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The effect of rotation and a buried magnetic field on stellar oscillations

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SUMMARY

Rotation, a magnetic field and any other symmetry-breaking agent raise the degeneracy of the resonant frequencies of stellar oscillations. Here a perturbation method is presented for calculating the resultant frequency modification caused by rotation and an internal magnetic field. The shift in the average multiplet frequency is also addressed. Only axisymmetric magnetic fields are considered explicitly, though the axis of symmetry is not constrained to coincide with the rotation axis. A short-wavelength asymptotic analysis derived from ray theory is also presented. The effects on high-order solar acoustic modes of various hypothetical angular velocity and magnetic field configurations are investigated using both methods of calculation. The asymptotic formulae provide a good estimate for the frequency splitting of five-minute modes when the field and the rotation vary sufficiently smoothly. On the other hand, a localized magnetic field, for example at the base of the convection zone, produces a characteristic oscillatory perturbation to the eigenfrequencies. We present our results in a form that we hope will be useful for comparison with future observations, and discuss them in the light of currently available splitting data.

1 INTRODUCTION

Resonant oscillations provide a probe of the interior of a star. Considerable effort is already being put into investigating the use of helioseismic data to improve our spherically symmetric models of the Sun and to learn about its internal rotation. One of the most intriguing aspects of the Sun is its magnetic activity, made manifest for example by surface features such as sunspots, whose number and distribution vary with the solar cycle. Little is known about the magnetic field in the interior, although it is widely believed that the field is maintained by a dynamo, with its seat possibly located near the base of the convection zone. It has also been suggested that the Sun's core may contain a strong magnetic field, perhaps the remnant of a primordial field. There is therefore considerable interest in the possibility that solar oscillations can be used to probe the magnetic field in the solar interior. Ultimately, limited information about the field structure in other stars might also be obtainable.

A magnetic field modifies the equilibrium structure of the star (including, in general, the shape of the surface), which changes the frequencies of oscillation. In addition, there is a direct effect on the frequencies arising from the perturbation to the Lorentz force caused by the wave as it propagates through the fluid. Similarly, there are indirect perturbations from modifications to the equilibrium state and direct perturbations from any other symmetry-breaking agent.

Before we can begin to make inferences about the field of any star we must be able to calculate the eigenfrequencies of a model with a given magnetic field. The principal aims of this paper are two-fold: first, to set out a method for calculating the effect on the eigenfrequencies of a prescribed large-scale magnetic field; and secondly to investigate the effect on different modes of oscillation for a variety of field geometries, presenting the results in sufficient detail for initial comparisons with future observations to be possible. We also discuss currently available solar observations in the light of our calculations.

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The normal modes of oscillation of a spherically symmetric stellar model have radial displacement

$$\mathrm{Re}[\xi(r)Y_l^m(\theta,\phi)\exp(-i\omega t)], \tag{1.1}$$

where Y_l^m is a spherical harmonic of degree l and azimuthal order m, (r, θ, ϕ) are spherical polar coordinates with respect to arbitrarily orientated axes with origin at the centre of the star, and t is time; ω is the corresponding eigenfrequency. By symmetry, ξ and ω cannot depend on m (the value of which is dependent on the orientation of the axes) and so they are determined only in terms of l and the order, n, of the mode. If a weak axisymmetric perturbation is introduced, then to a first approximation the displacement eigenfunction has the same form as before, provided the polar axis is taken along the axis of symmetry; ξ is still essentially independent of m, but the degeneracy of the eigenfrequencies is raised. Hence frequencies of modes of like l and n but different m are split. For the well-studied case of a star rotating slowly with angular velocity $\Omega(r, \theta)$, it is well known that

$$\omega_{nlm}-\omega_{nl0}=\frac{m\int_0^{\pi}\bar{\Omega}(\theta)|Y_l^m|^2\sin\theta d\theta}{\int_0^{\pi}|Y_l^m|^2\sin\theta d\theta} \tag{1.2}$$

to first order in Ω, where $\bar{\Omega}(\theta)$ is a depth average of Ω which depends on n and l (*cf.* for example, Cowling & Newing 1949; Ledoux 1951; Hansen, Cox & Van Horn 1977; Gough 1981; Brown 1985). To this level of approximation the average frequency for given n and l is unchanged by the rotation, though in general a weak perturbing force will shift the average frequencies as well as splitting the frequency multiplets.

Observations of the splitting of solar five-minute p-mode frequencies were first reported by Claverie *et al.* (1981), and subsequently by Gough (1982) and Hill, Bos & Goode (1982) (in both cases using the data of Bos & Hill 1983), Duvall & Harvey (1984), Duvall, Harvey & Pomerantz (1986), Brown (1985), Libbrecht (1986) and Brown & Morrow (1987). At fixed frequency the penetration depth of the oscillations decreases as l increases, while as $|m|$ increases at fixed l the mode is more nearly confined to the equatorial region. Thus when Duvall & Harvey (1984) measured the splitting between sectoral modes ($m=\pm l$) for a large number of values of l, it was possible to infer the variation with depth of the angular velocity near the equatorial plane (Duvall *et al.* 1984).

The splitting $\omega_{nlm}-\omega_{nl0}$ can be separated into one component that is odd in m and another that is even. The odd component arises from advection (and from Coriolis force, if the system is viewed from a rotating frame of reference), the effect of which changes sign when the direction of propagation of the wave is reversed. Centrifugal force, which does not distinguish between waves propagating in opposite directions, and any large-scale magnetic field or non-spherically symmetric distortion of the equilibrium state of the star, affect only the even component. Hence the data of Duvall & Harvey (1984) contain no information about magnetic fields. The possibility of inferring the latitudinal dependence of the rotation, and of observing a magnetic effect, is raised by the more complete data on the m dependence presented by Brown (1985, 1986), Duvall *et al.* (1986), Libbrecht (1986) and Brown & Morrow (1987). From the odd component of such data one might hope to determine the latitudinal and depth dependence of Ω. Then, provided Ω does not have a significant component that is antisymmetric about the equatorial plane [which from equation (1.2) does not contribute to the first-order frequency splitting], one may calculate the centrifugal contribution to the even component of the splitting. With this knowledge one can then hope to use the observed even component to make inferences about the magnetic field or other departures of the Sun's structure from spherical symmetry.

The consideration of the effect of a large-scale magnetic field on stellar oscillations is not new. References to early work may be found in the article by Ledoux & Walraven (1958; see also Unno *et al.* 1979 and Cox 1980). This work was advanced considerably by Goossens (1972), who included in his calculation the effect of the distortion from sphericity of the equilibrium state, which is of the same order as other leading terms in the magnetic perturbation to the eigenfrequencies. The work was simplified by specializing to a homopycnic stellar model; in general the calculation of the distortion is by no means trivial. We follow a method developed by Simon (1969) and Lebovitz (1970), and employed by Smeyers & Denis (1971), for calculating rotational distortion. In the magnetic context, some results using this method for a realistic solar model have previously been presented by Gough & Taylor (1984). When the field strength does not vanish at the surface of the star, singular perturbation theory must be used (Goossens, Smeyers & Denis 1976). We shall not include such complications; this seems not unreasonable in the case of the Sun, though it would probably not be adequate for considering stars with much stronger large-scale surface fields, such as some Ap stars [see the work by Biront *et al.* 1982; Roberts & Soward 1983; Campbell & Papaloizou 1986]. We also consider only briefly the effect of fibril fields (see Bogdan & Zweibel 1985; 1987; Zweibel & Bogdan 1986; Gough & Thompson 1988b).

Of course the Sun not only has a magnetic field but is also rotating. Even when (as we shall suppose) the magnetic field is axisymmetric, the two perturbations combine in a non-trivial way unless the axes of rotation and magnetic symmetry coincide

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(Dicke 1982a). Motivated by Isaak's (1982) suggestion of the possible detection of a magnetic effect in the data of Claverie *et al.* (1981), Dziembowski & Goode (1984, 1988) and Gough & Taylor (1984) obtained asymptotic estimates for the combined effect of rotation and a magnetic field for a realistic solar model. The present work is a further advance on these investigations, in that we include all terms up to second order in Ω and in magnetic field strength B, and we use numerical eigenfunctions rather than asymptotics. We also derive new asymptotic estimates valid when l/n is not necessarily small.

Rotation and a magnetic field not only split the frequency multiplet but also shift the average frequency of each multiplet. The shift arises both from the direct effect of the perturbed inertial and Lorentz forces on the waves, and because the unperturbed centrifugal and Lorentz forces change the spherically symmetric component of the structure of the star. The former cause was investigated asymptotically by Gough & Taylor (1984) and similar results were obtained for a plane-parallel stellar envelope model by Roberts & Campbell (1986). We present a more comprehensive discussion in Section 7.

Finally we discuss the available observational data in the light of our results.

2 THE PERTURBATION METHOD

To seek modes of oscillation with time dependence $e^{-i\omega t}$, where ω is constant, we suppose there exists a frame $\mathscr{S}$, rotating with respect to an inertial frame with steady angular velocity $\mathbf{\Omega}_c$, in which the structure of the non-oscillating star, its magnetic field $\mathbf{B}(\mathbf{r})$ and its velocity field $\mathbf{v}(\mathbf{r})$ are independent of time. We shall refer to this as the equilibrium state. For the purpose of elucidating the perturbation method, and in this section only, we express $\mathbf{\Omega}_c$, $\mathbf{v}$ and $\mathbf{B}$ in units of Ω_S, $\Omega_S R$ and $\bar{B}$, where Ω_S is a characteristic angular velocity (say the equatorial photospheric value of Ω), $\bar{B}$ is a characteristic magnetic field strength and R is the radius of the star. Further, we define dimensionless parameters $\varepsilon=\Omega_S(GM/R^3)^{-1/2}$ and $\delta=\bar{B}(\mu_0 GM^2/R^4)^{-1/2}$ which are presumed to be small compared with unity. Here M is the mass of the star, μ_0 is the magnetic permeability of the stellar material (which we presume to take the constant, vacuum value; in cgs units $\mu_0=4\pi$), and G is the gravitational constant. For example, using solar values and taking $\bar{B}=10^6$ G, $\varepsilon\approx 4.4\times10^{-3}$ and $\delta\approx 2.7\times10^{-3}$. It is supposed that $\mathbf{\Omega}_c$, $\mathbf{v}$ and $\mathbf{B}$ are nowhere much greater than unity. We shall assume that the velocity is wholly due to rotation about an axis parallel to $\mathbf{\Omega}_c$, so $\mathbf{v}+\mathbf{\Omega}_c\times\mathbf{r}=\mathbf{\Omega}\times\mathbf{r}$, say.

Following Lynden-Bell & Ostriker (1967) one may easily derive the linearized adiabatic oscillation equation in frame $\mathscr{S}$ under the Cowling approximation (Cowling 1941), which results from neglecting the Eulerian perturbation to the gravitational potential. Expressing the pressure p, density ρ, adiabatic sound speed c, position $\mathbf{r}$ and angular frequency ω in units of GM^2/R^4, M/R^3, $(GM/R)^{1/2}$, R and $(GM/R^3)^{1/2}$, respectively, and ignoring viscous forces, the oscillation equation is

$$\mathscr{L}\boldsymbol{\xi}+\rho\omega^2\boldsymbol{\xi}=\varepsilon\omega\mathscr{M}\boldsymbol{\xi}+\varepsilon^2\mathscr{N}\boldsymbol{\xi}+\delta^2\mathscr{B}\boldsymbol{\xi}, \tag{2.1}$$

where

$$\mathscr{L}\boldsymbol{\xi}=-\nabla[(p-\rho c^2)\nabla\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla p]+p\nabla(\nabla\cdot\boldsymbol{\xi})-\boldsymbol{\xi}\cdot\nabla(\ln\rho)\nabla p \tag{2.2}$$

$$\mathscr{M}\boldsymbol{\xi}=-2i\rho(\mathbf{\Omega}_c\times\boldsymbol{\xi}+\mathbf{v}\cdot\nabla\boldsymbol{\xi}) \tag{2.3}$$

$$\mathscr{N}\boldsymbol{\xi}=\rho[2\mathbf{\Omega}_c\times(\mathbf{v}\cdot\nabla\boldsymbol{\xi})-2\mathbf{\Omega}_c\times(\boldsymbol{\xi}\cdot\nabla\mathbf{v})-\boldsymbol{\xi}\cdot\nabla(\mathbf{v}\cdot\nabla\mathbf{v})+(\mathbf{v}\cdot\nabla)^2\boldsymbol{\xi}] \tag{2.4}$$

$$\mathscr{B}\,\xi=-\frac{\nabla\cdot(\rho\boldsymbol{\xi})}{\rho}(\nabla\times\mathbf{B})\times\mathbf{B}-(\nabla\times\mathbf{B}')\times\mathbf{B}-(\nabla\times\mathbf{B})\times\mathbf{B}' \tag{2.5}$$

and $\mathbf{B}'=\nabla\times(\boldsymbol{\xi}\times\mathbf{B})$ is the linearized Eulerian perturbation to $\mathbf{B}$ in the approximation of infinite conductivity. The real part of $\boldsymbol{\xi}(\mathbf{r})\exp(-i\omega t)$ is the displacement of a particle from its equilibrium position $\mathbf{r}$. The quantities p, ρ and c^2 refer to the equilibrium state.

In applying boundary conditions we shall adopt the approximation that there is a surface, which we shall call the surface of the star, above which lies an isothermal atmosphere in hydrostatic equilibrium. In our explicit application to the Sun, we apply the boundary condition in the chromosphere, our isothermal atmosphere being a representation of the high-temperature corona. The boundary condition is obtained by matching the eigenfunctions of our problem to the causal adiabatic oscillations of the atmosphere. The details are presented in the Appendix. For oscillations with frequencies well below the critical acoustic cut-off frequency ω_c (see, for example, Deubner & Gough 1984) at the chromospheric temperature minimum, the eigenfunctions in the interior of the star are quite insensitive to the temperature and position of the base of the corona. Indeed, for many practical purposes it is adequate to assume that pressure essentially vanishes on the surface. Then, as the magnetic field is presumed not to penetrate the surface, continuity of stress implies that the Lagrangian pressure perturbation, and consequently $\rho c^2\nabla\cdot\boldsymbol{\xi}$, must vanish. (We have indeed confirmed that, to the accuracy quoted, the numerical results presented in this paper are unchanged when this simpler boundary condition replaces that derived in the Appendix.) In addition we require that $\boldsymbol{\xi}$ be regular at $\mathbf{r}=\mathbf{0}$.

A typical p-mode solution of the non-rotating, non-magnetic star can be represented in the form

$$\boldsymbol{\xi}(\mathbf{r})=\boldsymbol{\xi}_0(\mathbf{r})\equiv\sum_{m=-l}^{l}c_m\boldsymbol{\xi}_{nlm}(\mathbf{r}), \tag{2.6}$$

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where, with respect to spherical polar coordinates (r, θ, ϕ),

$$\boldsymbol{\xi}_{nlm}(\mathbf{r})=\left[\xi(r)Y_l^m,\ \eta(r)\frac{\partial Y_l^m}{\partial\theta},\ \frac{\eta(r)}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\right]. \tag{2.7}$$

The amplitude functions ξ and η depend on n and l, but not on m. We omit subscripts for clarity. We choose to normalize the surface harmonics such that

$$\int_0^\pi d\theta\int_0^{2\pi} d\phi\, |Y_l^m(\theta,\phi)|^2 \sin\theta=1; \tag{2.8}$$

and we normalize ξ such that $\xi(R)=1$.

Lorentz and centrifugal forces distort the shape of the star. We wish to represent the oscillation eigensolution of the rotating, magnetic stellar model $\mathfrak{M}$ as a perturbation to the non-rotating, non-magnetic solution. Our expansion utilizes the fact that for the unperturbed state ($\varepsilon=0$, $\delta=0$) the operator $\mathscr{L}$, together with its associated boundary conditions, is very nearly Hermitian, and we wish to preserve this property for the distorted star. To this end, we find it convenient, following Simon (1969) and Lebovitz (1970), to map each point $\mathbf{r}$ in the distorted model $\mathfrak{M}$ to a point $\mathbf{x}$ in the spherical volume occupied by a corresponding spherically symmetric stellar model $\mathfrak{M}_0$ by means of a transformation $(r, \theta, \phi)\rightarrow(x, \theta, \phi)$ with:

$$x=[1+\varepsilon^2 h_\Omega(\mathbf{r})+\delta^2 h_B(\mathbf{r})]r+O[\max(\varepsilon^2,\delta^2)] \tag{2.9}$$

chosen in such a way that $x=R$ can be regarded as the surface of $\mathfrak{M}$. {In what follows we shall drop terms small compared with ε^2 or δ^2, without explicitly writing '$+O[\max(\varepsilon^2,\delta^2)]$'.} The precise forms we choose for the functions h_Ω and h_B, which depend on $\boldsymbol{\Omega}$ and $\mathbf{B}$, respectively, are given in Section 3.1. (The choice is by no means unique. Indeed, it is not even necessary to perform the transformation; it is merely a preference to retain simplicity in the boundary conditions at the minor expense of adding extra terms to the governing differential equations.)

