values of ξ we see that in the 2-D case the second integral term provides a zero contribution in the high-frequency limit. In the 3-D case, using the Taylor expansion of the logarithm, we obtain the following limit:

$$2\pi \int_0^\infty \xi^2 \Phi_n^{3D}(\xi)\, d\xi. \tag{49}$$

For Gaussian media, limits (47)–(49), after substitution in the corresponding equations for phase velocities, provide results (43)–(46). However, integrals (47)–(49) are not necessarily converging for arbitrary models of stochastic structures. For instance, in the case of exponential media, integrals (47)–(48) diverge due to too slow a rate of decay of fluctuation spectra in the high-frequency limit. Therefore, for such media the geometrical optics approximation of the velocity shift is singular. This is the same singularity as was found by Roth *et al.* [1993, see their eq. (17)] using the ray-peturbation-theory consideration.

We interpret this interesting fact in the following way. In exponential media, due to the fractal character of inhomogeneities (in the high-frequency range the fluctuation spectra have power-law form), there always exist significant fluctuations of a spatial scale smaller than or of the same order as the wavelength, no matter how small the wavelength may be. Therefore, the diffraction effects are always significant and ray theory in its extreme high-frequency approximation becomes singular. For such media, eqs (24) and (25) also have a singularity in the high-frequency limit. However, they provide velocity estimations for any finite frequency. To what extent such estimations are reliable depends on the validity range of our results, which we discuss later in this section.

Let us now consider the possibility of applying eqs (24) and (25) in the low-frequency range. Our theoretical consideration of $\sigma^2_{\chi\phi}$, providing the first integral terms in (24) and (25), is not valid for inhomogeneities of small spatial scale due to the approximations used in Appendix B in order to perform the integration in eq. (B1). Therefore, strictly speaking, expressions (24) and (25) are valid in the range $ka > 1$ only. However, in the low-frequency range $ka < 1$ the influence of $\sigma^2_{\chi\phi}$ on the phase velocity becomes vanishing in comparison with other factors. This can be acknowledged from the following short consideration of eq. (23). As is clear from eq. (19) the quantity $\sigma^2_{\chi\phi}$, at least in the weak fluctuation region, has the order $\mathcal{O}(\langle\varepsilon^2\rangle) = \mathcal{O}(\alpha L)$ and, therefore, in the low-frequency range it is proportional to ω^{m+1} in m-dimensional media (due to Rayleigh scattering). The quantity ϕ_c, however, is proportional to ω due to the frequency-independent part of the phase velocity of the coherent field (i.e. the static limit of this velocity). It is interesting to note that in the Rayleigh-scattering regime even the part of the phase ϕ_c corresponding to the frequency-dependent part of the coherent-field phase velocity has weaker frequency dependence than α. One can see this after a careful consideration of the low-frequency limits of the real and imaginary parts of the effective wavenumbers [given for example in formula (4.59), p. 138 of Rytov, Kravtzov & Tatarskii (1987) for 3-D media, and in formula (A14) of this paper for 2-D media]. Therefore, in the low-frequency range the term ϕ_c becomes dominant. The phase velocity tends to the velocity of the coherent wavefield. Our approximation for the coherent wavefield does not have any restriction limiting its validity range from below in the frequency domain. Therefore, the zero-frequency limit of eqs (24) and (25) should provide an exact result, and they can be used for other frequencies from the low-frequency range $ka < 1$, at least qualitatively. In the range $ka > 1$, eqs (24) and (25) again become quantitatively valid. The low-frequency limit of eqs (24) and (25) yields $v_{2D,3D} \to c_0$, i.e. the velocity shift becomes negative, $\delta v/v_0 \to -\sigma^2/2$. It is interesting to note that this limiting value of the phase velocity exactly coincides with the well-known effective-medium limit for acoustic media with constant density (see e.g. Sheng 1995, p. 78).

Validity range

The validity range of eqs (24) and (25) is restricted mainly by the three following basic assumptions that we made during our derivations. First, we assumed that the wavefield fluctuations are small. Using the estimations of the quantities $\langle\varepsilon^2\rangle$ and α, which we found in our discussion of the weak and strong fluctuation regions, we obtain $\langle\varepsilon^2\rangle \approx 2\alpha L = \mathcal{O}(\alpha^2 k^2 aL)$. Therefore, we can express the first limitation approximately by the inequality

$$\vartheta \equiv \sigma^2(ka)^2(L/a) < 1\,. \tag{50}$$

Above, we have shown that for the derivation of eq. (20) the assumption of weak fluctuations of the wavefield can be replaced by the assumption of a normal distribution of the quantities χ and ϕ in eq. (11). A short discussion on the probability distribution of the wavefield can be found in Ishimaru (1978, pp. 447–448). It seems that this assumption is relevant for seismic practice, and inequality (50) should not be considered as a very restrictive one. However, it must be taken into account that the Rytov approximation of the quantity $\sigma^2_{\chi\phi}$ requires that the wavefield fluctuations should not be too large, i.e. the quantity ϑ should not be too large in comparison with 1.

The second restriction,

$$ka > 1\,, \tag{51}$$

also relates to the derivation of the Rytov approximation of the quantity $\sigma^2_{\chi\phi}$. It arises due to the neglecting of the contribution of the backscattering in the wavefield fluctuations. We have, however, seen from the above discussion of the low-frequency limits of eqs (24) and (25) that this restriction is not very critical. These equations provide an exact low-frequency limit and, at least qualitatively, can also be used in the frequency range $ka < 1$.

Finally, the third restriction is related to the derivation of the phase of the coherent wavefield, where we used the Bourret approximation. The validity domain of this approximation can be roughly expressed as (Rytov *et al.* 1987, pp. 140–141)

$$\sigma^2(ka)^2 < 1\,. \tag{52}$$

A review of all three limitations of our theoretical results shows that the first restriction is the strongest. In the following we compare the theoretical predictions with results of numerical simulations in the very broad range of values of the quantity ϑ, $10^{-2} < \vartheta < 6$ (see Fig. 8), and we find good agreement between the theoretical and numerical results for $\vartheta < 4$. It is interesting to note that in the case of Gaussian media a comparison of our theoretical predictions with results of the FD solutions of the eikonal equation (Huygens method) from the paper of Roth *et al.* (1993) shows that the theory can provide a correct order of the velocity shift even for much larger values of ϑ.