The equilibrium quantities p, ρ, c^2 can be expanded about their values in $\mathfrak{M}_0$, so for example

$$p(r)=p_0(x)+\varepsilon^2 p_\Omega(\mathbf{x})+\delta^2 p_B(\mathbf{x}), \tag{2.10}$$

where $p_0(x)$ is the equilibrium pressure distribution in $\mathfrak{M}_0$. Writing vectors with respect to the unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ of the true spherical polar coordinate system (r, θ, ϕ), we define operators ∇_0, $\nabla_0\cdot$ and $\mathscr{L}_0$ according to

$$\nabla_0=\mathbf{e}_r\frac{\partial}{\partial x}+\mathbf{e}_\theta\frac{1}{x}\frac{\partial}{\partial\theta}+\mathbf{e}_\phi\frac{1}{x\sin\theta}\frac{\partial}{\partial\phi}, \tag{2.11}$$

$$\nabla_0\cdot\boldsymbol{\xi}=\frac{1}{x^2\sin\theta}\left[\frac{\partial}{\partial x}(x^2\sin\theta\,\mathbf{e}_r\cdot\boldsymbol{\xi})+\frac{\partial}{\partial\theta}(x\sin\theta\,\mathbf{e}_\theta\cdot\boldsymbol{\xi})+\frac{\partial}{\partial\phi}(x\mathbf{e}_\phi\cdot\boldsymbol{\xi})\right] \tag{2.12}$$

and

$$\mathscr{L}_0\boldsymbol{\xi}=-\nabla_0\{[p_0(x)-\rho_0(x)c_0^2(x)]\nabla_0\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla_0 p_0\}+p_0\nabla_0(\nabla_0\cdot\boldsymbol{\xi})-\boldsymbol{\xi}\cdot\nabla_0(\ln\rho_0)\nabla_0 p_0. \tag{2.13}$$

(Note that the partial derivatives with respect to θ and ϕ are now taken at constant x and not at constant r.) Then the function $\boldsymbol{\xi}_0(\mathbf{x})$, defined by

$$\boldsymbol{\xi}_0(\mathbf{x})=\xi(x)\mathbf{e}_r+\eta(x)\frac{\partial Y_l^m}{\partial\theta}\mathbf{e}_\theta+\frac{\eta(x)}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\mathbf{e}_\phi \tag{2.14}$$

[where ξ and η are the same functions as in equation (2.7)], and its associated eigenfrequency ω_0 satisfy the equation

$$\mathscr{L}_0\boldsymbol{\xi}_0+\rho_0\omega_0^2\boldsymbol{\xi}_0=\mathbf{0}. \tag{2.15}$$

The operator $\mathscr{L}$ may be written

$$\mathscr{L}\boldsymbol{\xi}=\mathscr{L}_0\boldsymbol{\xi}+\varepsilon^2\mathscr{L}_\Omega\boldsymbol{\xi}+\delta^2\mathscr{L}_B\boldsymbol{\xi}, \tag{2.16}$$

where $\mathscr{L}_\Omega$ and $\mathscr{L}_B$ are perturbations due to $\boldsymbol{\Omega}$ and $\mathbf{B}$, respectively. Their forms are made explicit in Section 3.1. Only the leading terms of operators $\mathscr{M}$, $\mathscr{N}$ and $\mathscr{B}$ are required. Thus equation (2.1) and the accompanying boundary conditions become

$$\left.\begin{aligned}&\mathscr{L}_0\boldsymbol{\xi}+\rho_0\omega^2\boldsymbol{\xi}=\varepsilon\mathscr{M}_0\boldsymbol{\xi}+\varepsilon^2(\mathscr{N}_0-\mathscr{L}_\Omega-\rho_\Omega\omega^2)\boldsymbol{\xi}+\delta^2(\mathscr{B}_0-\mathscr{L}_B-\rho_B\omega^2)\boldsymbol{\xi},\\&\qquad\boldsymbol{\xi}\text{ regular at }\mathbf{x}=\mathbf{0},\\&\mathbf{e}_r\cdot\nabla_0(\mathbf{e}_r\cdot\boldsymbol{\xi})-\kappa\mathbf{e}_r\cdot\boldsymbol{\xi}=O[\max(\varepsilon^2,\delta^2)]\qquad\text{at }x=R,\end{aligned}\right\}, \tag{2.17}$$

where κ is defined in the Appendix.

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We seek a solution that is a small perturbation about a mode $\boldsymbol{\xi}_0$ of the unperturbed equilibrium state:

$$\boldsymbol{\xi}=\boldsymbol{\xi}_0(\mathbf{x})+\boldsymbol{\Delta\xi}(\mathbf{x}), \qquad \omega=\omega_0+\Delta\omega, \tag{2.18}$$

where $\boldsymbol{\Delta\xi}$ and $\Delta\omega$ are $O[\max(\varepsilon, \delta^2)]$. The mode $\boldsymbol{\xi}_0$ is expressed as a linear combination of the degenerate eigenfunctions $\boldsymbol{\xi}_{nlm}$ as in equation (2.6). Accidental degeneracies between modes of different n and l are ignored in the present treatment; if such degeneracies exist it would be necessary to include the additional modes in the sum [*cf.* equation (2.6)]. Substituting equation (2.18) into (2.17) yields

$$(\mathscr{L}_0+\rho_0\omega_0^2)\boldsymbol{\Delta\xi}=-2\rho_0\omega_0(\Delta\omega)\boldsymbol{\xi}_0+\varepsilon\omega_0\mathscr{M}_0\boldsymbol{\xi}_0+\varepsilon^2(\mathscr{N}_0-\mathscr{L}_\Omega-\rho_\Omega\omega_0^2)\boldsymbol{\xi}_0+\delta^2(\mathscr{B}_0-\mathscr{L}_B-\rho_B\omega_0^2)\boldsymbol{\xi}_0-\rho_0(\Delta\omega)^2\boldsymbol{\xi}_0$$
$$-2\rho_0\omega_0(\Delta\omega)\boldsymbol{\Delta\xi}+\varepsilon\omega_0\mathscr{M}_0\boldsymbol{\Delta\xi}+\varepsilon(\Delta\omega)\mathscr{M}_0\boldsymbol{\xi}_0. \tag{2.19}$$

It is a straightforward matter to demonstrate by partial integration that

$$\langle\boldsymbol{\xi}^*_{nlm}\cdot(\mathscr{L}_0+\rho_0\omega_0^2)\boldsymbol{\Delta\xi}\rangle=\int_{x=R}\rho_0c_0^2\nabla_0\cdot(\boldsymbol{\Delta\xi})\,\boldsymbol{\xi}^*_{nlm}\cdot\mathbf{dS}, \tag{2.20}$$

where angular brackets $\langle\ldots\rangle$ denote

$$\int_{x<R}\ldots x^2\sin\theta\,dx\,d\theta\,d\phi,$$

namely integration over the volume of $\mathfrak{M}_0$. In deriving this equation one uses equations (2.13) and (2.15), the fact that p_0, ρ_0 and c_0 are functions only of x, and the property that $\rho_0c_0^2\nabla_0\cdot\boldsymbol{\xi}_{nlm}(\mathbf{x})$ is essentially zero at $x=R$. We now take the scalar product of $\boldsymbol{\xi}^*_{nlm}$ with equation (2.19), where the asterisk denotes the Hermitian conjugate. At the surface of the star $\nabla_0\cdot(\boldsymbol{\Delta\xi})=O[\max(\varepsilon,\delta^2)]$, $\rho_0c_0^2\ll R^{-3}\langle\rho_0c_0^2\rangle$ and, provided ω is less than and not nearly equal to ω_c, the eigenfunction is evanescent in the subsurface layers; thus (as can be confirmed by numerical integration) one can ignore surface integrals after application of the divergence theorem, and obtain to a good approximation:

$$2\omega_0\langle\rho_0\boldsymbol{\xi}^*_{nlm}\cdot\boldsymbol{\xi}_0\rangle\Delta\omega=\varepsilon\omega_0\langle\boldsymbol{\xi}^*_{nlm}\cdot\mathscr{M}_0\boldsymbol{\xi}_0\rangle+\varepsilon^2\langle\boldsymbol{\xi}^*_{nlm}\cdot(\mathscr{N}_0-\mathscr{L}_\Omega-\rho_\Omega\omega_0^2)\boldsymbol{\xi}_0\rangle+\delta^2\langle\boldsymbol{\xi}^*_{nlm}\cdot(\mathfrak{B}_0-\mathscr{L}_B-\rho_B\omega_0^2)\boldsymbol{\xi}_0\rangle-(\Delta\omega)^2\langle\rho_0\boldsymbol{\xi}^*_{nlm}\cdot\boldsymbol{\xi}_0\rangle$$
$$-2\omega_0\Delta\omega\langle\rho_0\boldsymbol{\xi}^*_{nlm}\cdot\boldsymbol{\Delta\xi}\rangle+\varepsilon\omega_0\langle\boldsymbol{\xi}^*_{nlm}\cdot\mathscr{M}_0(\boldsymbol{\Delta\xi})\rangle+\varepsilon\Delta\omega\langle\boldsymbol{\xi}^*_{nlm}\cdot\mathscr{M}_0\boldsymbol{\xi}_0\rangle, \qquad m=-l,\ldots,l. \tag{2.21}$$

To be specific we now select coordinates such that $\theta=0$ is the rotation axis. Let (x,θ',ϕ') be another set of spherical polar coordinates with $\theta'=0$ being the axis of symmetry of the magnetic field. Let the angle between the two axes be β. Define

$$\boldsymbol{\xi}_{nlm'}=\left[\xi(x)Y_l^{m'}(\theta',\phi'),\eta(x)\frac{\partial Y_l^{m'}}{\partial\theta'},\frac{\eta(x)}{\sin\theta'}\frac{\partial Y_l^{m'}}{\partial\phi'}\right]. \tag{2.22}$$

The vectors $\boldsymbol{\xi}_{nlm}$ and $\boldsymbol{\xi}_{nlm'}$ are related by

$$\boldsymbol{\xi}_{nlm'}=\sum_{m=-l}^{l}d^{(l)}_{mm'}(\beta)\boldsymbol{\xi}_{nlm}, \tag{2.23}$$

where the coefficients $d^{(l)}_{mm'}(\beta)$ relate spherical harmonics under rotation (Edmonds 1957). We choose the origins of ϕ and ϕ' such that the planes $\phi'=0$ and $\phi=0$ coincide. Then the $d^{(l)}_{mm'}(\beta)$ are real. From the symmetry properties of $d^{(l)}_{mm'}(\beta)$ one also has

$$\boldsymbol{\xi}_{nlm}=\sum_{m'=-l}^{l}d^{(l)}_{mm'}(\beta)\boldsymbol{\xi}_{nlm'}. \tag{2.24}$$

A leading approximation ω_1 to $\Delta\omega$ is obtained by neglecting terms on the right of equation (2.21) that are small compared with ε or δ^2: namely the $O(\varepsilon^2)$ term and the last four terms. Noting that by normalization the integral I defined by

$$I=\langle\rho_0|\boldsymbol{\xi}_{nlm}|^2\rangle \tag{2.25}$$

is independent of m, one obtains

$$\omega_1\mathbf{c}=\mathbf{Ac}+\mathbf{DBD}^{\mathrm{T}}\mathbf{c}, \tag{2.26}$$

where $\mathbf{c}$ has elements c_m [the coefficients in equation (2.6)], $\mathbf{D}$ is a matrix with elements $D_{mm'}=d^{(l)}_{mm'}(\beta)$ and $\mathbf{D}^{\mathrm{T}}$ is its transpose, and the matrices $\mathbf{A}$ and $\mathbf{B}$ are diagonal (this is implied by the axisymmetry of the magnetic and velocity fields, as may easily be

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seen by noting that the ϕ dependence of Y_l^m is simply $e^{im\phi}$) with elements

$$\left.\begin{aligned} A_{mm} &= (2I\omega_0)^{-1}\varepsilon\omega_0\langle \boldsymbol{\xi}^*_{nlm}\cdot \mathcal{M}_0 \boldsymbol{\xi}_{nlm}\rangle, \\ B_{m'm'} &= (2I\omega_0)^{-1}\delta^2\langle \boldsymbol{\xi}^*_{nlm'}\cdot(\mathfrak{B}_0 - \mathcal{L}_B - \rho_B\omega_0^2)\boldsymbol{\xi}_{nlm'}\rangle. \end{aligned}\right\} \tag{2.27}$$

Hence ω_1, $\mathbf{c}$ are the eigenvalues and eigenvectors of the matrix equation (2.26). [This is the eigenvalue problem of Dicke (1982a), Gough & Taylor (1984), Dziembowski & Goode (1985, 1986) and Kurtz & Shibahashi (1986).] From equation (2.26) one sees that, to this level of approximation, if $\delta = 0$

$$\boldsymbol{\xi}_0 = \boldsymbol{\xi}_{nlm}, \qquad \omega_1 = A_{mm}, \qquad (m = -l, \ldots, l) \tag{2.28}$$

where if $\varepsilon = 0$

$$\boldsymbol{\xi}_0 = \boldsymbol{\xi}_{nlm'}, \qquad \omega_1 = B_{m'm'}, \qquad (m' = -l, \ldots, l). \tag{2.29}$$

When ε is much larger than δ^2, one can expand the solutions of equation (2.26) in powers of $\varepsilon^{-1}\delta^2$, obtaining to first order.

$$\omega_1 \approx A_{mm} + \sum_{m'=-l}^{l} (D_{mm'})^2 B_{m'm'}, \tag{2.30}$$

with the eigenvector $\mathbf{c}$ being close to the corresponding eigenvector of the non-magnetic state. This approximation is valid provided there is no element $A_{m''m''}$, say, which differs from A_{mm} by as little as $O(\delta^2/\varepsilon)$. One should be aware that the validity of the approximation (2.30) may therefore depend on n and l. Notice also that even though the splitting viewed from an inertial frame may be dominated by rotation, it does not necessarily follow that rotation dominates magnetic effects in the frame $\mathscr{S}$ in which we must work.

Finally, we include all the remaining second-order rotation terms for the case $\varepsilon \gg \delta^2$. [Numerical results presented in Section 5 support the belief that for the Sun these terms are small for $l \le 100$. For the Sun, therefore, it is probably adequate to use equations (2.26), (2.27) and, if circumstances permit, equation (2.30).] We restore the five terms in equation (2.21) that were previously neglected, by making $O(\varepsilon)$ approximations $\varepsilon\boldsymbol{\xi}_1$ and $\varepsilon\omega_{\Omega 1}$ to $\boldsymbol{\Delta\xi}$ and $\Delta\omega$. The function $\boldsymbol{\xi}_1$ is obtained by solving equation (2.19) up to terms of order ε:

$$\mathcal{L}_0\boldsymbol{\xi}_1 + \rho_0\omega_0^2\boldsymbol{\xi}_1 = -2\rho_0\omega_0\omega_{\Omega 1}\boldsymbol{\xi}_{nlm} + \omega_0\mathcal{M}_0\boldsymbol{\xi}_{nlm}, \tag{2.31}$$

subject to $\boldsymbol{\xi}_1$ being regular at $x = 0$ and

$$\mathbf{e}_r\cdot\nabla_0(\mathbf{e}_r\cdot\boldsymbol{\xi}_1) - \kappa_0\mathbf{e}_r\cdot\boldsymbol{\xi}_1 = \kappa_1\mathbf{e}_r\cdot\boldsymbol{\xi}_0 \qquad \text{at } x = R \tag{2.32}$$

(see Appendix). We consider the calculation of $\boldsymbol{\xi}_1$ in Section 3.2. It is necessary, as a solubility condition, to evaluate the perturbed eigenvalue, also just to $O(\varepsilon)$ thus:

$$\omega_{\Omega 1} = (2I\omega_0)^{-1}\langle \boldsymbol{\xi}^*_{nlm}\cdot\mathcal{M}_0\boldsymbol{\xi}_{nlm}\rangle. \tag{2.33}$$

Substituting these first-order corrections into the previously neglected terms on the right-hand side of equation (2.21) yields an improved approximation to $\Delta\omega$, which in the rotating frame $\mathscr{S}$ can be written

$$\begin{aligned} \omega = \omega_0 + A_{mm} + \Sigma|D_{mm'}|^2B_{m'm'} + \varepsilon^2\langle \boldsymbol{\xi}^*_{nlm}\cdot(\mathcal{N}_0 - \mathcal{L}_\Omega - \rho_\Omega\omega_0^2)\boldsymbol{\xi}_{nlm}\rangle + \varepsilon^2(2\omega_0)^{-1}(\omega_{\Omega 1})^2 - \varepsilon^2 I^{-1}\omega_{\Omega 1}\langle\rho_0\boldsymbol{\xi}^*_{nlm}\cdot\boldsymbol{\xi}_1\rangle \\ + \varepsilon^2(2I)^{-1}\langle\boldsymbol{\xi}^*_{nlm}\cdot\mathcal{M}_0\boldsymbol{\xi}_1\rangle + \varepsilon^2\omega_{\Omega 1}(2I\omega_0)^{-1}\langle\boldsymbol{\xi}^*_{nlm}\cdot\mathcal{M}_0\boldsymbol{\xi}_{nlm}\rangle + O[\max(\varepsilon^2, \delta^2)]. \end{aligned} \tag{2.34}$$

In the inertial frame the frequency is augmented by $m\Omega_c$ due to the kinematic transformation from the rotating frame $\mathscr{S}$.

As is well known, when Ω is a function of r alone, $\omega_{\Omega 1}$ is proportional to m: explicitly

$$\begin{aligned} \omega_{\Omega 1} &= -m\Omega_c + \frac{m}{I}\int_0^R [(\xi - \eta)^2 + (L^2 - 2)\eta^2]\Omega(r)\rho r^2\, dr, \\ &\equiv -m\Omega_c + m\langle\langle\Omega\rangle\rangle, \end{aligned} \tag{2.35}$$

where $L^2 = l(l+1)$.

We refer to the term $-(2I\omega_0)^{-1}\delta^2\langle\boldsymbol{\xi}^*_{nlm'}\cdot(\mathcal{L}_B + \rho_B\omega_0^2)\boldsymbol{\xi}_{nlm'}\rangle$ in B_{mm} as the *distortional* contribution to B_{mm}, and we call that part of B_{mm} arising from $\mathscr{B}_0$ the *direct* contribution. (We shall also refer loosely to these contributions as the distortional and direct magnetic effects on the eigenfrequencies.) Similarly, we call the term $-(2I\omega_0)^{-1}\varepsilon^2\langle\boldsymbol{\xi}^*_{nlm}\cdot(\mathcal{L}_\Omega + \rho_\Omega\omega_0^2)\xi_{nlm}\rangle$ in equation (2.34) the distortional effect of rotation. All the other second-order rotational terms in equation (2.34) we refer to as the direct rotational contribution.

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3 METHOD OF CALCULATION

Having established the ordering of the expansion we now find it convenient to restore dimensions. The equations of Section 2 still hold, but with ε and δ formally set to unity, provided the right-hand side of equation (2.5) is divided by μ_0. In addition, we specialize to toroidal magnetic fields of the form

$$\mathbf{B}=\left[0,\ 0,\ a(r)\frac{d}{d\theta}P_k(\cos\theta)\right], \tag{3.1}$$

with respect to coordinates (r, θ, ϕ) now about the axis of symmetry of **B**, poloidal fields of the form

$$\mathbf{B}=\left[k(k+1)\frac{b(r)}{r^2}P_k(\cos\theta),\ \frac{1}{r}\frac{db}{dr}\frac{d}{d\theta}P_k(\cos\theta),\ 0\right], \tag{3.2}$$

and angular velocity distributions that are independent of latitude: that is,

$$\mathbf{\Omega}=\Omega(r)\mathbf{e}_z, \tag{3.3}$$

where $\mathbf{e}_z$ is a unit vector along the axis of rotation. Here $P_k(\cos\theta)$ is a Legendre polynomial in $\cos\theta$ of degree k, and a, b and Ω are functions of r which will be specified for the numerical calculations below. The axis of symmetry of the magnetic field need not coincide with the axis of rotation. In that case, in order for the frame $\mathscr{S}$ to exist, as we have presumed, Ω must be uniform where $\mathbf{B}\neq\mathbf{0}$. When the axes of **B** and $\mathbf{\Omega}$ coincide, Ω need be uniform only where $b\neq 0$. Of course, there are more general configurations of **B** and $\mathbf{\Omega}$ for which frames $\mathscr{S}$ exist, but we do not consider them here.

The Lorentz force due to a field of the form in equations (3.1) or (3.2) may be written as

$$\mathbf{F}=\rho(r)\sum_{\lambda=0}^{2k}\left[f_{r\lambda}(r)P_\lambda,\ f_{\theta\lambda}(r)\frac{dP_\lambda}{d\theta},\ 0\right], \tag{3.4}$$

where the sum is over even values of λ only. For each value of λ, the term in the sum (equation 3.4) gives rise to a perturbation of each of p, ρ, c^2, etc. proportional to $P_\lambda(\cos\theta)$. To $O(|\mathbf{B}|^2)$ each value of λ may be considered separately, and its contributions to $\mathscr{L}_B$ and ρ_B we call $\mathscr{L}_{B\lambda}$ and $\rho_{B\lambda}P_\lambda(\cos\theta)$, respectively. The contribution to matrix element B_{mm} from $\mathscr{L}_{B\lambda}$ and $\rho_{B\lambda}$ is

$$-(2I\omega_0)^{-1}\langle\boldsymbol{\xi}^*_{nlm}\cdot(\mathscr{L}_{B\lambda}+\rho_{B\lambda}P_\lambda\omega_0^2)\boldsymbol{\xi}_{nlm}\rangle. \tag{3.5}$$

The angular integrals may be performed analytically. It may easily be shown that expression (3.5) is equal to an integral over r (which is independent of m) multiplied by $Q_{\lambda lm}$, where

$$Q_{\lambda lm}=\int_{-1}^{1}P_\lambda(z)[P_l^m(z)]^2\,dz\bigg/\int_{-1}^{1}[P_l^m(z)]^2\,dz. \tag{3.6}$$

Thus the distortional contribution to B_{mm} is a sum of terms each proportional to $Q_{\lambda lm}$, for $\lambda=0, 2,\ldots, 2k$. Similarly, for a field of the form in equations (3.1) or (3.2) the direct contributions to B_{mm} is also a sum of terms proportional to $Q_{\lambda lm}$ ($\lambda=0, 2,\ldots, 2k$). Thus we may write

$$B_{mm}=\sum_{\lambda=0\,(\text{even})}^{2k}I_\lambda^{\text{mag}}Q_{\lambda lm}, \tag{3.7}$$

where I_λ^{mag} depends on n and l, and the strength and geometry of the magnetic field, but not on m. The integrals I_λ^{mag} must in general be computed numerically.

The centrifugal force is also of the form (3.4), with $k=1$ for latitudinally independent rotation. [More generally, if $\Omega(r)$ in equation (3.3) were replaced by $\Omega(r,\theta)=\Sigma_{j=1}^{N}\Omega_j(r)\cos^j\theta$, the appropriate value of k would be $k=N+1$. The sum would be over even values of λ if Ω were symmetric or antisymmetric about the equatorial plane, and over both odd and even values in the more general case.] The second-order rotation terms may be written in the form

$$\sum_{\lambda=0,2}I_\lambda^{\text{rot}}Q_{\lambda lm}, \tag{3.8}$$

and equation (2.34) may be written as

$$\omega=\omega_0+A_{mm}+\sum_{\lambda\text{ even}}I_\lambda Q_{\lambda lm}. \tag{3.9}$$

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It is a useful property of the integrals $Q_{\lambda lm}$ that

$$\sum_{m'=-l}^{l} \{d^{(l)}_{mm'}(\beta)\}^2 Q_{2jlm} = P_{2j}(\cos\beta) Q_{2jlm}. \tag{3.10}$$

Thus, when equations (2.34) and (3.9) are valid,

$$I_\lambda = I_\lambda^{\rm rot} + P_\lambda(\cos\beta) I_\lambda^{\rm mag}. \tag{3.11}$$

Another useful property is that when $l \gg \lambda$

$$Q_{2jlm} \approx \frac{(-1)^j (2j)!}{2^{2j}(j!)^2} P_{2j}(m/L), \tag{3.12}$$

where

$$L = \sqrt{l(l+1)}. \tag{3.13}$$

(See Appendix 2 of Edmonds 1957.) Thus the coefficients I_λ are simply related to the splitting coefficients quoted by observers (Duvall, Harvey & Pomerantz 1986; see also Section 8).

Many of the terms in equations (2.27) and (2.34) are straightforward to calculate. In particular, $\omega_{\Omega 1}$ is the familiar linear splitting term due to advection and the Coriolis force in frame $\mathscr{S}$, given by equation (2.35) in the case of latitudinally independent rotation, and $m\Omega_c$ is the kinematic effect of the rotation of the frame: $\omega_{\Omega 1} + m\Omega_c$ is simply the right-hand side of equation (1.2). The evaluation of the terms involving $\mathscr{N}_0$ and $\mathscr{B}_0$, which arise from the direct effect on the wave of the perturbed centrifugal and Lorentz forces, presents no difficulty in principle. The remaining second-order rotation terms in equation (2.34), excluding the distortion terms and those involving ξ_1, sum to $\frac{1}{2}(\omega_{\Omega 1})^2/\omega_0$.

3.1 Non-spherically symmetric distortion

We follow closely the method of Lebovitz (1970). In the presence of a small perturbing force $\mathbf{F}$ the equations of hydrostatic equilibrium of the self-gravitating star are

$$\nabla p = \rho \nabla \Phi + \mathbf{F} \equiv \rho \mathbf{g} + \mathbf{F}, \tag{3.14}$$

$$\nabla^2 \Phi = -4\pi G \rho, \tag{3.15}$$

with

$$\Phi \to 0 \text{ as } |\mathbf{r}| \to \infty, \tag{3.16}$$

where Φ is the gravitational potential and $\mathbf{g}$ the gravitational acceleration. To the order to which we are working the distortion due to the Lorentz and centrifugal forces may be considered separately. To illustrate the method we consider the Lorentz force. For the fields considered this takes the form of equation (3.4). To order $|\mathbf{B}|^2$ each value of λ can be considered separately. The transformation (coordinate 2.9) is chosen to be

$$x = [1 + h_{B\lambda}(r) P_\lambda(\cos\theta)] r, \tag{3.17}$$

with θ and ϕ unchanged, and the equilibrium quantities p, ρ, etc. expanded in the form

$$p = p_0(x) + p_{B\lambda}(x) P_\lambda(\cos\theta). \tag{3.18}$$

Working throughout with respect to the original base vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ the gradient operator is

$$\nabla = \nabla_0 + \nabla_{B\lambda}, \tag{3.19}$$

where ∇_0 is given by equation (2.11) and

$$\nabla_{B\lambda} = \mathbf{e}_r \frac{d}{dx}(x h_{B\lambda}) P_\lambda \frac{\partial}{\partial x} + \mathbf{e}_\theta \left(h_{B\lambda} \frac{dP_\lambda}{d\theta} \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} \right) + \mathbf{e}_\phi \frac{h_{B\lambda} P_\lambda}{x \sin\theta} \frac{\partial}{\partial \theta}. \tag{3.20}$$

The divergence is

$$\nabla \cdot \boldsymbol{\xi} = \nabla_0 \cdot \boldsymbol{\xi} + \nabla_{B\lambda} \cdot \boldsymbol{\xi}, \tag{3.21}$$

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where $\nabla_0\cdot$ is given by equation (2.12) and

$$\nabla_{B\lambda}\cdot\boldsymbol{\xi}=\frac{1}{x^2}\frac{d}{dx}(xh_{B\lambda})P_\lambda\frac{d}{dx}(x^2\xi)Y_l^m-2\frac{dh_{B\lambda}}{dx}P_\lambda\xi Y_l^m-L^2\frac{h_{B\lambda}}{x}P_\lambda Y_l^m+h_{B\lambda}\frac{dP_\lambda}{d\theta}\frac{d\eta}{dx}\frac{\partial Y_l^m}{\partial\theta}. \tag{3.22}$$

The operator $\mathscr{L}$ is given by

$$\mathscr{L}\boldsymbol{\xi}=\mathscr{L}_0\boldsymbol{\xi}+\mathscr{L}_{B\lambda}\boldsymbol{\xi} \tag{3.23}$$

where $\mathscr{L}_0$ is given by equation (2.13) and

$$\begin{aligned}\mathscr{L}_{B\lambda}\boldsymbol{\xi}=&\nabla_0[(\rho_{B\lambda}c_0^2+2\rho_0c_0c_{B\lambda})P_\lambda\nabla_0\cdot\boldsymbol{\xi}+\boldsymbol{\xi}\cdot\nabla_0(p_{B\lambda}P_\lambda)]-\nabla_0(p_{B\lambda}P_\lambda)\nabla_0\cdot\boldsymbol{\xi}+\nabla_0(\rho_0c_0^2\nabla_{B\lambda}\cdot\boldsymbol{\xi}+\boldsymbol{\xi}\cdot\nabla_{B\lambda}p_0)\\&+\nabla_{B\lambda}(\rho_0c_0^2\nabla_0\cdot\boldsymbol{\xi}+\boldsymbol{\xi}\cdot\nabla_0p_0)-(\nabla_{B\lambda}p_0)\nabla_0\cdot\boldsymbol{\xi}-(\nabla_0p_0)\nabla_{B\lambda}\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla_0(\rho_{B\lambda}P_\lambda/\rho_0)\nabla_0p_0-(\boldsymbol{\xi}\cdot\nabla_0\ln\rho_0)\nabla_0(p_{B\lambda}P_\lambda)\\&-(\boldsymbol{\xi}\cdot\nabla_0\ln\rho_0)\nabla_{B\lambda}p_0-(\boldsymbol{\xi}\cdot\nabla_{B\lambda}\ln\rho_0)\nabla_0p_0.\end{aligned} \tag{3.24}$$

The photospheric distortion amplitude $h_{B\lambda}(R)$ is such that the stellar surface maps to $x=R$, but there remains much freedom in choosing $h_{B\lambda}(r)$. We determine $h_{B\lambda}$ by choosing surfaces of constant x to be surfaces of constant pressure (i.e. $p_{B\lambda}=0$) and by taking $h_{B\lambda}(x)=h_{B\lambda}(R)$ for $x>R$. Expressing ∇ and ∇^2 in terms of $\mathbf{x}$, equations (3.14) and (3.15) together yield three scalar equations for each non-zero value of λ, enabling $\Phi_{B\lambda}$, $\rho_{B\lambda}$, $h_{B\lambda}$ to be found. However, hydrostatic equilibrium is insufficient to determine the spherically symmetric component of the structure. For $\lambda=0$ the horizontal component of equation (3.14) is automatically satisfied, and provides no constraint relating Φ_{B0}, ρ_{B0} and h_{B0}. Consequently, only two non-trivial equations remain. As the spherically symmetric distortion does not affect the fine splitting, however, we postpone consideration of this issue until Section 7.

For $\lambda\neq0$, an inhomogeneous differential equation and boundary conditions for $h_{B\lambda}(x)$ are obtained by matching Φ on to the vacuum potential in $x>R$ and requiring that the transformation (2.9), (3.17) be regular at $x=0$. In terms of $u_\lambda\equiv xh_{B\lambda}(x)$:

$$\left.\begin{aligned}&\mathscr{K}u_\lambda=\mathscr{G}(\mathbf{x})\qquad 0<x<R\\&u_\lambda\text{ bounded as }x\to0\\&x\frac{du_\lambda}{dx}+(\lambda-1)u_\lambda=\frac{-x^3}{GMq_0(x)}\left[x\frac{d}{dx}f_{\theta\lambda}(x)+(\lambda+2)f_{\theta\lambda}(x)\right]\text{ as }x\to R-\end{aligned}\right\} \tag{3.25}$$

where

$$\mathscr{K}\equiv\frac{d^2}{dx^2}+\left[\frac{8\pi\rho_0x^2}{Mq_0}-\frac{2}{x}\right]\frac{d}{dx}-\frac{(\lambda-1)(\lambda+2)}{x^2}, \tag{3.26}$$

$$\mathscr{G}\equiv\frac{4\pi x^2}{GMq_0^2}\rho_0\left[f_{r\lambda}-\frac{d}{dx}(xf_{\theta\lambda})\right]+\frac{\lambda(\lambda+1)x}{GMq_0}f_{\theta\lambda}-\frac{1}{GMq_0}\frac{d}{dx}\left[x^2\frac{d}{dx}(xf_{\theta\lambda})\right], \tag{3.27}$$

$$q_0(x)=M^{-1}\int_0^x 4\pi\rho_0(x')x'^2\,dx'. \tag{3.28}$$

This determines u_λ and hence $h_{B\lambda}$. The choice $p_{B\lambda}=0$ implies

$$\rho_{B\lambda}=\frac{\rho_0x^2}{GMq_0}\left[f_{r\lambda}-\frac{d}{dx}(xf_{\theta\lambda})\right], \tag{3.29}$$

and the perturbation to the sound speed is

$$c_{B\lambda}=\tfrac{1}{2}c_0\left[\left(\frac{\partial\ln\gamma}{\partial\ln\rho}\right)_p-1\right]\frac{\rho_{B\lambda}}{\rho_0}, \tag{3.30}$$

where γ is the adiabatic exponent $(\partial\ln p/\partial\ln\rho)_s$, the partial derivative being taken at constant specific entropy s. The perturbation $\mathscr{L}_{B\lambda}$ to $\mathscr{L}$ is then given by equation (3.24) in terms of $h_{B\lambda}$, $\rho_{B\lambda}$ and $c_{B\lambda}$. Then

$$\mathscr{L}_B=\Sigma\mathscr{L}_{B\lambda},\qquad \rho_B=\Sigma\rho_{B\lambda}P_\lambda. \tag{3.31}$$

The component of the distortion resulting from the centrifugal force is calculated in an analogous manner.

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Note that the transformation $\mathbf{r}\to\mathbf{x}$ affects the calculation only of the distortion terms. Therefore, in the other terms we continue to work in untransformed space if confusion is unlikely. In addition, we shall often use r instead of x to denote a radial dummy variable of integration even in the distortion terms.

3.2 Perturbation to the eigenfunction

The first-order perturbation $\boldsymbol{\xi}_1$ to the eigenfunction $\boldsymbol{\xi}$ due to rotation has previously been calculated for g-modes and toroidal modes in the special case of uniform rotation (Berthomieu *et al.* 1978; Provost, Berthomieu & Rocca 1981). Indeed, these authors have taken the expansion to an even higher order than we require. However, here we allow Ω to vary with depth. The equation (2.31) for $\boldsymbol{\xi}_1$ is

$$\mathscr{L}_0\boldsymbol{\xi}_1+\rho_0\omega_0^2\boldsymbol{\xi}_1=-2\rho_0\omega_0\omega_{\Omega 1}\boldsymbol{\xi}_0+\omega_0\mathscr{M}_0\boldsymbol{\xi}_0 \tag{3.32}$$

with the boundary conditions that $\boldsymbol{\xi}_1$ be regular at $r=0$ and the condition (2.32) at $r=R$. To the solution $\boldsymbol{\xi}_1$ may be added any multiple of $\boldsymbol{\xi}_0$. (This merely affects the amplitude of the solution and, of course, leaves the frequency perturbation unchanged.) We normalize our solution by imposing the condition $\mathbf{r}\cdot\boldsymbol{\xi}_1=0$ at $r=R$.