Let us now consider the following seismologically relevant situation: $\lambda/a \approx 1$ (for example, the wavelength $\lambda = 10$ km and the correlation distance $a = 10$ km, which is a possible order of the correlation distances in the lithosphere), and a velocity fluctuation of 2 per cent. In this case the restriction $\vartheta < 4$ implies that our theoretical results are applicable for $L/a < 250$.

Frequency dependence of the phase velocity

Because in our approximation the function $S(ka, L/a)$ from eq. (40) has a universal form independent of σ^2 we can consider the normalized quantity $\delta v/(v_0\sigma^2)$ as a function of two arguments, ka and L/a, to gain an impression of the frequency dependence of the phase velocity. We shall, however, be more specific, in order to gain an idea of the possible orders of magnitude of the velocity shift.

In the following we consider the frequency dependences of the velocity shift for 2-D and 3-D Gaussian and exponential media with $\sigma = 0.05$ and $L/a < 6$. In the next section we use these parameters for some of our numerical illustrations of the velocity shift in 2-D exponential media. Further, we plot the velocity shift in 3-D exponential and Gaussian media for $L/a \leq 250$ and $\sigma = 0.02$. In spite of different values of σ, the ka- and L/a-dependences of all these curves are universal.

Figs 1 and 2 show the phase velocities versus the normalized frequency ka in the case of 2-D and 3-D Gaussian random media. These curves show a typical dispersion behaviour similar to frequency dependences of the phase velocities in 1-D media (Shapiro & Hubral 1995). The most important difference is that now the phase velocity is also travel-distance-dependent. This dependence becomes more or less unimportant in the frequency range $ka \lesssim 5$. With increasing L/a the dispersion

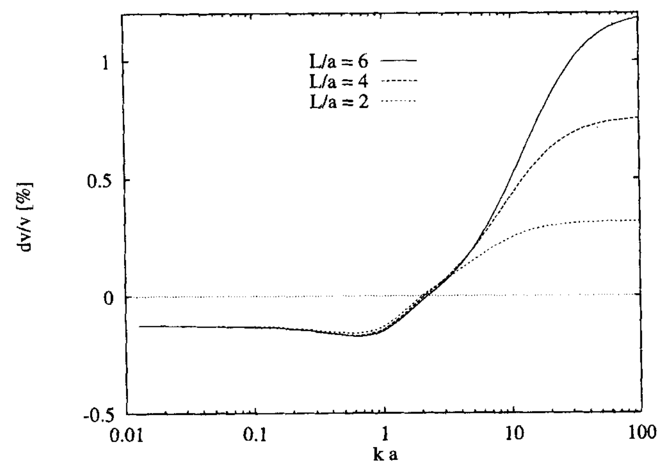

Figure 1. Velocity shift of an initially plane wave as a function of frequency at different distances L, produced by fluctuations of 5 per cent and a correlation length of 60 m in a 2-D Gaussian medium.

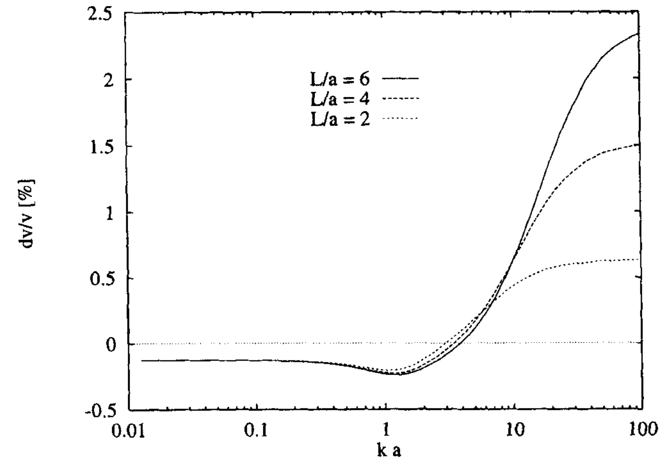

Figure 2. Velocity shift of an initially plane wave as a function of frequency at different distances L, produced by fluctuations of 5 per cent and a correlation length of 60 m in a 3-D Gaussian medium.

becomes larger. It is important for the frequency range $1 < ka \lesssim 10\,L/a$. As is clear from the discussion above, in the frequency range $ka < 1$ features of the frequency dependence of the phase velocities are controlled by those of the phase velocity of the mean field. In turn, the behaviour of the mean-field phase velocity is closely related to the features of the frequency dependence of the scattering cross-sections.

The corresponding curves for exponential media are similar to those for Guassian media. However, eqs (24) and (25) have singularities in the high-frequency limit, and, therefore, for a given relation L/a the velocity shift does not reach a constant value in the limit of infinite ka (Fig. 3). A general similarity between the curves in the 2-D and 3-D cases is seen.

The next two plots of the velocity shift in 3-D Gaussian and exponential media (Figs 4 and 5) are especially interesting: they show this quantity in the intermediate-frequency range $1 < ka < 100$ for increasing L/a. It is clearly seen that if ka is not too large the velocity shift will decrease with increasing

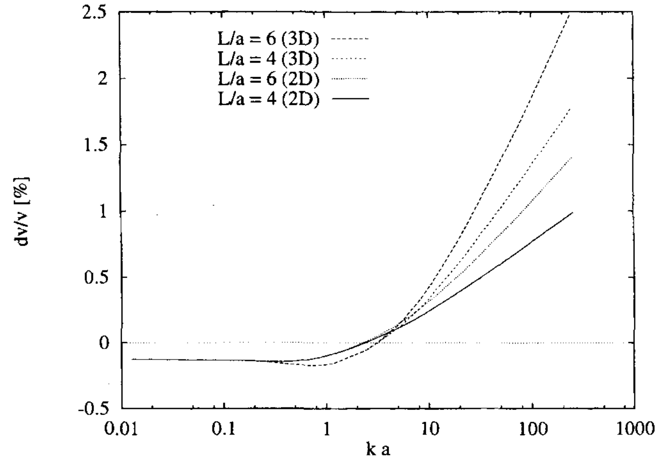

Figure 3. Velocity shift of an initially plane wave as a function of frequency at different distances L, produced by fluctuations of 5 per cent and a correlation length of 60 m in an exponential medium in two dimensions and three dimensions. Due to the broad range of the argument in each decade 10^n, only the ticks corresponding to $1, 2, 5, 8 \times 10^n$ are shown along the abscissa.