For any $\boldsymbol{\xi}$, $\mathbf{r}\cdot\nabla\times\mathscr{L}_0\boldsymbol{\xi}=0$. Further, since $\boldsymbol{\xi}_0$ is a non-trivial (i.e. spheroidal) mode of a spherically symmetric star, $\mathbf{r}\cdot\nabla\times\boldsymbol{\xi}_0=0$. Thus equation (3.32) implies

$$\rho_0\omega_0^2\mathbf{r}\cdot\nabla\times\boldsymbol{\xi}_1=\omega_0\mathbf{r}\cdot\nabla\times(\mathscr{M}_0\boldsymbol{\xi}_0). \tag{3.33}$$

Hence, as has been pointed out by Simon (1969) and others, $\boldsymbol{\xi}_1$ must in general have a toroidal part:

$$\boldsymbol{\xi}_1=\boldsymbol{\xi}_{\mathrm{S}}+\boldsymbol{\xi}_{\mathrm{T}}, \tag{3.34}$$

where $\boldsymbol{\xi}_{\mathrm{T}}$ is toroidal and $\boldsymbol{\xi}_{\mathrm{S}}$ is spheroidal (see, for example, Cox 1980). For simplicity we consider only latitudinally independent rotation. From equations (3.32) and (2.3)

$$\boldsymbol{\xi}_{\mathrm{T}}=\sum_{\lambda=l\pm 1}\zeta_\lambda(r)\left(0,\frac{1}{\sin\theta}\frac{\partial Y_\lambda^m}{\partial\phi},-\frac{\partial Y_\lambda^m}{\partial\theta}\right), \tag{3.35}$$

where $\theta=0$ is now taken to be the rotation axis, and

$$\left.\begin{aligned}\zeta_{l-1}&=\frac{2i\Omega}{l\omega_0}\left[\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\right]^{1/2}[\xi+(l+1)\eta]\\ \zeta_{l+1}&=\frac{-2i\Omega}{(l+1)\,\omega_0}\left[\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}\right]^{1/2}(\xi-l\eta).\end{aligned}\right\} \tag{3.36}$$

Had Ω been considered to vary with latitude, say

$$\Omega=\sum_{k=1}^{N}\Omega_k(r)\cos^k\theta,$$

then the same procedure would in general have produced for each k non-zero ζ_λ for $\lambda=l-k-1, l-k+1, l-k+3, \ldots, l+k+1$ (with the indices λ bounded below so that $\lambda\geq|m|$), by repeated use of standard identities for $\cos\theta\, P_l^m(\cos\theta)$ and $\sin\theta\, d[P_l^m(\cos\theta)]/d\theta$ (see Abramowitz & Stegun 1964).

The spheroidal part $\boldsymbol{\xi}_{\mathrm{S}}$ may be found to be of the form

$$\boldsymbol{\xi}_{\mathrm{S}}=\left[\xi_{\mathrm{S}}(r)Y_l^m,\ \eta_{\mathrm{S}}(r)\frac{\partial Y_l^m}{\partial\theta},\ \frac{\eta_{\mathrm{S}}}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\right], \tag{3.37}$$

where ξ_{S} and η_{S} satisfy the differential equations

$$\frac{d\xi_{\mathrm{S}}}{dr}+\left(U+\frac{2}{r}+\frac{d\ln\rho}{dr}\right)\xi_{\mathrm{S}}-\left(\frac{L^2}{r}-\frac{\omega_0^2}{c^2}r\right)\eta_{\mathrm{S}}=-\frac{2m\Omega}{L^2\omega_0}\frac{r\omega_0^2}{c^2}(\xi+\eta)+\frac{2[m\Omega-(\omega_{\Omega 1}+m\Omega_c)]}{\omega_0}\frac{r\omega_0^2}{c^2}\eta \tag{3.38}$$

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$$\frac{d\eta_S}{dr}-\left(\frac{1}{r}+\frac{U}{\rho\omega_0^2}\frac{dp}{dr}\right)\xi_S-\left(U-\frac{1}{r}\right)\eta_S=\frac{[2(\omega_{\Omega 1}+m\Omega_c)-m\Omega]}{\omega_0}\left[\frac{\xi}{r}+\left(U-\frac{1}{r}\right)\eta-\frac{d\eta}{dr}\right]$$

$$+\frac{2m\Omega}{\omega_0}[\eta-L^{-2}(\xi+\eta)]+\frac{2m\Omega}{L^2\omega_0}\left[\frac{L^2\eta}{r}+\left(U-\frac{1}{r}-\frac{d}{dr}\right)(\xi+\eta)\right] \tag{3.39}$$

with ξ_S regular as $r\to 0$ and $\xi_S(R)=0$. Here

$$U=\frac{1}{\gamma}\frac{d\ln p}{dr}-\frac{d\ln \rho}{dr}. \tag{3.40}$$

Again, had Ω been considered to take the form $\Sigma\Omega_k(r)\cos^k\theta$, then each k-component would in general have given rise to extra terms in $\boldsymbol{\xi}_S$, of the form (3.37) but with l replaced by $\lambda=l-k,\ l-k+2,\ \dots,\ l+k$.

The contributions from $\boldsymbol{\xi}_1$ to equation (2.34) are

$$\langle\rho_0\boldsymbol{\xi}^*_{nlm}\cdot\boldsymbol{\xi}_1\rangle=\int_0^R\rho_0(r)(\xi_S\xi+L^2\eta_S\eta)r^2\,dr \tag{3.41}$$

and (using the fact that the velocity $\mathbf{v}$ is parallel to $\mathbf{e}_\phi$)

$$\langle\boldsymbol{\xi}^*_{nlm}\cdot\mathscr{M}_0\boldsymbol{\xi}_1\rangle=\langle\boldsymbol{\xi}^*_1\cdot\mathscr{M}_0\xi_{nlm}\rangle^*=2m\int(\Omega-\Omega_c)\rho_0(\xi_S\xi+L^2\eta_S\eta)r^2\,dr-2m\int\Omega\rho_0(\xi_S\eta+\eta_S\xi+\eta_S\eta)r^2\,dr$$

$$+\int\frac{4\Omega^2\rho_0}{\omega_0}\left\{\frac{(l+1)(l+2)}{(2l+1)(2l+3)}\left[1-\frac{m^2}{(l+1)^2}\right](\xi-l\eta)^2+\frac{l(l-1)}{(2l-1)(2l+1)}\left(1-\frac{m^2}{l^2}\right)[\xi+(l+1)\eta]^2\right\}r^2\,dr. \tag{3.42}$$

Note that since $\omega_{\Omega 1}$ is proportional to m, so are ξ_S and η_S. Thus, apart from an m-independent contribution to equation (3.42) from $\boldsymbol{\xi}_T$ in the third integral on the right-hand side, the direct (i.e. excluding distortional terms) second-order rotational contribution to equation (2.34) is proportional to m^2. Thus when expressed in terms of the coefficients $Q_{\lambda lm}$, which are of order unity even for large values of l, the direct second-order contribution to the coefficients I_λ typically tends to become more important as l increases.

Note also that $\boldsymbol{\xi}_1$ is independent of Ω_c, as it should be. Also, although equation (3.42) has an explicit dependence on Ω_c, it can be shown that the total direct second-order rotational contribution to the frequency does not depend on Ω_c, nor does the distortional term. Indeed, this must be so, because the frequency depends on Ω_c only through the first-order kinematic contribution $-m\Omega_c$.

4 SHORT-WAVELENGTH ASYMPTOTICS

Before presenting numerical results, we discuss how to obtain asymptotic estimates of the frequency splitting due to the distortion of the equilibrium state, to advection by rotation and to the perturbed Lorentz force. Our treatment generalizes that of Gough & Taylor (1984), who presented asymptotic formulae appropriate only to low-degree modes. Our results reduce to theirs if one formally sets L/ω to zero. Throughout this section, the zero subscripts on quantities relating to the non-rotating, non-magnetic state are dropped.

It is useful to record the asymptotic formula for the high-frequency p modes of a non-magnetic non-rotating star:

$$\pi(n+\alpha)\approx\int_{r_t}^R\frac{\omega}{c}\left(1-\frac{L^2c^2}{\omega^2r^2}\right)^{1/2}dr \tag{4.1}$$

where r_t is the lower turning point, at which $(1-L^2c^2/\omega^2r^2)$ vanishes. The quantity α is a weak function of ω, and to a first approximation can be regarded as being constant. This formula can be derived either by applying the Liouville–Green (JWKB) method to equation (2.15) and its boundary conditions (e.g. Deubner & Gough 1984; Gough 1989), or by regarding the mode as a constructive interference pattern of resonant propagating waves (e.g. Christensen-Dalsgaard *et al.* 1985; Gough 1986b). In the discussion that follows we adopt the latter view. Thus we consider the mode to be a superposition of locally plane waves which, in the spherically symmetric hydrostatic case, have wavenumber

$$\mathbf{k}=(k_r,k_y,k_\phi)=\left[\left(\frac{\omega^2}{c^2}-\frac{L^2}{r^2}\right)^{1/2},\quad\left(\frac{L^2}{r^2}-\frac{m^2}{r^2\sin^2\theta}\right)^{1/2},\quad\frac{m}{r\sin\theta}\right] \tag{4.2}$$

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and satisfy the ordinary acoustic dispersion relation

$$\omega^2 = k^2 c^2, \tag{4.3}$$

where $k = |\mathbf{k}|$ and now $L = l + \frac{1}{2}$. (This k is distinct from the integer k used to define the magnetic geometry elsewhere in the paper.)

4.1 Perturbation theory

The phase integral (4.1) is simply the integral of $k = \omega c^{-1}$ along a ray, and satisfies a variational principle, related to Fermat's principle, which under small variations of the equilibrium state permits the integral to be evaluated correct to first order without perturbing the expressions for k_θ and k_ϕ in equation (4.2), provided that the perturbation to the frequency resulting from the perturbation to the dispersion relation (4.3) is taken into account in the evaluation of k_r. This is analogous to the use of zero-order eigenfunctions in Section 2 to calculate first-order perturbations. The result is that any perturbation δc to c modifies the phase integral by an amount proportional to the magnitude of the perturbation weighted by the time the (unperturbed) wave spends in the corresponding part of the star (Gough 1989). Thus, for example, in the case of a spherically symmetric perturbation δc to the sound speed, the fractional frequency change is given by

$$\frac{\delta\omega}{\omega} = \frac{\delta_c\omega}{\omega} \equiv \frac{1}{S}\int_{r_t}^{R} \frac{\delta c}{c}\frac{k}{k_r}\frac{dr}{c} = \frac{1}{S}\int_{r_t}^{R} \frac{\delta c}{c}\left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{-1/2}\frac{dr}{c}, \tag{4.4}$$

where

$$S(\omega/L) = \int_{r_t}^{R}\left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{-1/2}\frac{dr}{c} \tag{4.5}$$

and we have used equations (4.2) and (4.3) for the unperturbed rays.

This simple result can evidently also be obtained by formally perturbing equation (4.1) to first order (*cf.* Christensen-Dalsgaard 1986; Christensen-Dalsgaard, Gough & Perez Hernandez 1988). However, when the perturbation is not spherically symmetric, the ray theory must be considered in somewhat more detail. Rays of the unperturbed star that constitute a mode of degree l and azimuthal order m all lie in planes through the centre of the star the normals of which are at an angle $\theta = \sin^{-1}(m/L)$ out of the equatorial plane of the spherical polar coordinates (e.g. Keller & Rubinow 1960; see also Gough 1986b). For perturbations δc that are axisymmetric with respect to the coordinate axis, all such planes are equivalent. In such a plane define polar coordinates $(r, \tilde{\phi})$. Then $\cos\theta = \cos\Theta\cos\tilde{\phi}$, with a suitable choice of the origin of $\tilde{\phi}$. The effect of a perturbation $\delta c(r, \cos\theta)$ on the eigenfrequency is then given by equation (4.4) but with $\delta c/c$ replaced by its average over $\tilde{\phi}$:

$$\frac{\overline{\delta c}}{c} \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{\delta c}{c}\,d\tilde{\phi} = \frac{1}{\pi}\int_{-\cos\Theta}^{\cos\Theta}\frac{\delta c}{c}\frac{d\cos\theta}{(\cos^2\Theta - \cos^2\theta)^{1/2}}. \tag{4.6}$$

Note that in the limit of large l, the weight function

$$[P_l^m(\cos\theta)]^2 \Big/ \int_{-1}^{1}[P_l^m(\mu)]^2\,d\mu$$

[for example in the definition (3.6) of $Q_{\lambda lm}$] reduces to the weight function $\pi^{-1}(\cos^2\Theta - \cos^2\theta)^{-1/2}$ in equation (4.6). This is one way of approaching the result (3.12).

For non-axisymmetric perturbations, $\delta c/c$ must also be appropriately averaged over ϕ.

Equations (4.4)–(4.6) are valid for a genuine scalar perturbation δc that is not associated with a distortion of the surface of the star. If the star is distorted, as is generally the case when rotation or a magnetic field is present, due account must also be taken of the change in the time it takes the wave to travel over the modified distance. Thus, if $x = r[1 + h(\mathbf{r})]$ is a modified radial coordinate chosen (as in Section 3) to be constant on surfaces of constant pressure, and δc is the perturbed sound speed at fixed x, then in addition to the perturbation (4.4) there is a contribution

$$\frac{\delta_h\omega}{\omega} = \frac{1}{S}\int_{r_t}^{R}\left[\left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)\frac{d}{dr}(r\bar{h}) + \frac{L^2c^2}{\omega^2 r^2}\bar{h}\right]\left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{-1/2}\frac{dr}{c} \tag{4.7}$$

to $\delta\omega/\omega$. (Here r rather than x has been used as the dummy integration variable.) This formula can be derived by formally

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perturbing equation (4.1) at constant $c(x)$. Once again, the overbar denotes an average over the rays, of the form of equation (4.6) [with h replacing $(\delta c/c)$].

4.2 First-order rotational splitting

In a frame rotating rigidly with the angular velocity $\Omega(r, \theta)$ (for some r and θ), locally the acoustic waves still satisfy asymptotically the local dispersion relation (equation 4.3); the waves are simply advected by the rotational flow. Thus, on transforming back to an inertial frame, the dispersion relation becomes

$$(\omega - m\Omega)^2 = k^2 c^2, \tag{4.8}$$

which, to first order in Ω/ω, is mathematically equivalent to a relative perturbation $\delta c/c$ to the sound speed of magnitude $m\Omega/\omega$. Substituting into equation (4.4), this yields for the first-order rotational frequency perturbation

$$\delta_\Omega \omega = \frac{m}{S}\int_{r_t}^{R} \bar{\Omega}\left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{dr}{c}, \tag{4.9}$$

where, once again, the overbar denotes the average (4.6) over co-latitude (*cf.* Christensen-Dalsgaard 1988). This formula generalizes the result obtained by Gough (1984) for sectoral modes.

4.3 Lorentz splitting

A weak slowly varying magnetic field **B** modifies the dispersion relation (equation 4.3), which becomes

$$\omega^2 = k^2(c^2 + v_{A\perp}^2) \tag{4.10}$$

where $v_{A\perp}^2 = v_A^2 \sin^2\psi$, $v_A = (\mu_0 \rho)^{-1/2} B$ is the Alfvén speed, μ_0 being the magnetic permeability of the vacuum, and ψ is the angle between the magnetic field and the direction of propagation of the wave. As with advection by rotation, the influence of the field can be represented as a sound-speed perturbation, except that now the magnitude of the perturbation depends on the direction of propagation of the wave. The perturbation to the frequency arising directly from the influence of the Lorentz force on the wave is once again given by equation (4.4), which becomes

$$\frac{\delta_L \omega}{\omega} = \frac{1}{S}\int_{r_t}^{R} \frac{\overline{(B\sin\psi)^2}}{2\mu_0 \rho c^2}\left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{dr}{c}. \tag{4.11}$$

It is straightforward to show that for a toroidal field $(0, 0, B_\phi)$, equation (4.6) becomes

$$\overline{(B\sin\psi)^2} = \frac{1}{\pi}\int_{-\cos\Theta}^{\cos\Theta} B_\phi^2 \left[1 - \frac{m^2 c^2}{\omega^2 r^2 (1-\cos^2\theta)}\right] \frac{d\cos\theta}{(\cos^2\Theta - \cos^2\theta)^{1/2}}; \tag{4.12}$$

and for a poloidal field $(B_r, B_\theta, 0)$

$$\overline{(B\sin\psi)^2} = \frac{1}{\pi}\int_{-\cos\Theta}^{\cos\Theta} \left[\frac{L^2 c^2}{\omega^2 r^2} B_r^2 + \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\,\frac{\cos^2\Theta - \cos^2\theta}{1-\cos^2\theta}\right) B_\theta^2\right] \frac{d\cos\theta}{(\cos^2\Theta - \cos^2\theta)^{1/2}} \tag{4.13}$$

(Gough 1989). Thus, in particular, the direct contributions I_λ^{dir} to the integrals I_λ^{mag} can be approximated by

$$I_\lambda^{dir} \approx \frac{\omega}{2S}\int_{r_t}^{R} \frac{a^2}{\mu_0 \rho c^2} F_0 \left(1 + \frac{L^2 c^2}{\omega^2 r^2} F_1\right)\left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{dr}{c} \tag{4.14}$$

for the toroidal field (3.1), where the coefficients $F_0(k, \lambda)$ and $F_1(k, \lambda)$ come from evaluating the angular integrals in equation (4.11) in the limit of large l. In this case, for example, they may thus be found by equating coefficients of m/L and of $Lc/\omega r$ in the identity

$$\sum_{\lambda\ \mathrm{even}} F_0 \left\{1 + F_1 \frac{L^2 c^2}{\omega^2 r^2}\right\} \frac{(-1)^{\lambda/2}\lambda!}{2^\lambda (\lambda/2)!^2} P_\lambda(m/L) \equiv \frac{1}{\pi}\int_{-\cos\Theta}^{\cos\Theta} \left(\frac{d}{d\theta} P_k(\cos\theta)\right)^2 \left[1 - \frac{(m^2/L^2)}{\sin^2\theta}\frac{L^2 c^2}{\omega^2 r^2}\right] \frac{d\cos\theta}{(\cos^2\Theta - \cos^2\theta)^{1/2}}. \tag{4.15}$$

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Similarly, for the poloidal field (3.2),

$$I_\lambda^{\mathrm{dir}} \approx \frac{\omega}{2S}\int_{r_t}^{R}\left\{\frac{(b'/r)^2}{\mu_0\rho c^2}F_0\left(1+\frac{L^2c^2}{\omega^2r^2}F_1\right)+\frac{[k(k+1)b/r^2]^2}{\mu_0\rho c^2}\frac{L^2c^2}{\omega^2r^2}F_2\right\}\left(1-\frac{L^2c^2}{\omega^2r^2}\right)^{-1/2}\frac{dr}{c}, \tag{4.16}$$

where the prime denotes differentiation with respect to the argument r. A few values of F_0, F_1 and F_2, which come from evaluating the angular integrals in equation (4.11) in the limit of large l, are given in Tables 1 and 2.

5 ROTATIONAL SPLITTING: NUMERICAL RESULTS

The main focus of this paper is the effect on oscillations of magnetic fields. However, as has been shown, to decipher any magnetic effect from the even component of the observed splitting it is necessary to be able to calculate the second-order effect of rotation. Accordingly, we have calculated the effect of rotation for a few latitudinally independent angular velocity profiles. The equilibrium solar model we have used for the computations in this and the following section is Model A of Christensen-Dalsgaard, Gough & Morgan (1979).

The magnitudes of the individual second-order rotational splitting terms in equation (2.34), apart from the distortional term, depend on the angular velocity of the reference frame $\mathscr{S}$ and on the normalization of ξ. However, writing their sum as $\omega_{\Omega 2}^{(0)}+\omega_{\Omega 2}^{(2)}Q_{2lm}$, such that $\omega_{\Omega 2}^{(0)}$ and $\omega_{\Omega 2}^{(2)}$ are independent of m, the coefficients $\omega_{\Omega 2}^{(0)}$ and $\omega_{\Omega 2}^{(2)}$ are also independent of frame and normalization. The distortional splitting is $\omega_\Omega^{\mathrm{dist}}Q_{2lm}$. Tables 3–5 give the rotational splitting coefficients for three examples of $\Omega(r)$. The first is an angular velocity (Fig. 1) which is qualitatively similar to the equatorial rotation rate inferred from the splitting of sectoral-mode frequencies by Duvall *et al.* (1984). The second is uniform rotation at the observed surface equatorial rotation rate Ω_S inferred by Howard & Harvey (1970). The third is

$$\Omega(r)=\begin{cases}\Omega_S(R/r)^2 & r>r_c\equiv 0.7R\\ \Omega_S(R/r_c)^2 & r\le r_c\end{cases} \tag{5.1}$$

as considered by Gough & Taylor (1984). Although it is almost certainly inappropriate for the Sun, it is included here as an example of a rotation somewhat different from the first two, being faster throughout the radiative interior.

The effect of centrifugal distortion, $\omega_\Omega^{\mathrm{dist}}$, is found to vary little with degree and is approximated well by the asymptotic estimate. The small variations from one mode to the next are due primarily to the differences in ω_0, for $\omega_\Omega^{\mathrm{dist}}/\omega_0$ varies quite smoothly. Because the star is oblate, the traveltime along the ray path is greater for a sectoral mode, which is confined near the equatorial plane, than it is for a mode that propagates in other latitudes. Hence the frequency of a sectoral mode is lower than those of other modes of the same order and degree. As Q_{2lm} is least for $|m|=l$, we therefore expect $\omega_\Omega^{\mathrm{dist}}$ to be positive. This might not be the case if the internal distortion was a more complicated function of depth, as is evident from the discussion pertaining to magnetic distortion in Section 6.

Gough & Taylor (1984) argued that for low-degree modes $\omega_{\Omega 1}$ is generally of order Ω and $\omega_\Omega^{\mathrm{dist}}$ is of order $(\Omega^2R^3/GM)\omega_0$, while the other second-order terms, arising from first-order terms in the equation of motion, are of order Ω^2/ω_0 and are

Table 1. Coefficients F_0 and F_1 required for equation (4.14), for a toroidal field of the form (3.1) with $k=1$ or $k=2$.

	$k=1$		$k=2$		
	$\lambda=0$	$\lambda=2$	$\lambda=0$	$\lambda=2$	$\lambda=4$
F_0	2/3	−2/3	6/5	6/7	−72/35
F_1	−1/2	−2	−1/2	1	−4/3

Table 2. Coefficients F_0, F_1 and F_2 required for equation (4.16), for a poloidal field of the form (3.2) with $k=1$ or $k=2$.

	$k=1$		$k=2$		
	$\lambda=0$	$\lambda=2$	$\lambda=0$	$\lambda=2$	$\lambda=4$
F_0	2/3	−2/3	6/5	6/7	−72/35
F_1	−1/2	1	−1/2	−2	1/3
F_2	1/3	2/3	1/5	2/7	18/35

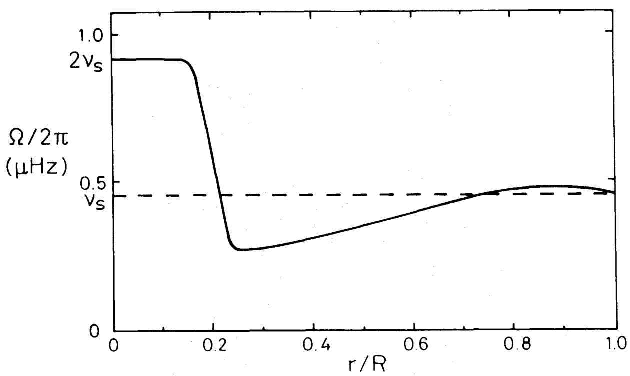

Figure 1. Rotation rate $\Omega(r)$ (expressed as a cyclic frequency measured in μHz) used to compute the results in Table 3, plotted as a function of the fractional radius. The rotation law is qualitatively similar to that inferred from frequency splitting by Duvall *et al.* (1984). For comparison, the dashed horizontal line is at the value of ν_s, the observed surface equatorial rotation rate.

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Table 3. Rotational splitting coefficients computed for the rotation depicted in Fig. 1. All columns except the first two are in nanohertz.

l	ν_0 (mHz)	$(m^{-1}\omega_{\Omega 1}+\Omega_c)/2\pi$	$\omega_\Omega^{dist}/2\pi$	Asympt. $\omega_\Omega^{dist}/2\pi$	$\omega_{\Omega 2}^{(0)}/2\pi$	$\omega_{\Omega 2}^{(2)}/2\pi$
2	3.57	442.	43.6	43.6	0.061	-0.032
3	3.49	437.	42.9	42.8	0.025	0.095
4	3.54	430.	43.6	43.7	0.003	0.187
6	3.50	415.	43.5	43.5	-0.014	0.244
8	3.59	407.	44.7	44.7	-0.016	0.242
10	3.54	407.	44.2	44.2	-0.034	0.315
15	3.59	411.	45.1	45.0	-0.095	0.559
20	3.49	418.	44.1	43.7	-0.182	0.917
25	3.52	424.	44.7	44.2	-0.276	1.29
30	3.53	431.	45.2	44.6	-0.361	1.64
40	3.51	443.	45.8	44.9	-0.456	2.02
50	3.59	450.	48.8	47.8	-0.465	2.05
65	3.53	455.	50.1	48.8	-0.520	2.28
80	3.58	456.	51.6	49.8	-0.631	2.72

Table 4. Rotational splitting coefficients computed for uniform rotation at a rate of 442.5 nHz. All columns except the first two are in nanohertz.

l	ν_0 (mHz)	$(m^{-1}\omega_{\Omega 1}+\Omega_c)/2\pi$	$\omega_\Omega^{dist}/2\pi$	Asympt. $\omega_\Omega^{dist}/2\pi$	$\omega_{\Omega 2}^{(0)}/2\pi$	$\omega_{\Omega 2}^{(2)}/2\pi$
2	3.57	441.	47.6	47.2	0.055	-0.010
4	3.54	441.	47.5	47.2	0.037	0.066
6	3.50	441.	47.4	47.0	0.020	0.141
8	3.59	441.	48.8	48.4	0.002	0.208
10	3.54	442.	48.3	47.8	-0.016	0.283
15	3.59	442.	49.3	48.8	-0.059	0.453
20	3.49	442.	48.4	47.6	-0.105	0.644
25	3.52	442.	49.3	48.3	-0.148	0.815
30	3.53	442.	50.0	49.0	-0.191	0.986
40	3.51	442.	51.2	49.8	-0.278	1.34
50	3.59	442.	54.7	53.2	-0.355	1.64
65	3.53	442.	56.4	54.4	-0.485	2.16
80	3.58	442.	58.0	55.4	-0.598	2.61

Table 5. Rotational splitting coefficients computed for the rotation given by equation (5.1). All columns except the first two are in nanohertz.

l	ν_0 (mHz)	$(m^{-1}\omega_{\Omega 1}+\Omega_c)/2\pi$	$\omega_\Omega^{dist}/2\pi$	Asympt. $\omega_\Omega^{dist}/2\pi$	$\omega_{\Omega 2}^{(0)}/2\pi$	$\omega_{\Omega 2}^{(2)}/2\pi$
2	3.57	668.	48.2	48.8	0.254	-0.583
4	3.54	686.	48.0	48.8	0.208	-0.455
6	3.50	684.	48.1	48.5	0.158	-0.273
8	3.59	684.	49.5	49.9	0.139	-0.205
10	3.54	682.	48.8	49.3	0.098	-0.043
15	3.59	680.	49.6	50.1	0.025	0.244
20	3.49	674.	49.1	49.0	-0.202	1.14
25	3.52	668.	49.8	49.5	-0.427	2.03
30	3.53	661.	50.2	50.2	-0.789	3.47
40	3.51	651.	52.1	52.5	-2.51	10.3
50	3.59	602.	62.4	65.2	-10.0	40.2
65	3.53	548.	59.9	61.3	-7.00	28.2
80	3.58	524.	59.9	61.5	-6.20	24.9

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therefore negligible for five-minute modes. Our results bear this out for low-degree modes, for the angular velocities we have considered. Indeed, $\omega_{\Omega 2}^{(2)}$ is much smaller than $\omega_{\Omega}^{\text{dist}}$ for all values of l considered for the first two examples. For the third, $\omega_{\Omega 2}^{(2)}$ reaches a magnitude comparable with that of $\omega_{\Omega}^{\text{dist}}$ for $l \approx 50$ before decreasing with higher l. One should note, however, that although the term $\omega_{\Omega 2}^{(2)}$ is typically small, the individual contributions to it are not necessarily so; indeed they are frame-dependent [though $\omega_{\Omega 2}^{(2)}$ is not]. The above argument might suggest that for five-minute modes the second-order advection terms should always be small compared with the distortion term. However, while the distortional effect is proportional to Q_{2lm} (for latitudinally independent rotation), which is of order unity, the other second-order rotation terms include a natural m^2 dependence which causes $\omega_{\Omega 2}^{(0)}$ and $\omega_{\Omega 2}^{(2)}$ to be greater, at higher degree, than the above argument might suggest.

Note that if the rotation is uniform (Table 4), then the second-order distortional frequency perturbation can be greater than the linear rotational perturbation in frame $\mathscr{S}$. In this case, the distortional contribution should be included in matrix **A** (equation 2.27) because this contribution will have as important an effect as the linear rotation terms on the mode structure as determined by equation (2.26) for the rotating, magnetic star.

6 MAGNETIC SPLITTING: NUMERICAL RESULTS

Distributed core field

We have applied the method described in Sections 2 and 3 above to calculate the effects of fields of the form (3.1) and (3.2). Gough & Taylor (1984) considered fields of the form (3.1) in which

$$a(r) = (1+\sigma)(1+\sigma^{-1})^{\sigma} B_0 (r/r_0)^2 [1-(r/r_0)^2]^{\sigma} \quad (r<r_0), \tag{6.1}$$

with $\sigma = 10 r_0 + 1$, and $a = 0$ elsewhere. The function $a(r)$ has a maximum at $r = r_m \equiv (1+\sigma)^{-1/2} r_0$. This field approximates that invoked by Dicke (1982b) to explain the Princeton solar oblateness measurements (Dicke 1976, 1979, 1982b; see also Dicke, Kuhn & Libbrecht 1985, 1987; Kuhn, Libbrecht & Dicke 1985). Fig. 2 shows quantities which the asymptotics suggest are important. [Except for an angle-dependent factor, p_m is the magnetic pressure. More precisely, $p_m = a^2/(2\mu_0)$.] These are displayed for $k=2$, $r_0 = 0.7R$ and $B_0 = 10^7$ G. For this field, Figs 3–5 show $I_2^{\text{mag}}/\omega_0$, $I_4^{\text{mag}}/\omega_0$ (Figs 4 and 5) and the separate contributions to $I_2^{\text{mag}}/\omega_0$ from the direct effect, the h-distortion and the other distortion terms (Fig. 3). Asymptotic estimates are also shown. The results are plotted against the position of the lower turning point, as the divergence at $r = r_t$ of the geometric factor $(1 - L^2 c^2/\omega^2 r^2)^{-1/2}$ in the integrands of the asymptotic estimates implies that the splitting coefficients are particularly sensitive to conditions near $r = r_t$. Because of the simple form of p_m/p_0, it is easy to see the similarity between this and the direct contribution to $I_2^{\text{mag}}/\omega_0$. The latter does not, of course, drop off at small values of r_t as much as $(p_m/p_0)(r)$ does, because the modes are also sensitive to conditions above the lower turning point. As one expects, the asymptotic analysis provides a better estimate for higher frequency modes, which have shorter local wavelengths. Further results for fields of this form but with different values of r_0 are presented in Table 6.

Figs 6 and 7 show the splitting coefficients for a dipole ($k=1$) poloidal field of the form (equation 3.2). The flux function $b(r)$ was rather arbitrarily chosen such that $k(k+1)b(r)/r$ is equal to the function $a(r)$ given by equation (6.1) with $r_0 = 0.7R$ and $B_0 = 10^7$ G (and $b(r) = 0$ for $r > r_0$). Once again the asymptotic estimates approximate the numerical results quite well. These two examples are sufficient to show that a magnetic field can produce frequency splitting which varies with l in a rather complicated way. Moreover, different fields can produce quite varied behaviour of the splitting coefficients. It seems likely, however, that the effect of the distortion above the magnetic region is small, so that the presence of a buried magnetic field will affect appreciably only those modes whose lower turning points are located at least as deeply as the field.

Fields concentrated near the base of the convection zone

The fields considered so far vary on sufficiently large length scales for the asymptotic estimates to work well for five-minute modes. We have also considered some rather narrowly confined toroidal fields at the base of the convection zone, given by equation (3.1) with

$$a(r) = B_0 [1-(r-r_0)^2/d^2]^2 \quad |r-r_0| < d \tag{6.2}$$

and $a(r)$ zero elsewhere, for example with $k=2$, $r_0 = 0.7R$, $d = 0.05R$ and $B_0 = 10^7$ G. The width of the layer is similar to the local wavelength of a five-minute p mode at the base of the convection zone. For such a field, or one even more tightly confined, the frequency perturbation depends on the spatial phase of the oscillation (Vorontsov 1988; Gough & Thompson 1988a). For low-degree five-minute modes whose lower turning points lie well beneath the base of the convection zone $r = r_c$, the oscillation phase at $r = r_c$ is insensitive to the value of l and depends primarily on ω. The displacement eigenfunction of the wave is essentially vertical and proportional to $r^{-1}(\rho c)^{-1/2} \sin(\omega\tau + \Phi)$, where Φ (which is distinct from the Φ in Section 3) is a phase

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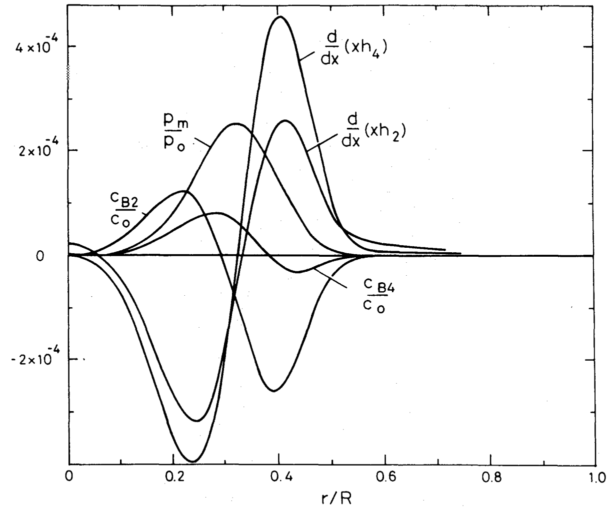

Figure 2. The functions $d(xh_\lambda)/dx$, $c_{B\lambda}/c_0$ for $\lambda=2$, 4 and p_m/p_0, for a toroidal field given by equations (3.1) and (6.1), as described in the text, with $B=10^7$ G, $k=2$ and $r_0=0.7R$.