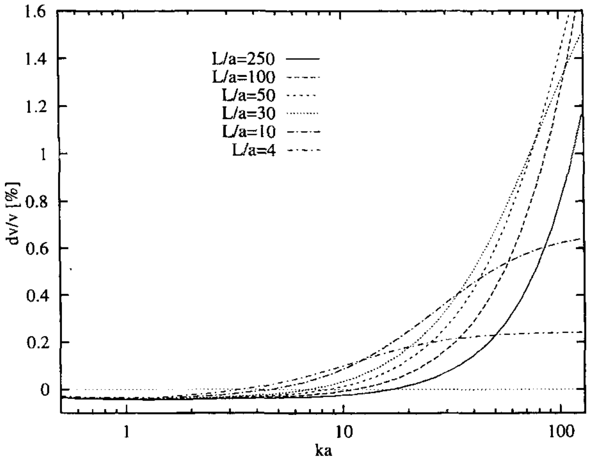

Figure 4. Velocity shift of an initially plane wave as a function of frequency at different distances L, produced by fluctuations of 2 per cent in a 3-D Gaussian medium.

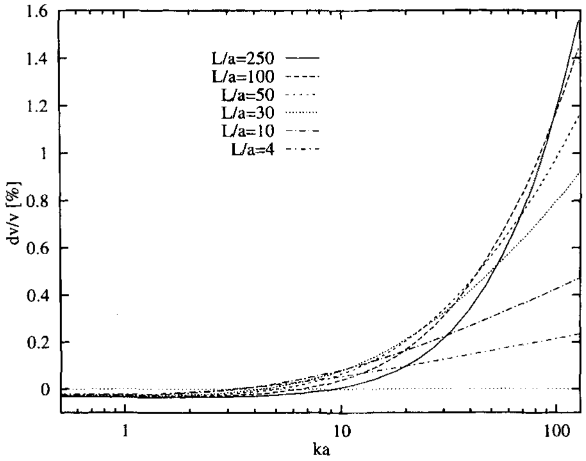

Figure 5. Velocity shift of an initially plane wave as a function of frequency at different distances L, produced by fluctuations of 2 per cent in a 3-D exponential medium.

travel distances. This can be explained by an increasing role of the diffraction effects in the intermediate-frequency range. The low- and intermediate-frequency diffraction tends to slow down the wavefield (for instance, as we have shown above the low-frequency limit of the velocity shift is always negative). The larger the travel distance, the more effective the diffraction (e.g. due to an increasing Fresnel zone). With increasing frequency for a given value of L/a the effect of diffraction decreases and the velocity shift starts to increase with increasing travel distance, because the wavefield enters the geometrical-optics regime of propagation. In this regime the more optimal ray paths are more probable in the case of larger travel distances.

THEORETICAL VERSUS NUMERICAL RESULTS

Strictly speaking, numerical computations of the phase velocities require an unwrapped (i.e. continuous) phase function of the time-harmonic transmitted wavefield. However, procedures for a smooth phase reconstruction of the numerically computed wavefields for a random medium are principally unstable. Moreover, due to practical reasons, numerical simulations in 2-D and 3-D are usually not performed in the frequency domain, but in the time domain (this is the case for our study). In such a situation the problem of numerical simulation of the phase velocity is hardly resolvable from the problem of picking first arrivals [this was also clearly pointed out by Wielandt & Friederich (1996)].

Some additional restrictions of the numerical simulations arise from the following practical arguments. The theoretical computations (Figs 1 and 3) show that we are dealing here with effects of the order of one per cent in the velocity. Therefore, the numerical model should be large enough to make the effect visible. However, the model should not be too large in order to provide a reasonable computational time and the possibility of simulating wave propagation in many realizations of a model. Finally, the model should not be too large, so that in spite of the increasing phase and amplitude fluctuations of the wavefield, first arrivals can be picked safely.

Practically this means that we should not go too far into the strong-fluctuation region of the wavefield.

All these factors together strongly restrict the suitable range of the parameters σ, ka and L/a and make the problem of numerical simulations of the phase velocity in random media fairly challenging. In this section we provide some numerical illustrations of our analytical results for the phase velocity. An exhaustive numerical test of the theory is, however, beyond the scope of this paper.

In addition to our own numerical computations, we compared our theoretical velocity shift computed for 2-D Gaussian and exponential media in the case of $\sigma = 5$ per cent (Figs 1 and 3) with the FD numerical simulations performed by Roth *et al.* (1993, Figs 7 and 8, bottom). We found a satisfactory agreement between these results (now shown here) in a rather broad frequency range, $6 < ka < 50$.

The aim of our numerical computations was to compare the theoretical and synthetic results for a possibly broad range of the parameter ϑ [see inequality (50)] controlling the validity of our approximation. In the examples given below values of the quantity ϑ are restricted to the interval $10^{-2} < \vartheta < 6$.

We compared our analytical results (eq. 24) with finite-difference solutions of the 2-D wave equation using the numerical codes of Kneib & Kerner (1993). Our model of a random medium has an exponential correlation function and a Gaussian probability distribution. Except for the parameters of the model, we reproduce here the same scheme of numerical simulation as in Shapiro & Kneib (1993). More details about our numerical simulations can be found in Schwarz (1995).