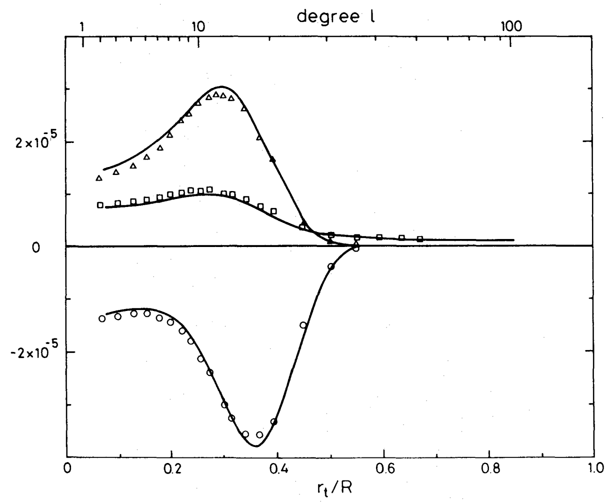

Figure 3. For the same field as in Fig. 2: contributions to the coefficients $I_2^{\rm mag}/\omega_0$ from the direct effect (△), the distortion terms involving h_2 (□) and the remaining distortion terms (○), for modes with frequencies closest to 4 mHz. Asymptotic estimates of these quantities are shown with solid lines. The results are plotted against lower turning point r_t and against degree l.

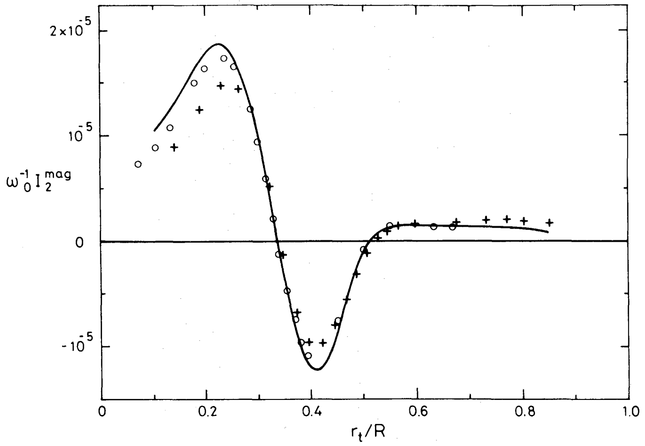

Figure 4. For the same field as in Fig. 2: coefficients $I_2^{\rm mag}/\omega_0$ for modes with frequencies closest to 2 mHz (+) and 4 mHz (○). Asymptotic estimates are shown with a solid line.

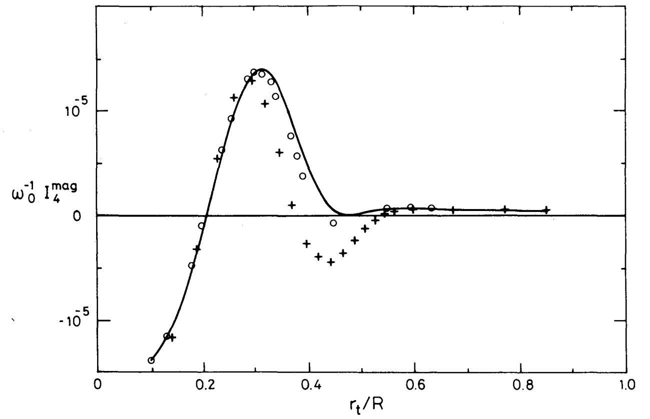

Figure 5. For the same field as in Fig. 2: coefficients $I_4^{\rm mag}/\omega_0$ for modes with frequencies closest to 2 mHz (+) and 4 mHz (○). Asymptotic estimates are shown with a solid line.

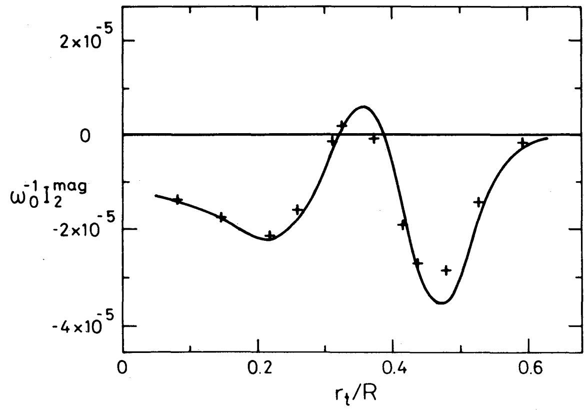

Figure 6. For a poloidal field given by equations (3.2) and (6.1), as described in the text, with $B_0=10^7$ G, $k=2$ and $r_0=0.7R$: coefficients $I_2^{\rm mag}/\omega_0$ for modes with frequencies closest to 3.5 mHz. Asymptotic estimates are shown with a solid line.

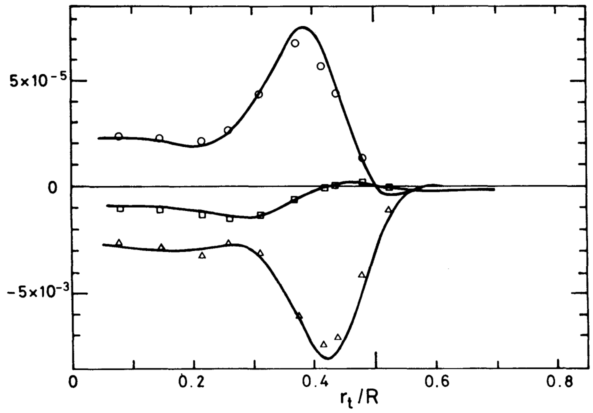

Figure 7. For the same field as in Fig. 6: contributions to the coefficients $I_2^{\rm mag}$ from the direct effect (△), the distortion terms involving h_2 (□) and the remaining distortion terms (○), for modes with frequencies closest to 3.5 mHz. Asymptotic estimates of these quantities are shown with solid lines.

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Table 6. Computed values of $h_\lambda(R)$ and coefficients I_λ^{mag} for various $l=2$ modes. The fields are given by equations (3.1) and (6.1), as described in the text, with $B_0=10^7$ G, $k=2$ and the values of r_0 indicated. The fields have maximum strength $r=r_m$. The values of the coefficients $I_\lambda^{\text{mag}}/2\pi$ are quoted in nanohertz. Because the effect of the spherically symmetric distortion has not been calculated for this field (for reasons discussed in the text), only the *direct* magnetic contribution is included for the case $\lambda=0$. We have designated this contribution to the splitting coefficient I_0^{dir}.

	n	ν_{nl} (mHz)	$10^6 I_0^{dir}/\omega_0$	$10^6 I_2^{mag}/\omega_0$	$10^6 I_4^{mag}/\omega_0$	$10^7 h_2$	$10^7 h_4$
$r_m = 0.134$	16	2.47	0.826	1.94	0.88	-7.19	-0.082
$(r_0 = 0.3)$	18	2.74	1.06	1.43	0.22		
	20	3.01	1.27	1.20	-0.42		
	22	3.29	1.42	1.39	-0.87		
	24	3.56	1.48	1.66	-1.13		
$r_m = 0.189$	16	2.47	5.60	5.09	-2.98	-29.1	-0.774
$(r_0 = 0.5)$	18	2.74	5.61	5.15	-3.58		
	20	3.01	5.61	5.13	-4.02		
	22	3.29	5.65	5.08	-4.36		
	24	3.56	5.69	5.02	-4.64		
$r_m = 0.233$	16	2.47	18.4	8.16	-13.0	-75.7	-3.45
$(r_0 = 0.7)$	18	2.74	18.5	8.02	-13.7		
	20	3.01	18.5	7.88	-14.2		
	22	3.29	18.7	7.73	-14.7		
	24	3.56	18.8	7.58	-15.1		

determined by conditions near the surface of the star and τ is acoustic depth:

$$\tau(r)=\int_r^R c^{-1}\,dr. \tag{6.3}$$

Thus the phase at the base of the convection zone, and hence the frequency splitting coefficients, are oscillatory functions of ω. This is clearly seen in Figs 8–10. As the mode frequency increases, the vertical wavenumber increases, so that greater cancellation takes place in the averaging over the magnetic region. Hence the frequency perturbation is less sensitive to the phase of the mode; the amplitude of the oscillation visible in the figures diminishes with increasing frequency (see also Gough & Thompson 1988a). The dominant term in the direct effect is proportional to $\sin^2(\omega\tau+\Phi)$, appropriately averaged over the magnetic layer (Vorontsov 1988). The dominant distortion term comes from $\langle\boldsymbol{\xi}^*\cdot\rho_B\omega_0^2\boldsymbol{\xi}\rangle$ and at least in the $\lambda=2$ case, considered here, this dominates the direct effect, being greater by a factor of order $\omega_0 c/g$ where g is the magnitude of the gravita-

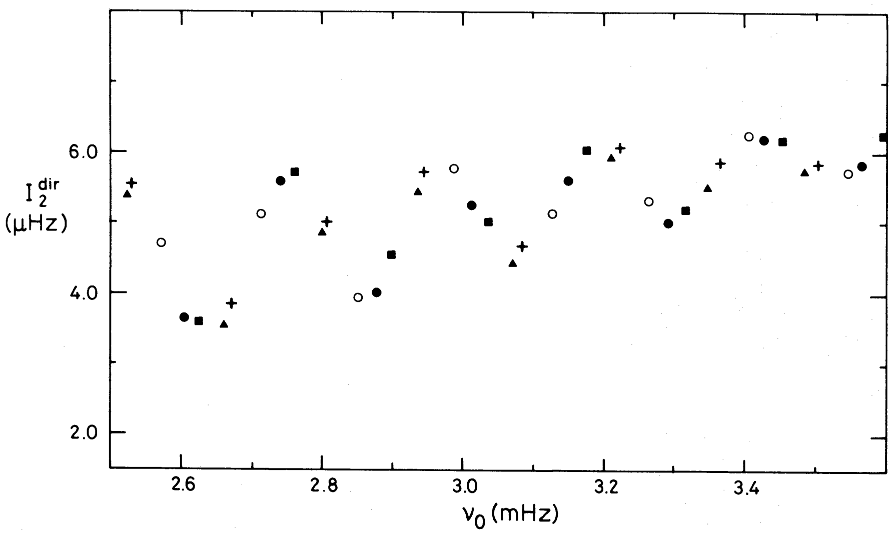

Figure 8. For a toroidal field given by equations (3.1) and (6.2), as described in the text, with $B_0=10^7$ G, $r_0=0.7R$ and $d=0.05R$ and $k=2$: coefficients I_2^{dir} plotted against cyclic frequency for $l=2$ (●), 3 (▲), 4 (○), 5 (■) and 6 (+).

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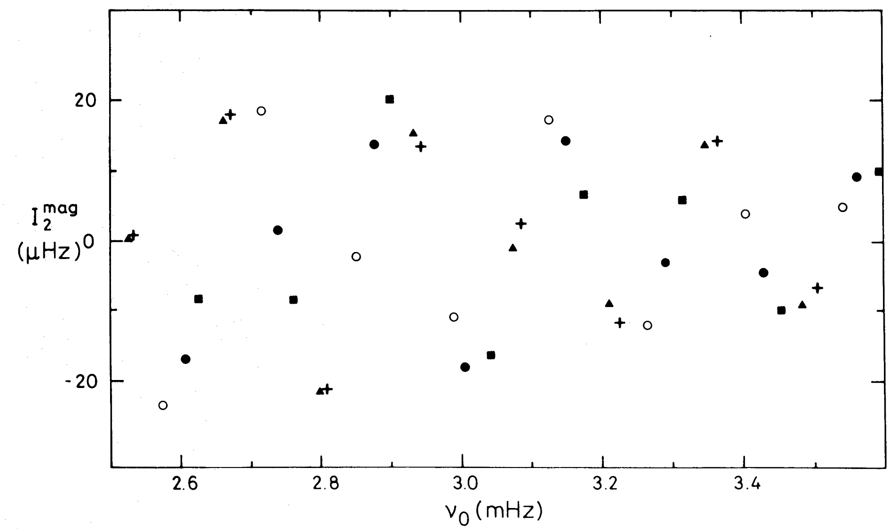

Figure 9. As for Fig. 8, but showing the coefficients I_2^{mag}.

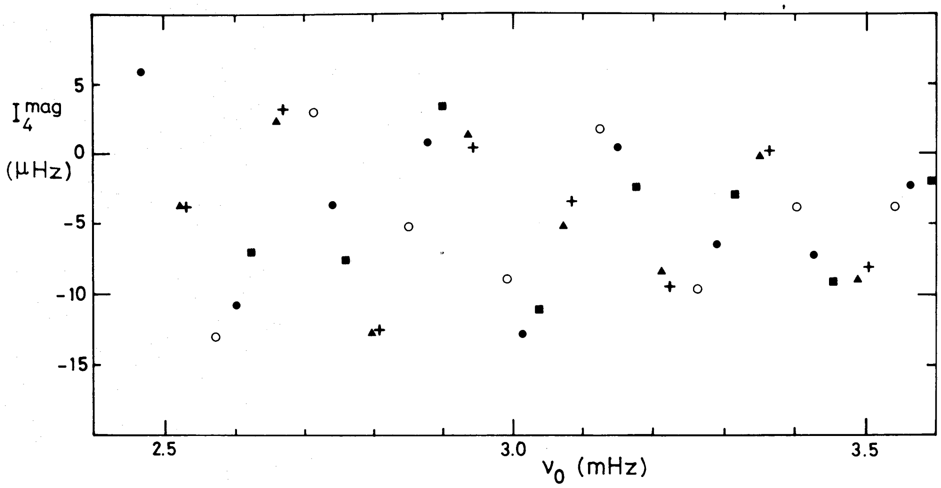

Figure 10. As for Fig. 8, but showing the coefficients I_4^{mag}.

tional acceleration, evaluated in the thin layer. This distortional term is proportional to $\sin(\omega\tau+\Phi)\cos(\omega\tau+\Phi)$, averaged over the layer, the cosine arising because ρ_B contains a derivative of $a(r)$: to obtain an estimate of the splitting coefficients for a confined field it is convenient to transfer this derivative to $\boldsymbol{\xi}$, by integration by parts (see the discussion below of kernels). This term averages to zero in the limit $\omega \to \infty$ (when any continuous field becomes slowly varying) and so it contributes nothing to the short-wavelength asymptotic estimates of Section 4. It may now be seen that the cycle length of the oscillation in Figs 8–10 should be $[2\tau(r_0)]^{-1}$, taking Φ to be constant. Thus, the cycle length gives the acoustic depth (and hence by comparison with solar models the actual depth) at which the perturbing magnetic layer is situated. Actually, Φ varies a little with ω, so the cycle length is modified, but this can in principle be taken into account. Of course, any highly localized perturbation will have a similar oscillatory signature.

Figs 11 and 12 show I_2^{mag} and I_4^{mag} together with their (phase-independent) asymptotic estimates, for $l=30$ and $l=40$. Of course, the asymptotic formulae approximate the splitting coefficients more poorly than was the case for less localized fields, because the basic assumption of a slowly varying perturbation is not valid in the present case. Nevertheless, it is instructive to consider why the asymptotic estimates deviate from the numerical results. When the lower turning point lies beneath the magnetic layer, the numerical results oscillate about the estimates because of the spatial phase dependence, as discussed above. When the turning point is in the vicinity of the magnetic layer, the deviation between the asymptotic estimates and the numerical results is even more pronounced. This is because of the nature of the sensitivity of a mode near its lower turning point. While the integrands in the asymptotic formulae (4.7) and (4.14) are infinite (albeit integrable) at $r=r_{\text{t}}$, the actual eigenfunctions are finite there. When the field is slowly varying, the two methods give similar results. However, if the magnetic field and its derivatives

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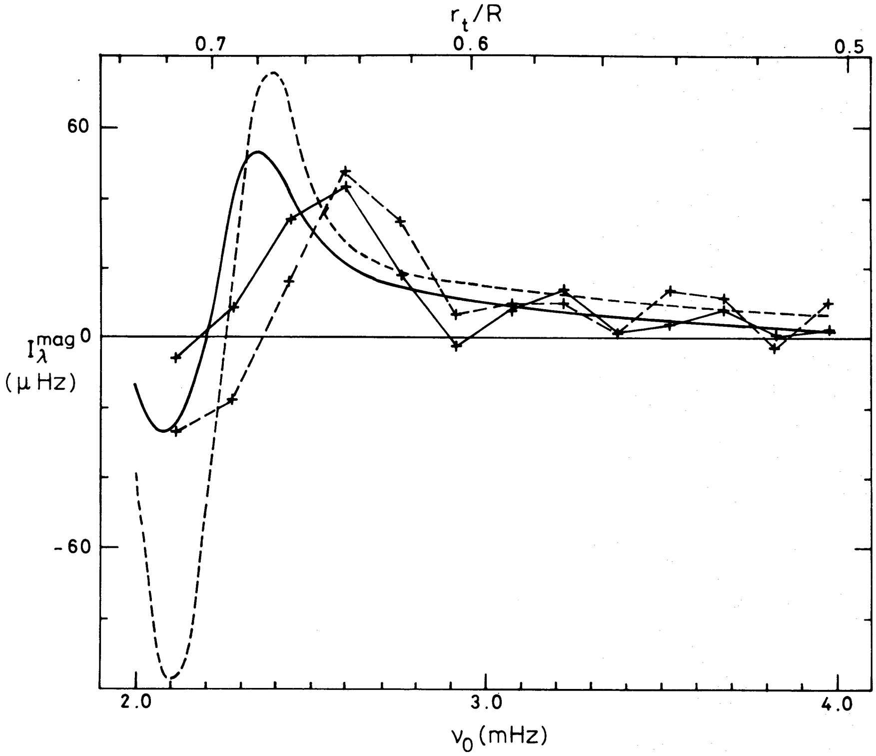

Figure 11. For the same field as in Fig. 8: coefficients I_2^{mag} (joined by dashed straight lines) and I_4^{mag} (continuous lines) for modes with degree $l=30$. The dashed and continuous curves represent phase-independent asymptotic estimates for I_2^{mag} and I_4^{mag}, respectively.