Fig. 6 shows a sketch of the numerical experiment. The initially plane wave has the signature of the first derivative of a Gaussian pulse. In Fig. 7 the synthetic seismogram is shown for a medium with $a = 60$ m and $\sigma = 0.05$. The wavefield has travelled 260 m in the inhomogeneous medium. The line of the picked arrivals and the straight line of the averaged traveltime are also shown. As the picking criterion we used the first zero-crossing after the first significant extremum of the waveform (i.e. the first minimum in the case of Fig. 7).

Figs 8 and 9 show a comparison of the theoretical predictions

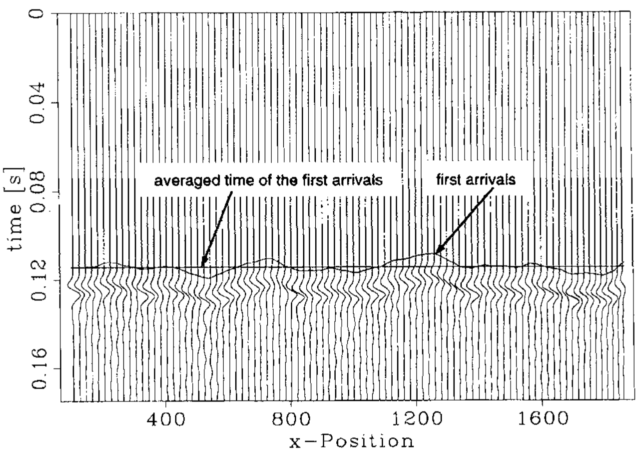

Figure 7. Synthetic seismogram for a realization of an exponential random medium with $a = 60$ m, $c_0 = 2688$ m s^{-1}, $\sigma = 0.05$ and $\lambda/a = 0.6$. The wavefield has travelled 50 m of the homogeneous reference medium and 260 m of the inhomogeneous medium.

and the results of numerical experiments. In both figures the velocity shift $\delta v/v_0$ is plotted versus the normalized travel distance L/a. For a given travel distance and a given realization of the medium the traveltimes were obtained at 100 receiver points distributed parallel to the z-axis. The numerical values of the velocity shift are obtained by averaging traveltimes computed in eight realizations of the random medium. The values of the velocity shift in individual realizations of the model fluctuate strongly from their average. Generally, we observed the same order of fluctuations as shown in Roth *et al.* (1993, Figs 7 and 8). In the best cases we reached a standard deviation of the velocity shift of about 20 per cent of the average values.

Fig. 8 shows the velocity shift versus L/a for different variances of the velocity fluctuations. The dominant frequency of the wavefield is 80 Hz in this series of simulations. Fig. 9 shows the velocity shift $\delta v/v_0$ for different dominant frequencies of the wavefield. Now the standard deviation of the velocity

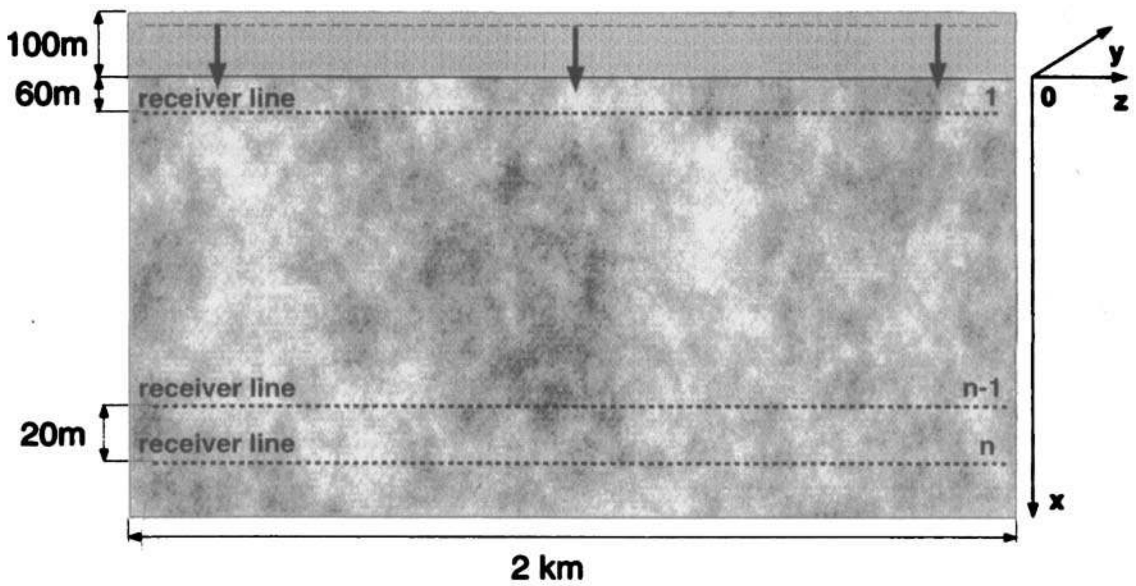

Figure 6. Sketch of the numerical simulations.

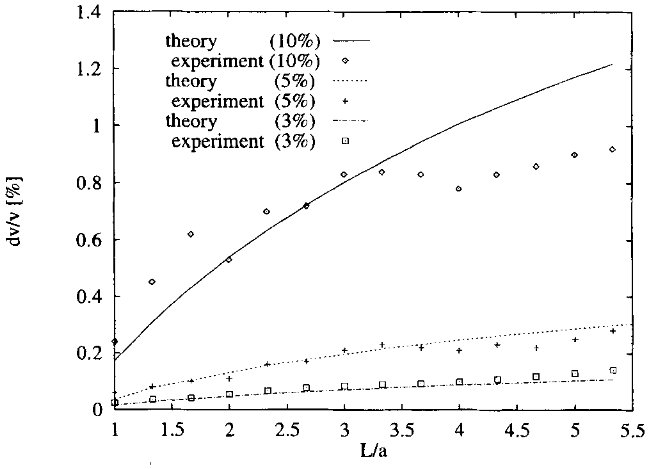

Figure 8. Velocity shift of an initially plane wave in a 2-D exponential random medium versus normalized travel distance for different standard deviations of fluctuations and constant frequency ($\lambda/a = 0.6$).