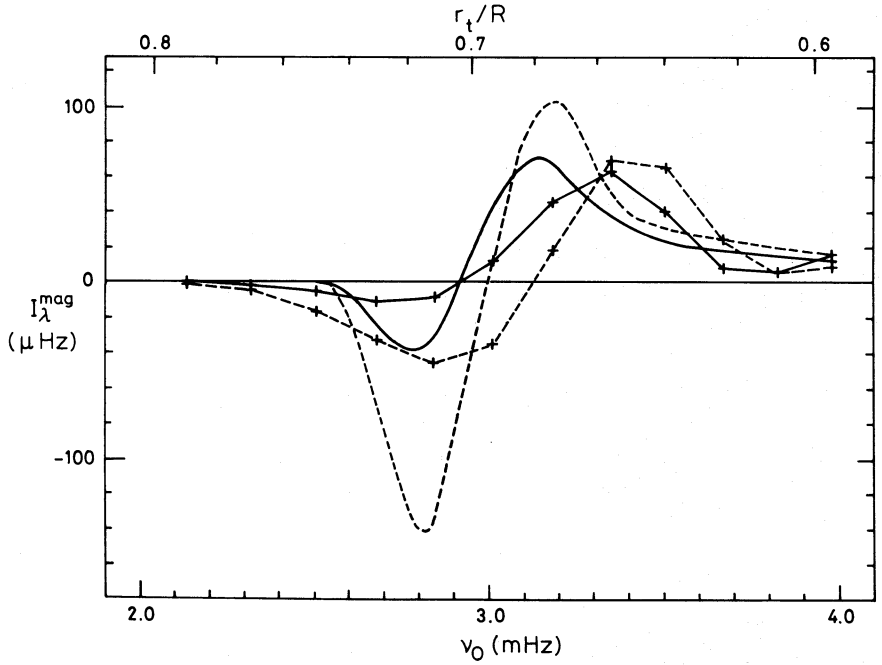

Figure 12. As for Fig. 11, but for modes with degree $l=40$.

vary sufficiently rapidly, the asymptotic estimates for modes with lower turning point in the vicinity of the field also vary rapidly, while the numerical results in this case tend to average out the extreme variations of the asymptotic estimates. This is clearly visible in Figs 11 and 12 for modes with lower turning point around $r=0.7R$. Note also that the peak in the numerical results is to the right of the peak in the asymptotic estimates. This is because the modes are actually most sensitive not at the radius where $L^2c^2/\omega^2r^2=1$ but at a point slightly above it. Near the turning point the radial displacement may be represented by an Airy

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function (e.g. Olver 1974; Unno *et al.* 1979) and this has its first maximum not at zero but at a point inside the oscillatory domain. Similarly, the direct perturbation (Fig. 3) peaks to the left of the maximum of p_m/p_0. Finally, we remark that even when r_t is above the magnetic region, there is some frequency perturbation. This arises in part from the distortion of the outer layers, but this effect is negligible compared with the perturbation when the turning point is lower. As is evident in Fig. 12, as r_t approaches the magnetic region from above, the numerical results show the effect of the magnetic field before the asymptotics do. This is because the modes actually have small but non-zero amplitude in the evanescent domain beneath the lower turning point, and hence some sensitivity to conditions in this region, whereas the simple asymptotics give the region no weight at all.

It should be mentioned that, provided the field vanishes at the surface of the star, the direct magnetic contribution to the splitting may be written

$$\int M_{ij} B_i B_j \, dV, \tag{6.4}$$

where M_{ij} is a symmetric kernel which is a function of the equilibrium state and the oscillation eigenfunction, and the integration is over the volume of the star (Dziembowski & Goode 1984). In the case of a toroidal field of the form (3.1), each whole magnetic splitting coefficient I_λ^{mag} may similarly be written in terms of a kernel:

$$\frac{I_\lambda^{\mathrm{mag}}}{\omega_0} = \int_0^R K_\lambda^{(n,l)}(r)(p_m/p_0)c^{-1}\,dr \bigg/ \int_0^R c^{-1}\,dr. \tag{6.5}$$

The kernels $K_\lambda^{(n,l)}$ depend also, of course, on k. The factor c^{-1} in the numerator is written explicitly because the acoustic radius is the natural independent variables with which to describe acoustic modes. The denominator is present to make $K_\lambda^{(n,l)}$ dimensionless. Some kernels are shown in Fig. 13, with r_t indicated by an arrow. They differ from the corresponding smoothed kernels published by Dziembowski & Goode (1988) because they contain the oscillatory structure of the eigenfunction. Throughout most of the propagating region, their envelope shapes are determined largely by the factor $\omega c/g$ in the dominant distortional term, as discussed above. For large values of n/l this term is approximately

$$-(\lambda+\tfrac{1}{2})\int_0^\pi \left[\left(\frac{dP_k}{d\theta}\right)^2 + k(k+1)P_k^2\right] P_\lambda \sin\theta\, d\theta \cdot \frac{\omega_0 c}{\gamma g}\left(1-\frac{L^2c^2}{\omega^2 r^2}\right)^{-1/2} \sin 2(\omega_0\tau+\Phi). \tag{6.6}$$

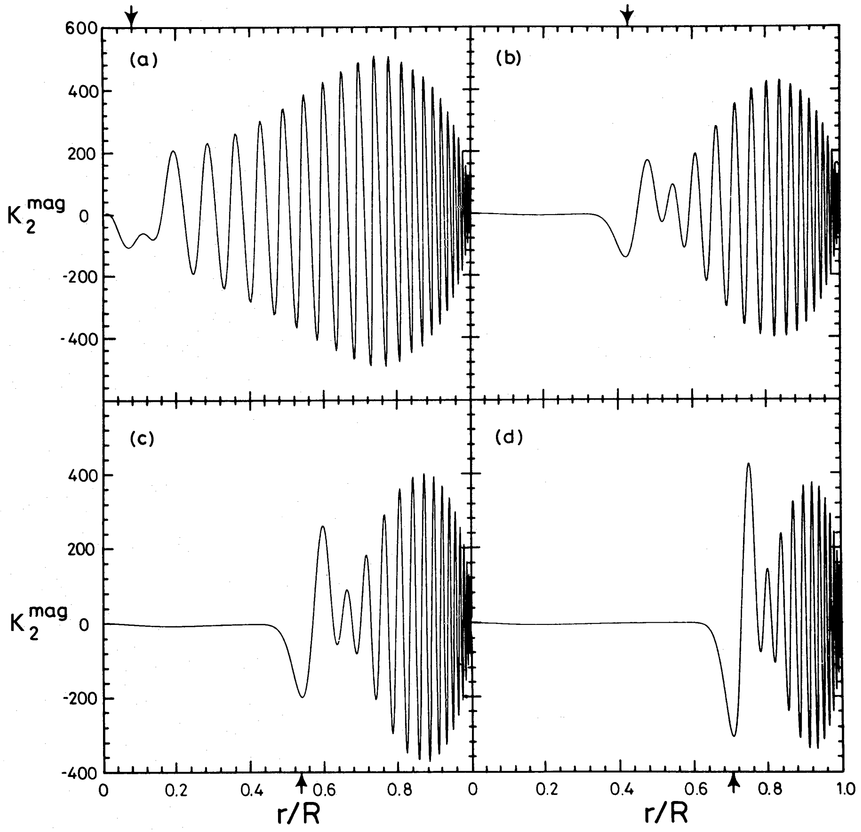

Figure 13. Dimensionless kernels $K_2^{\mathrm{mag}} = (GM/R^3)^{-1/2}\omega_0 K_2^{(n,l)}$ for I_2^{mag}, as defined by equation (6.5), for a toroidal field of the form (3.1) with $k=2$. The examples shown are for: (a) $l=2$, $n=24$, $\nu=3.57$ mHz; (b) $l=20$, $n=18$, $\nu=3.49$ mHz; (c) $l=30$, $n=16$, $\nu=3.53$ mHz; (d) $l=50$, $n=13$, $\nu=3.59$ mHz. The position of the lower turning point r_t is indicated by an arrow.

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The leading direct contribution to the kernel comes from the two r-derivatives of ξ in the term $(-1/\mu_0)(\nabla\times\mathbf{B}')\times\mathbf{B}$ in $\mathscr{B}\boldsymbol{\xi}$ (equation 2.5) and can easily be shown to be

$$(\lambda+\tfrac{1}{2})\int_0^\pi \left(\frac{dP_k}{d\theta}\right)^2 P_\lambda \sin\theta\, d\theta \cdot \frac{1}{2\gamma}\left(1-\frac{L^2c^2}{\omega^2r^2}\right)^{-1/2}\sin^2(\omega_0\tau+\Phi). \tag{6.7}$$

It is clear from Fig. 13 that the magnitude of the kernels is much greater than their mean value. This is evident also from the kernels published recently by Dziembowski (1988). Thus a smoothly varying field tends to produce a much smaller frequency perturbation than a tightly confined field does, as in the former case the oscillations in the kernel are averaged out to a great extent whereas in the latter case they are not. This is the reason that our frequency perturbations for a tightly confined field (Figs 9–12) greatly exceed those obtained for smoother field configurations (Fig. 3–7; see also Dziembowski & Goode 1988).

7 OVERALL FREQUENCY SHIFT DUE TO A MAGNETIC FIELD

So far we have considered only the splitting of (degenerate) frequency multiplets of modes of like order and degree. In general, rotation and a magnetic field also shift the average frequency of each multiplet. Provided the weighting given to each mode is independent of m, the shift of the multiplet average frequency is just I_0, because

$$\sum_{m=-l}^{l} Q_{\lambda lm}=0 \qquad (\lambda\neq 0). \tag{7.1}$$

The evaluation of the absolute frequency of any particular mode requires a knowledge of the coefficients I_λ for all λ, including $\lambda=0$.

The contributions to I_0, as to any I_λ, may be considered to derive from one of two origins: the distortion of the equilibrium state and the perturbed Lorentz force (or centrifugal force in the case of rotation). The latter, direct contributions arise in a straightforward manner from the terms involving $\mathscr{N}_0$ and $\mathscr{B}_0$ in equation (2.34), and the magnetic term is explicitly included in the asymptotic discussion of Section 4. In the case of a toroidal field, a low-degree asymptotic estimate of the direct effect was given by Gough & Taylor (1984), and equation (4.14) reduces to their value if L/ω is formally set to zero. A similar low-degree estimate was obtained by Roberts & Campbell (1986).

Some care is required in specifying what is meant by the distortional contribution to I_0, because in general the inclusion of rotation or a magnetic field will change a model's radius and luminosity. We shall insist here that the model is calibrated to have the same radius and luminosity as the non-rotating, non-magnetic model. The effect on the frequencies of any variation in the radius and luminosity is therefore to be considered as distinct from I_0.

7.1 Calculating the spherically symmetric component of the distortion

As discussed above, the spherically symmetric component of a perturbing force in general changes the radius and luminosity of the star. For a standard non-rotating, non-magnetic solar model, the observed radius R and luminosity L_R are reproduced by adjusting the model parameters, in particular the mixing-length parameter α [which is unrelated to the constant α in equation (4.1)] and the initial hydrogen abundance X_0. Our approach has been to compute the spherically symmetric distortion to a calibrated model of the present Sun, by adjusting α and scaling the present hydrogen abundance $X(r)$ by a constant factor to obtain the correct radius and luminosity. A force in the energy-generating core will generally change the present hydrogen abundance there in a complicated position-dependent way. To calculate this would require assumptions about the magnetic field or rotation at all previous ages [*cf.* Chitre, Ezer & Stothers (1973) concerning a magnetic perturbation, and Demarque, Mengel & Sweigart (1973) on a rotational perturbation to the solar structure]. For this reason we restrict attention in Section 7.2 to a magnetic field at the base of the convection zone, for which our treatment of the perturbed hydrogen profile is probably adequate.

To compute the perturbation we regarded the Sun as being divided into two zones, an inner (radiative) region $r<r_*$ including the energy generating core, and an outer region $r>r_*$ including the convection zone. In the inner region we perturbed the equations of stellar structure in the presence of a small spherically symmetric force $F(r)\mathbf{e}_r$ to obtain equations for the relative Lagrangian perturbations

$$\frac{\delta r}{r},\ \frac{\delta p}{p},\ \frac{\delta T}{T},\ \frac{\delta L_r}{L_r}$$

to radius, pressure, temperature and luminosity respectively, together with equations for the perturbations to density ρ, opacity

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$\tilde{\kappa}$ and energy generation rate $\tilde{\varepsilon}$:

$$\frac{1}{M}\frac{d}{dq}\left(\frac{\delta p}{p}\right)=\frac{GMq}{4\pi r^4 p}\left(4\frac{\delta r}{r}+\frac{\delta p}{p}\right)+\frac{F}{4\pi r^2 \rho p}, \tag{7.2}$$

$$\frac{1}{M}\frac{d}{dq}\left(\frac{\delta r}{r}\right)=\frac{-1}{4\pi r^3 \rho}\left(3\frac{\delta r}{r}+\frac{\delta \rho}{\rho}\right), \tag{7.3}$$

$$\frac{1}{M}\frac{d}{dq}\left(\frac{\delta T}{T}\right)=\frac{3\tilde{\kappa}L_r}{64\pi^2 acr^4 T^4}\left(4\frac{\delta r}{r}+4\frac{\delta T}{T}-\frac{\delta\tilde{\kappa}}{\tilde{\kappa}}-\frac{\delta L_r}{L_r}\right), \tag{7.4}$$

$$\frac{1}{M}\frac{d}{dq}\left(\frac{\delta L_r}{L_r}\right)=\frac{\tilde{\varepsilon}}{L_r}\left(\frac{\delta\tilde{\varepsilon}}{\tilde{\varepsilon}}-\frac{\delta L_r}{L_r}\right), \tag{7.5}$$

where the independent variable q is defined as in equation (3.28), with the subscripts omitted, and three equations of the form

$$\frac{\delta\theta}{\theta}=\left(\frac{\partial \ln \theta}{\partial \ln p}\right)_{X,T}\frac{\delta p}{p}+\left(\frac{\partial \ln \theta}{\partial \ln T}\right)_{p,X}\frac{\delta T}{T}+\left(\frac{\partial \ln \theta}{\partial \ln X}\right)_{T,p}\frac{\delta X}{X}, \tag{7.6}$$

where θ is each of ρ, $\tilde{\kappa}$ and $\tilde{\varepsilon}$. The constant $(\delta X/X)$ is an eigenvalue.

At $m=0$, regularity conditions

$$\frac{\delta r}{r}=0, \qquad \frac{\delta L_r}{L_r}=0 \tag{7.7}$$

were imposed. At the matching point $r=r_*$, $(\delta L_r/L_r)$ was set to zero, and three other conditions were applied by demanding continuity of p, T and m with the solution in the outer region $r>r_*$. The solution in the outer region was obtained by integrating the equations of stellar structure inwards from the surface, using $q=1$, $L_r=L_R$ and $r=R$ as photospheric conditions, the atmosphere above the photosphere being assumed to be in hydrostatic equilibrium with its temperature being related to optical depth according to the Harvard–Smithsonian Reference Atmosphere (Gingerich *et al.* 1971). Once again, a perturbing force $F(r)\mathbf{e}_r$ was added to the hydrostatic equation beneath the photosphere. The matching between the inner and outer regions, and the determination of the eigenvalues $(\delta X/X)$ and α, were achieved by Newton–Raphson iteration.

7.2 Numerical results

We have considered the specific example of a force F derived from the quadrupole toroidal field (3.1) with $a(r)$ given by equation (6.2) and with $k=2$, $B_0=10^7$G, $r_0=0.7R$, $d=0.05R$. The equilibrium model for $r<r_*$ was Model A of Christensen-Dalsgaard *et al.* (1979). Approximate partial derivatives of $\tilde{\varepsilon}$ were obtained from the simple formula $\tilde{\varepsilon}=\tilde{\varepsilon}_0\rho X^2 T^{-2/3}\exp[-(T_0/T)^{1/3}]$ with $T_0=3.86\times10^{10}$ K for the p-p reaction (e.g. Clayton 1968). The radius of the boundary between the inner and outer regions was chosen to be $r_*=0.6R$; thus F differed from zero only in the outer region. Partial derivatives at constant r_* were obtained by varying α and X by ±1 per cent. The values thus obtained were

$$\frac{\partial \ln p}{\partial \ln \alpha}=1.14 \qquad \frac{\partial \ln T}{\partial \ln \alpha}=0.192 \qquad \frac{\partial \ln m}{\partial \ln \alpha}=-0.0927,$$

$$\frac{\partial \ln p}{\partial \ln X}=-2.87 \qquad \frac{\partial \ln T}{\partial \ln X}=-0.543 \qquad \frac{\partial \ln m}{\partial \ln X}=0.222,$$

and the changes to $\ln p$, $\ln T$ and $\ln m$ at $r=r_*$, when the field was inserted at standard X, α were

$\Delta \ln p=2.93\times10^{-3}$, $\Delta \ln T=4.79\times10^{-4}$, $\Delta \ln m=-1.04\times10^{-4}$.