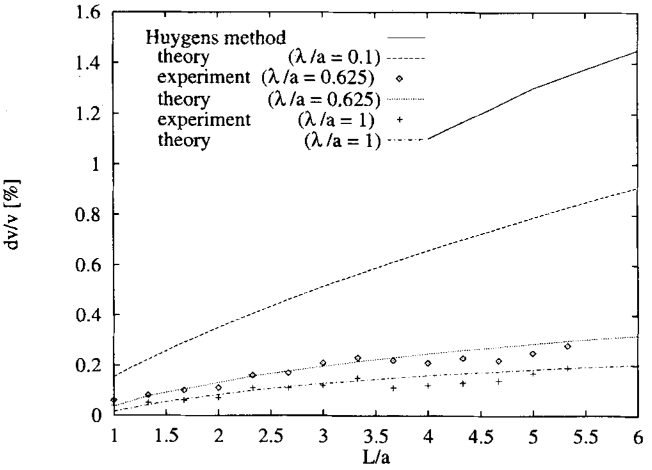

Figure 9. Velocity shift of an initially plane wave in a 2-D exponential random medium versus normalized travel distance for different frequencies and a constant standard deviation of fluctuations (5 per cent). The curve computed by the Huygens method has been taken from Fig. 8 of Roth *et al.* (1993).

fluctuations is 5 per cent. Good agreement between theoretical and numerical results is observed. For comparison, the results of the high-frequency asymptotic calculations of Roth *et al.* (1993) are also shown in Fig. 9. The major importance of the frequency dependence of the effect is evident.

Finally, Fig. 10 provides the theoretical results in the case of 3-D inhomogeneous exponential media that have the same statistical parameters as the 2-D models described above. This figure should be compared with Fig. 8. One can see a considerable increase of the velocity shift in 3-D media in comparison with 2-D media.

CONCLUSIONS

In 2-D and 3-D random media the phase velocity is not only frequency-dependent, but travel-distance-dependent as well. This dependence becomes unimportant in the frequency range which can be approximately given by the inequality $ka \lesssim 5$. Moreover, in the low-frequency range the velocity shift

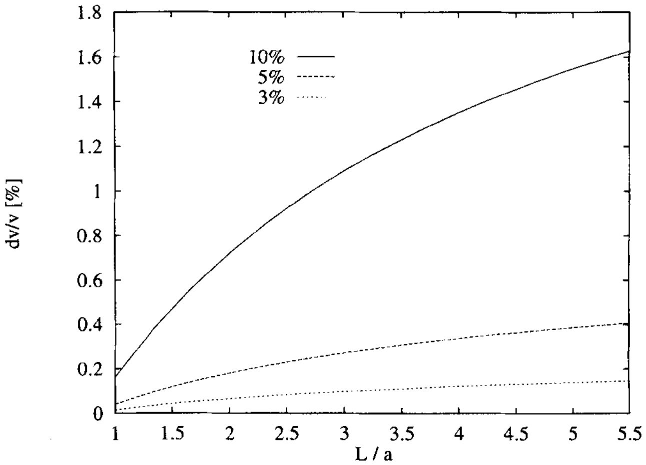

Figure 10. Velocity shift of initially plane waves in 3-D exponential media for different variances of velocity fluctuations of $\lambda/a = 0.6$.

becomes negative. With increasing L/a the dispersion becomes larger. It is important for the intermediate-frequency range, which can be approximately expressed as $1 < ka \lesssim 10\, L/a$. In realistic cases the shift of the observed phase velocity relative to the reciprocal averaged slowness is smaller than 2 per cent. The frequency dependence of the velocity shift can be very strong. Therefore, the results of ray-perturbation theory and other high-frequency approximations may be non-applicable in relevant frequency- and travel-distance domains. Moreover, ray-perturbation theory is not able to describe the effect of fractal-like inhomogeneities. Here we have presented a theory that describes the frequency- and travel-distance dependences of the phase velocity of the transmitted wavefield. The low-frequency limit of our analytical results coincides with the known effective-medium limit of the phase velocity in statistically isotropic inhomogeneous fluids with constant densities. In the high-frequency limit our results coincide with the result previously obtained by ray-perturbation theory. Our theory is, however, also valid in the case of media with non-differentiable autocorrelation functions (e.g. media with exponential autocorrelation functions or fractal media). Its results agree well with numerical simulations.

ACKNOWLEDGMENTS

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APPENDIX A: THE PHASE OF THE COHERENT WAVEFIELD

In this appendix we derive a formula for the phase ϕ_c of the coherent wavefield in two dimensions, according to the calculation of Rytov *et al.* (1987) for a 3-D medium.

The averaged Green's function $G(\mathbf{r}, \mathbf{r}_0)$ for a random medium is given by the so-called Dyson equation [see Rytov *et al.* (1987), eq. (4.28), p. 131]:

$$G(\mathbf{r}, \mathbf{r}_0) = G_0(\mathbf{r}, \mathbf{r}_0) + \iint G_0(\mathbf{r}, \mathbf{r}_1) Q(\mathbf{r}_1, \mathbf{r}_2) G(\mathbf{r}_2, \mathbf{r}_0)\, d^2\mathbf{r}_1 d^2\mathbf{r}_2, \tag{A1}$$

where G_0 is the Green's function for a homogeneous reference medium and the integration is performed over the entire space. The generally unknown quantity Q is called the kernel-of-mass operator.

By definition, the Green's function of the homogeneous reference medium G_0 satisfies

$$(\Delta + k^2) G_0(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0). \tag{A2}$$

Inserting eq. (A1) into eq. (A2), we obtain

$$\Delta G(\mathbf{r}, \mathbf{r}_0) + k^2 G(\mathbf{r}, \mathbf{r}_0) - \int Q(\mathbf{r}, \mathbf{r}') G(\mathbf{r}', \mathbf{r}_0)\, d^2\mathbf{r}' = \delta(\mathbf{r} - \mathbf{r}_0). \tag{A3}$$

In a statistically homogeneous medium, $Q(\mathbf{r}, \mathbf{r}')$ depends only on $\mathbf{r} - \mathbf{r}'$. This is also valid for $G_0(\mathbf{r}, \mathbf{r}')$. Therefore, from eq. (A1) it is clear that $G(\mathbf{r}, \mathbf{r}')$ has the same form.

Now we can solve eq. (A3) by using the following Fourier transforms:

$$G_0(\mathbf{r} - \mathbf{r}_1) = \int g_0(\boldsymbol{\kappa})\, e^{i\boldsymbol{\kappa}(\mathbf{r} - \mathbf{r}_1)}\, d^2\boldsymbol{\kappa}, \tag{A4}$$

$$G(\mathbf{r} - \mathbf{r}_1) = \int g(\boldsymbol{\kappa})\, e^{i\boldsymbol{\kappa}(\mathbf{r} - \mathbf{r}_1)}\, d^2\boldsymbol{\kappa}, \tag{A5}$$

$$Q(\mathbf{r} - \mathbf{r}_1) = \int q(\boldsymbol{\kappa})\, e^{i\boldsymbol{\kappa}(\mathbf{r} - \mathbf{r}_1)}\, d^2\boldsymbol{\kappa}. \tag{A6}$$

By taking the Fourier transform of eq. (A2), we find $g_0(\kappa)$:

$$g_0(\kappa) = \frac{1}{4\pi^2(k^2 - \kappa^2 + io)}, \tag{A7}$$

where $k = |\mathbf{k}|$ and $\kappa = |\boldsymbol{\kappa}|$. An infinitesimal absorption io $(k \rightarrow k + io/2k)$ is introduced in order to automatically satisfy the radiation conditions in the Green's functions after taking the integrals above (see Rytov *et al.* 1987, p. 132).

Now $g(\kappa)$ can be found by taking the Fourier transform of eq. (A3) and inserting eq. (A7):

$$g(\kappa) = \frac{1}{4\pi^2[k^2 - \kappa^2 + io - 4\pi^2 q(\kappa)]}. \tag{A8}$$

Substituting this into eq. (A5) and taking into account eq. (A6), we obtain the following expression for $G(\mathbf{r} - \mathbf{r}_0)$:

$$G(\mathbf{r} - \mathbf{r}_0) = \frac{1}{4\pi^2} \int \frac{e^{i\boldsymbol{\kappa}(\mathbf{r} - \mathbf{r}_0)}\, d^2\boldsymbol{\kappa}}{k^2 - \kappa^2 + io - \int Q(\mathbf{r}')\, e^{-i\boldsymbol{\kappa}\mathbf{r}'}\, d^2\mathbf{r}'} \tag{A9}$$

$$= \frac{1}{2\pi} \int_0^\infty \frac{J_0(\kappa|\mathbf{r} - \mathbf{r}_0|)\kappa\, d\kappa}{k^2 - \kappa^2 + io - 2\pi \int Q(r') J_0(\kappa r') r'\, dr'}, \tag{A10}$$

where eq. (A10) is obtained by assuming statistical isotropy of the medium, and $r = |\mathbf{r}|$.

In order to specify the quantity Q, we use the Bourret approximation that keeps only the first term of the perturbation series expansion of $Q(r)$:

$$Q(\mathbf{r} - \mathbf{r}_0) = 4k^4 G_0(\mathbf{r} - \mathbf{r}_0) B_n(\mathbf{r} - \mathbf{r}_0), \tag{A11}$$

where $B_n(\mathbf{r} - \mathbf{r}_0)$ is the autocorrelation function of the velocity fluctuations:

$$B_n(\mathbf{r} - \mathbf{r}_0) = \langle n(\mathbf{r}_0) n(\mathbf{r}) \rangle. \tag{A12}$$

Our eq. (A11) differs from the analogous eq. (4.37) of Rytov *et al.* (1987) by the factor 4, because in our paper we work with velocity fluctuations rather than with fluctuations of the squared refractive index, which are twice as large as the former.

Substituting eq. (A11) into eq. (A10) and taking into account the form of the Green's function for eq. (A2) in 2-D media [see, e.g. eq. (A2-2) from Shapiro & Kneib (1993)] we obtain the following results:

$$G(\mathbf{r} - \mathbf{r}_0) = \frac{1}{2\pi} \int_0^\infty \frac{J_0(\kappa|\mathbf{r} - \mathbf{r}_0|)\kappa\, d\kappa}{k_{\mathrm{eff}}^2 - \kappa^2 + io}, \tag{A13}$$

with

$$k_{\mathrm{eff}} \approx k\left[1 + i\pi k^2 \int_0^\infty H_0^1(kr') B(r') J_0(kr') r'\, dr'\right], \tag{A14}$$

where H_0^1 is a Hankel function and we neglect terms of higher than second order of velocity fluctuations in the brackets.

A comparison of the average Green's function (e.g. eq. A9) with the Green's function of a homogeneous medium [e.g. eq. (A4) along with eq. (A7)] immediately shows that the quantity $k_{\rm eff}$ in eq. (A13) plays the role of an effective wavenumber of the coherent wavefield.

The only remaining problem now is to simplify eq. (A14). Using the Fourier transform $\Phi_n^{2D}(\xi)$ of the function $B_n(r)$ and integrating over r', we obtain

$$k_{\rm eff}=k\left[1+4\pi k^2\int_{2k}^{\infty}\frac{\Phi_n^{2D}(\xi)\,d\xi}{\sqrt{\xi^2-4k^2}}+4i\pi k^2\int_0^{2k}\frac{\Phi_n^{2D}(\xi)\,d\xi}{\sqrt{4k^2-\xi^2}}\right]. \tag{A15}$$

The imaginary part of the effective wavenumber results only in an attenuation of the coherent wavefield. The phase ϕ_c, in turn, is given by $\phi_c=\mathscr{R}\{k_{\rm eff}\}L$. Therefore, the result for $\phi_c-\phi_0$ becomes

$$\phi_c-\phi_0=4\pi k^3 L\int_{2k}^{\infty}\frac{\Phi_n^{2D}(\xi)\,d\xi}{\sqrt{\xi^2-4k^2}}. \tag{A16}$$

By an analogous calculation, for a 3-D medium we obtain

$$\phi_c-\phi_0=\pi L k^2\int_0^{\infty}\ln\left(\frac{2k+\xi}{2k-\xi}\right)^2\Phi_n^{3D}(\xi)\xi d\xi, \tag{A17}$$

which corresponds to eq. (4.63) from Rytov *et al.* (1987).

APPENDIX B: CROSS-VARIANCE OF PHASE AND AMPLITUDE FLUCTUATIONS

In this appendix we derive an expression for $\sigma_{\chi\phi}^2$ in the 2-D case, following the known method of derivation of $\sigma_{\chi\phi}^2$ in the 3-D case [pp. 351–358, eqs (17.23)–(17.53) of Ishimaru (1978)]. In many of its details our derivation also repeats the derivation of the autocorrelation functions of χ and ϕ given for 2-D media in Appendix 1 of Shapiro & Kneib (1993). Therefore, referring to these two works for a detailed discussion, we try to keep our derivation below as short as possible.

The first-order Rytov approximation of the quantity $\chi+i\phi$ provides

$$\chi(\mathbf{r})+i\phi(\mathbf{r})\approx\psi_1(\mathbf{r})\equiv 2k^2\int_{V'}G(\mathbf{r}-\mathbf{r}')n(\mathbf{r}')\frac{u_0(\mathbf{r}')}{u_0(\mathbf{r})}d^2\mathbf{r}', \tag{B1}$$

where $u_0(\mathbf{r})=e^{ikx}$.

Now we make the assumption that the correlation radius is of the order of or larger than the wavelength. In this case scattering is confined within an angle of order λ/a in the forward direction, so we can neglect backscattering. The integration in eq. (B1) can be limited to the interval $0\le x'\le x$. For the Green's function we can now assume $|z-z'|\ll|x-x'|$ and obtain the following in two dimensions:

$$G(r-r')\approx\frac{i}{4}e^{i(-\pi/4)}\left(\frac{2}{\pi k|x-x'|}\right)^{1/2}\times\exp\left(ik\left[x-x'+\frac{(z-z')^2}{2(x-x')}\right]\right). \tag{B2}$$

Next we describe the random medium fluctuation $n(r)$ in the space–wavenumber domain (see Ishimaru 1978, Appendix A1):

$$n(x,z)=\int_{-\infty}^{\infty}e^{i\xi z}\,dv(x,\xi). \tag{B3}$$

Inserting eq. (B3) into eq. (B1) and integrating over z' gives

$$\psi_1(L,z)=ik\int_0^L dx'\int_{-\infty}^{\infty}dv(x',\xi)\,e^{i\xi z}\exp\left[-i\frac{\varepsilon^2}{2k}(L-x')\right] \tag{B4}$$

The corresponding complex-conjugate quantity reads

$$\psi_1^*(L,z)=-ik\int_0^L dx'\int_{-\infty}^{\infty}dv^*(x',\xi)\,e^{-i\xi z}\exp\left[i\frac{\varepsilon^2}{2k}(L-x')\right]$$
$$=ik\int_0^L dx'\int_{-\infty}^{\infty}dv(x',\xi)\,e^{i\xi z}\exp\left[i\frac{\varepsilon^2}{2k}(L-x')\right]. \tag{B5}$$

In eq. (B5) the identity $dv(x',\xi)=dv^*(x',-\xi)$ (valid for real functions n) and the substitution $\xi\to-\xi$ were used.

Taking into account eq. (B1), we can use the following representations of the functions χ and ϕ:

$$\chi(\mathbf{r})=\frac{1}{2}(\psi_1+\psi_1^*), \tag{B6}$$

$$\phi(\mathbf{r})=\frac{1}{2i}(\psi_1-\psi_1^*). \tag{B7}$$

We now calculate the cross-correlation function $B_{\chi\phi}$ of the quantities χ and ϕ. For this, we find the product $\chi(\mathbf{r}')\phi(\mathbf{r}'')$ using the right-hand sides of eqs (B6) and (B7) and taking into account eqs (B4) and (B5). The points $\mathbf{r}'$ and $\mathbf{r}''$ have the coordinates (L,z') and (L,z'') respectively. After statistical averaging we obtain the following:

$$B_{\chi\phi}(\Delta z)=k^2\int_{-\infty}^{\infty}d\xi\int_0^L dx'\int_0^L dx''\,e^{i\xi z}\sin\left[\frac{\xi^2}{2k}(L-x')\right]\times\cos\left[\frac{\xi^2}{2k}(L-x'')\right]F(|\Delta x|,\xi). \tag{B8}$$

Here we defined $\Delta x=x'-x''$ and $\Delta z=z'-z''$. Furthermore, we used the 1-D version of the known relationship for statistically isotropic media (Ishimaru 1978, eqs (17-43) and (A-21), p. 356 and p. 516, respectively):

$$\langle dv(x',\xi')\,dv(x'',\xi'')\rangle=F(|\Delta x|,\xi')\delta(\xi'-\xi'')\,d\xi', \tag{B9}$$

where the function F is the 1-D Fourier transform of the autocorrelation function of the velocity fluctuations:

$$F(|\Delta x|,\xi)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle n(x,z)n(x+\Delta x,z+\Delta z)\rangle\,e^{i\xi\Delta z}\,d\Delta z. \tag{B10}$$

We now introduce the new variable $\eta=\frac{1}{2}(x'+x'')$. Using now η and Δx as the new integration variables in eq. (B8) we peform the integration over Δx from $-\infty$ to ∞. It is possible because we assume vanishing autocorrelation of the velocity fluctuations for large Δx. The integration yields:

$$B_{\chi\phi}(\Delta z)=\pi k^2\int_0^L d\eta\int_{-\infty}^{\infty}d\xi\,e^{i\xi\Delta z}\sin\left[\frac{\xi^2}{k}(L-\eta)\right]\Phi_n^{2D}(\xi) \tag{B11}$$

$$=2\pi k^3\int_0^{\infty}d\xi\frac{\cos(\xi\Delta z)}{\xi^2}\left(1-\cos\frac{\xi^2 L}{k}\right)\Phi_n^{2D}(\xi), \tag{B12}$$

where we used the fact that for statistically isotropic media the 2-D Fourier transform of the autocorrelation function of the velocity fluctuations Φ_n^{2D} is obtained from F by the integral $\Phi_n^{2D}(\xi) = 1/(2\pi)\int_{-\infty}^{\infty} d\Delta x F(|\Delta x|, \xi)$ [see eq. (A27) of Ishimaru (1978)]. Finally, using $\sigma_{\chi\phi}^2 = B_{\chi\phi}(0)$, we obtain the desired quantity:

$$\sigma_{\chi\phi}^2 = 4\pi k^3 \int_0^\infty d\xi \frac{\sin^2(\xi^2 L/2k)}{\xi^2} \Phi_n^{2D}(\xi). \tag{B13}$$

By an analogous calculation one can get the result in the 3-D case [see Ishimaru (1978) eq. (17-53) for $\varrho = 0$]:

$$\sigma_{\chi\phi}^2 = 4\pi^2 k^3 \int_0^\infty d\xi \frac{\sin^2(\xi^2 L/2k)}{\xi} \Phi_n^{3D}(\xi). \tag{B14}$$

For a point source in a 2-D random medium the corresponding results are obtained by the substitution of the cylindrical incident wave $u_0(\mathbf{r}) \propto (1/|\mathbf{r}|)^{1/2} \exp(ik|\mathbf{r}|)$ into eq. (B1). After this, the chain of calculations described above must be repeated. The result reads:

$$\sigma_{\chi\phi}^2 = 2\pi k^2 \int_0^L d\eta \int_0^\infty d\xi \sin\left(\frac{\eta(L-\eta)}{kL}\xi^2\right) \Phi_n^{2D}(\xi). \tag{B15}$$

In the case of a point source in a 3-D random medium we use the corresponding result of Ishimaru (1978, p. 378, eq. 18-7 in the case $\varrho = 0$):

$$\sigma_{\chi\phi}^2 = 2\pi^2 k^2 \int_0^L d\eta \int_0^\infty \xi d\xi \sin\left(\frac{\eta(L-\eta)}{kL}\xi^2\right) \Phi_n^{3D}(\xi). \tag{B16}$$

APPENDIX C: QUANTITY $\sigma_{\chi\phi}^2$ IN 2-D GAUSSIAN MEDIA

In this Appendix we calculate the quantity $\sigma_{\chi\phi}^2$ for 2-D Gaussian media. The first part of the calculation is straightforward: by substituting $x \equiv \xi^2$ in eq. (B13), taking Φ_n^{2D} from eq. (29) and using $2\sin^2(xL/2k) = 1 - \cos(xL/k)$ we obtain

$$\sigma_{\chi\phi}^2 = \frac{\sigma^2 a^2 k^3}{4} \int_0^\infty \frac{1-\cos(xL/k)}{x^{3/2}} e^{-xa^2/4}\, dx \tag{C1}$$

$$= \frac{\sigma^2 a^2 k^3}{4}\left(\int_0^\infty \frac{e^{-xa^2/4}}{x^{3/2}}\, dx - \int_0^\infty \frac{e^{-xa^2/4}}{x^{3/2}} \cos\left(\frac{xL}{k}\right) dx\right). \tag{C2}$$

Using tables of definite integrals (Gradshteyn & Ryzhik 1983) one can find the solutions for the two integrals in eq. (C2). We obtain:

$$\sigma_{\chi\phi}^2 = \frac{\sigma^2 a^2 k^3}{4}\left[\left(\frac{a^2}{4}\right)^{1/2} \Gamma\left(-\frac{1}{2}, \frac{a^2}{4}u\right) - \frac{1}{2}\left(\frac{a^2}{4} + i\frac{L}{k}\right)^{1/2} \Gamma\left(-\frac{1}{2}, \left(\frac{a^2}{4} + i\frac{L}{k}\right)u\right) - \frac{1}{2}\left(\frac{a^2}{4} - i\frac{L}{k}\right)^{1/2} \Gamma\left(-\frac{1}{2}, \left(\frac{a^2}{4} - i\frac{L}{k}\right)u\right)\right] \tag{C3}$$

In this equation Γ is the incomplete gamma function,

$$\Gamma(\alpha, u) \equiv \int_u^\infty e^{-t} t^{\alpha-1}\, dt. \tag{C4}$$

The exact solution of eq. (C3) is the limiting value for $u \to 0$. Using

$$\Gamma(\alpha+1, u) = \alpha\Gamma(\alpha, u) + u^\alpha e^{-u} \tag{C5}$$

and

$$\Gamma\left(\frac{1}{2}, u^2\right) = \sqrt{\pi}(1 - \Phi(u)), \tag{C6}$$

where $\Phi(u)$ is the probability integral, we are able to perform $u \to 0$ and obtain the following:

$$\sigma_{\chi\phi}^2 = \frac{\sigma^2 a^2 k^3}{4}\sqrt{\frac{L}{k}}\sqrt{\pi}\left[-2\left(\frac{a^2 k}{4L}\right)^{1/2} + \left(\frac{a^2 k}{4L} + i\right)^{1/2} + \left(\frac{a^2 k}{4L} - i\right)^{1/2}\right]. \tag{C7}$$

The imaginary parts of the last two terms compensate each other. After adding the real parts of the last two terms we obtain result (34).