The values of $\delta X/X$ and $\delta\alpha/\alpha$ required to obtain a perturbed solar model calibrated to the correct mass, luminosity and radius were

$$\frac{\delta X}{X}=1.20\times10^{-4} \qquad \frac{\delta\alpha}{\alpha}=-1.82\times10^{-3}.$$

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Fig. 14 shows the distortional and direct contributions to $I_0^{\rm mag}$ for (a) $l=2$ modes with 2.5 mHz $<\nu<$ 3.6 mHz and (b) modes of varying degree with frequencies near 3.5 mHz. It is readily apparent that the distortional effect is negligible compared with the direct effect on the modes, in contrast with the non-spherically symmetric perturbation (Figs 8–10) where the distortional contribution dominates the direct effect. It is interesting to note also that in the matrix element B_{mm}, which is a sum of $I_\lambda^{\rm mag} Q_{\lambda lm}$

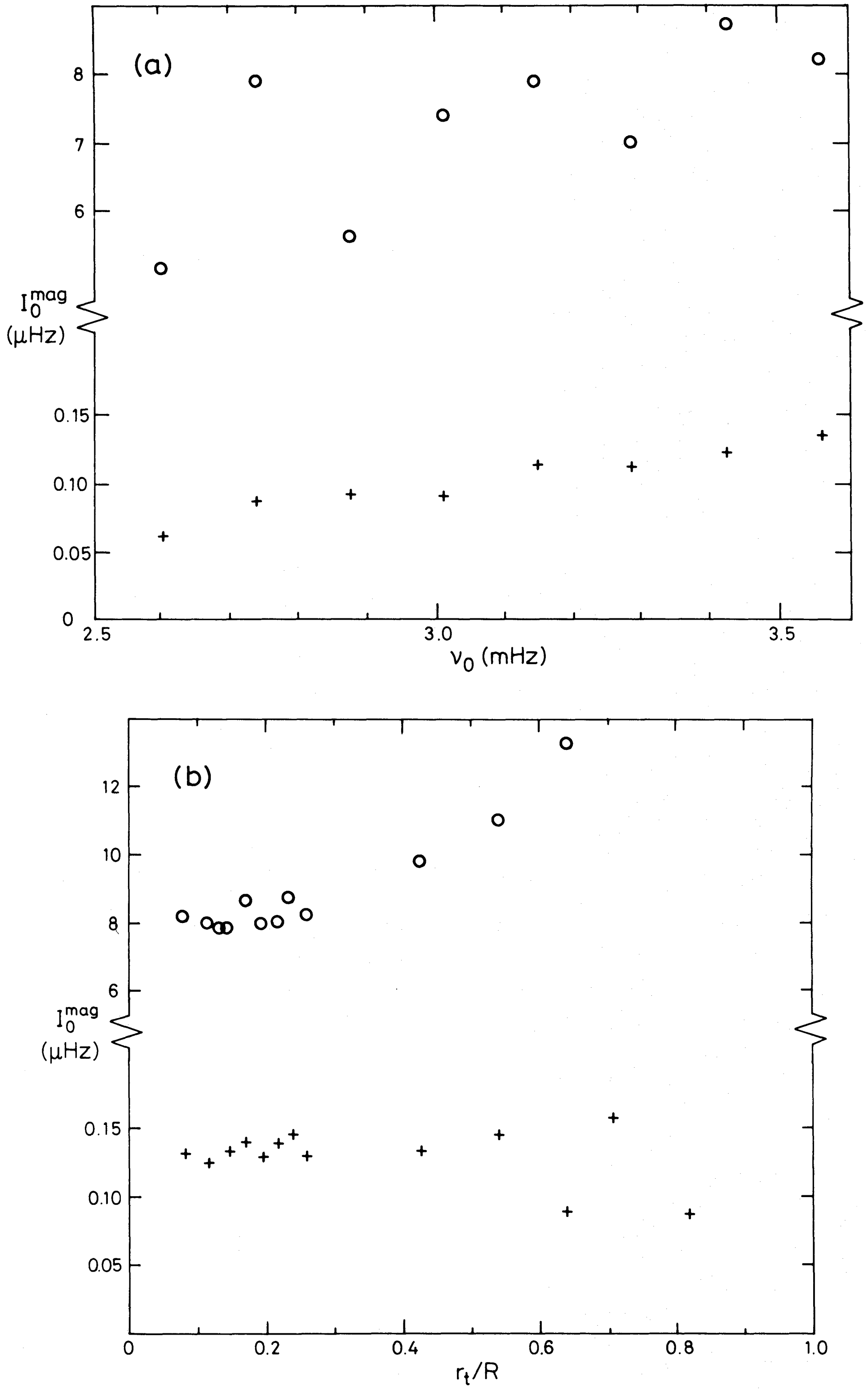

Figure 14. For a toroidal field given by equations (3.1) and (6.2), as described in Section 6, with $B_0=10^7$G, $r_0=0.7R$ and $d=0.05R$ and $k=2$: (a) Direct (○) and distortional (+) contributions to $I_0^{\rm mag}$ for various $l=2$ modes, (b) Direct (○) and distortional (+) contributions to $I_0^{\rm mag}$ for various modes with frequencies closest to 3.5 mHz.

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over all λ, the non-spherically symmetric distortion has an effect of greater magnitude than the spherically symmetric distortion. This is largely because the effect of the latter is diminished by the recalibration of the star.

An alternative method of calculation confirms the magnitude of the effect of the spherically symmetric distortion. In principle it is a less reliable method because it involves finding the relatively small difference between two quantities, namely the frequencies of two models. A term $F(r)\mathbf{e}_r$ was added to the equation of hydrostatic support in Christensen-Dalsgaard's (1982) stellar structure programme, and a 'magnetic model' of the present Sun was then computed (as a static calculation at the present solar age), by adjusting the standard hydrogen profile (by a uniform scaling $\delta X/X$) and the mixing length parameter α to reproduce the observed solar radius and luminosity. The required values of $\delta X/X$ and $\delta\alpha/\alpha$ were

$$\frac{\delta X}{X}=1.25\times10^{-4}\quad \frac{\delta\alpha}{\alpha}=-1.85\times10^{-3},$$

in good agreement with the previous method. Frequencies for the magnetic model and for a standard model were computed and their difference taken. A typical value of $\delta\omega/\omega$ for 3-mHz modes was thus found to be 3×10^{-5}, in reasonable accord with the results due to distortion shown in Fig. 14(a).

It may be noticed that the results for deeply penetrating modes in Fig. 14(b) appear to be irregular. With better sampling of points these results would exhibit an oscillatory behaviour, as in the non-spherically symmetric case. In particular, the direct perturbation would be seen to be given approximately by equation (6.7) in conjunction with equation (6.5).

8 DISCUSSION

Observed frequency splittings have been quoted in terms of coefficients a_i:

$$\nu_{nlm}-\nu_{nl0}=L\sum_i a_i(n,l)P_i(-m/L), \tag{8.1}$$

where $\nu\equiv\omega/2\pi$ is the cyclic frequency. [The precise definition of the coefficients differs from one author to another; the representation (8.1) is typical.] Except for small values of l, we see from equation (3.12) that the even coefficients a_{2k} are related approximately to our coefficients I_λ by

$$La_{2k}=\frac{(-1)^k(2k)!}{2^{2k}(k!)^2}I_{2k}. \tag{8.2}$$

Of course, equation (8.1) is not the only possible representation of the data. For the component of the splitting that is even in m, it would be more natural not to take out a factor L. In addition, because the odd component comes from the linear effects of rotation (assuming effects smaller than second order to be negligible) it can be seen from the asymptotic equation (4.9) that it would be more natural to take a factor m, not L, out of this component. Then, in particular, if the rotation rate is expressed as

$$\Omega=\Omega_0(r)+\Omega_2(r)P_2(\cos\theta)+\Omega_4(r)P_4(\cos\theta)+\dots \tag{8.3}$$

and the splitting that is an odd function of m is represented as

$$m\sum_{k\,\mathrm{even}} c_k(n,l)P_k(m/L), \tag{8.4}$$

an application of equation (3.12) gives that each coefficient c_k depends principally only on Ω_k (Durney, Hill & Goode 1988).

How does the presence of more than one asymmetry affect the interpretation of the splitting coefficients? The interpretation is in principle a relatively simple matter when slow rotation (about a unique axis) is the only symmetry-breaking agent. As already discussed, the odd coefficients a_i then arise from linear effects, and the even coefficients come from second-order terms (principally centrifugal distortion for moderate values of l, according to the results of Section 5). When a second symmetry-breaking perturbation is introduced, such as a magnetic field or a latitudinally dependent component to the structure, this picture is not greatly changed provided the star still has an axis of symmetry and its structure is independent of time. A co-rotating frame $\mathscr{S}$ may be introduced, as in Section 2, but this is not necessary because even in an inertial frame the star's structure is independent of time. The new perturbation contributes only to the even coefficients and the frequency perturbations arising from the two symmetry-breaking agents are additive. In both these cases the azimuthal order m is still a well-defined quantum number.

However, even if the second perturbation is axisymmetric, if its axis does not coincide with the axis of rotation the interpretation of the coefficients a_i is more difficult. This is because m is no longer a well-defined quantum number. It is a

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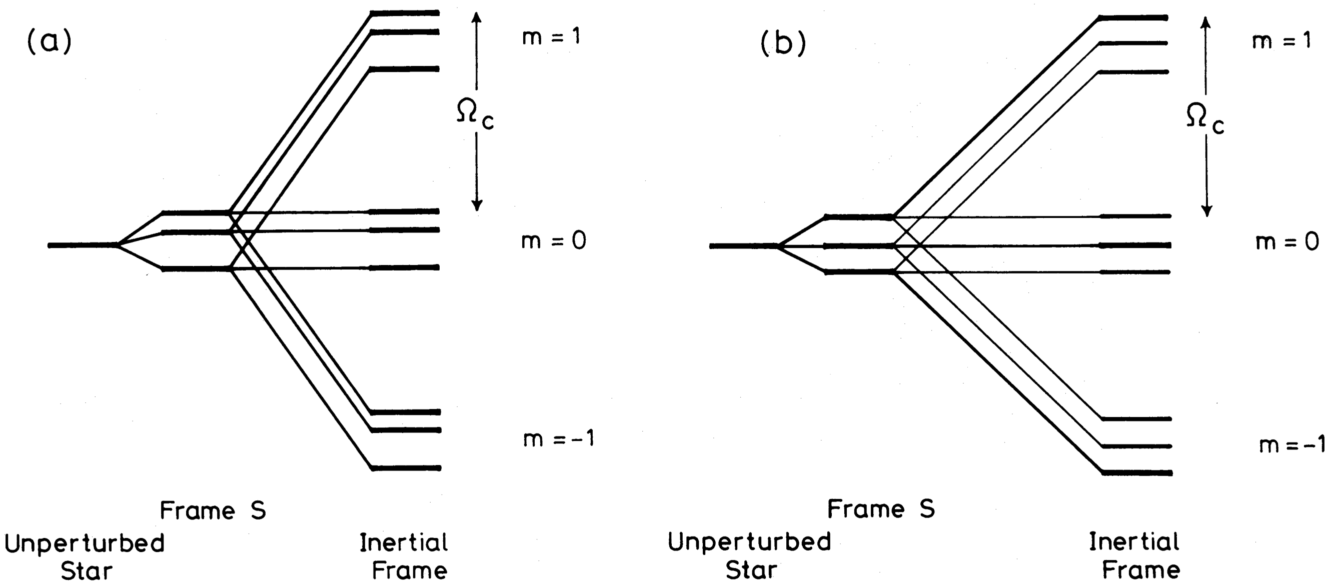

Figure 15. Schematic diagram showing (for the case $l=1$) how the m-degeneracy in the non-rotating, non-magnetic star (left of diagram) is raised by rotation and a magnetic field so that in the corotating frame $\mathscr{S}$ there are $(2l+1)$ distinct eigenfrequencies (centre of diagram). In the general case (a) an observer in an inertial frame sees $(2l+1)^2$ frequencies (right of diagram). However, when rotational effects dominate other asymmetries in frame $\mathscr{S}$, as in case (b), $2l(2l+1)$ of the modes have very small amplitudes (shown faint), so the observer is likely to see only $(2l+1)$ frequencies (provided, of course, the observing technique can detect modes of any geometry).

necessary condition for the existence of normal modes of oscillation that a frame $\mathscr{S}$ exists in which the structure of the star is stationary. It is in this frame that one must solve the matrix eigenvalue problem (2.26). In general there is no longer an axis orientation in which each eigenfunction corresponds to a single value of m. In particular, referred to a polar axis aligned with the rotation axis each zero-order eigenfunction $\boldsymbol{\xi}_0$ is a linear combination of the $(2l+1)$ functions $\boldsymbol{\xi}_{nlm}$ $(m=-l, \ldots, l)$. Viewed from an inertial frame each $\boldsymbol{\xi}_{nlm}$ has its frequency incremented by $m\Omega_c$. Thus there appear to be $(2l+1)^2$ eigenfrequencies corresponding to given n and l when viewed from an inertial frame. This is illustrated for $l=1$ in Fig. 15(a). Now the observer sees in his power spectrum not one $l=1$, $m=1$ mode (at fixed n) but three. The relative amplitudes of the peaks in each triplet will differ, of course, since the amplitudes of the different $\boldsymbol{\xi}_{nlm}$ forming any one $\boldsymbol{\xi}_0$ will not in general be equal, and their relative magnitudes will vary from one $\boldsymbol{\xi}_0$ to another. This is exemplified in the case when rotation dominates other non-spherically symmetric effects (as calculated in frame $\mathscr{S}$). Then each $\boldsymbol{\xi}_0$ is well approximated by a single $\boldsymbol{\xi}_{nlm}$. This case is illustrated in Fig. 15(b). In a sense, each m value still has a triplet of frequencies, but the peaks in the power spectrum which correspond to two of the frequencies have negligible amplitudes. This case corresponds to equation (2.30), and then the quantum number m is practically well defined. The interpretation of the coefficients a_i is as for the axisymmetric case, except that the non-rotational contributions to the even coefficients are weighted by $P_\lambda(\cos\beta)$, in accordance with equation (3.11).

It has been implicitly assumed in the analyses of Brown (1985, 1986), Brown & Morrow (1987), Libbrecht (1986) and Duvall *et al.* (1986) that the appropriate picture of the fine splitting of the solar p-mode spectrum is Fig. 15(b), but it should always be borne in mind that for some modes Fig. 15(a) may be more appropriate. This will be the case if the matrix elements B_{mm} (*cf.* equation 2.27) are roughly speaking at least as big as some of the elements A_{mm}. Taking the second perturbation to be a magnetic field, this gives the rough criterion for Fig. 15(b) to be inapplicable:

$$\omega_0\langle\langle v_A^2/c^2\rangle\rangle \gtrsim [\langle\langle\Omega\rangle\rangle-\Omega_c]\,m \qquad (m\neq 0). \tag{8.5}$$

Here «...» denotes an appropriate mode-dependent integral over the star, as, for example, in equation (2.35), and v_A is the Alfvén speed. If the rotation were uniform, «Ω»/Ω would differ from unity by an amount C, as a result of the Coriolis force (e.g. Ledoux 1951). For solar five-minute modes of low and intermediate degree, C is typically of the order 10^{-2}. Thus the Coriolis contribution is much smaller than the contribution from differential rotation. Taking 40 nHz as a conservative estimate of «Ω» $-\,\Omega_c$ (*cf.* Fig. 1), we obtain about $(40m)$ nHz for the right-hand side of inequality (8.5) (expressed in terms of cyclic frequency). For a core magnetic field of magnitude 10^7 G, as is considered in Section 6, the left-hand side could be as large as 50 nHz for a deeply penetrating mode. Thus it is feasible for a magnetic field to influence significantly the structure of low-degree solar p modes. Current observations would be inadequate to resolve multiplets of like m, such as those illustrated in Fig. 15(a), if the spread were only 50 nHz. However, the long data sets envisaged in the near future will change that state of affairs for sufficiently long-lived modes. The fact that each multiplet has identical frequency spacing would aid their detection.

Of course, a magnetic field is not the only asymmetry that might be present in addition to rotation. Other asymmetries will

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have very similar effects. Latitudinal variations in the solar structure may have measurable effects (Gough & Thompson 1988b; Kuhn 1988a, b). Such effects have also been considered in the case of Ap stars (Dolez, Gough & Vauclair 1988). Whether one can distinguish seismologically between a latitudinally dependent structure and an internal magnetic field is an important question. Perhaps the anisotropic nature of the direct magnetic perturbation will indeed make such a distinction possible, though it seems likely that given a finite set of data with observational errors one could always contrive some non-magnetic internal structure to explain it. Thus one may have to decide which is the most reasonable explanation of the data on physical grounds.

Even without detailed calculation, our interpretation of the splitting coefficients is guided by simple principles. The n and l dependence of the data is determined by the variation with depth of the underlying cause; the m/L dependence by its variation with latitude. Asymptotically one expects each splitting coefficient to vary like

$$S^{-1}(\omega/L)\{f_1(\omega/L)+f_2(\omega)\} \tag{8.6}$$

(*cf.* Gough 1986a; Christensen-Dalsgaard *et al.* 1988), where the function $f_1(\omega/L)$ arises from the variation of the asymmetry with depth and $f_2(\omega)$ comes from the asymmetry near the surface. This is essentially equation (4.4), with the additional possibility of splitting arising from causes near the surface – for example the latitudinal distribution of magnetic fibrils (Bogdan & Zweibel 1985; Zweibel & Bogdan 1986; Gough & Thompson 1988b). Thus from such a decomposition as (8.6) one may hope to separate the effects of asymmetries confined to the surface regions from those at greater depths. With the present averaged data, in which the n dependence of the coefficients a_i is suppressed by averaging over frequency, this decomposition is not possible. However, it is still possible to say something useful about the depth dependence. If the asymmetry were confined to great depths, it would be possible to perceive a decrease in La_i with increasing degree l. If, on the other hand, the perturbation were confined to a layer much more shallow than the penetration depth $R-r_t$ of the modes, the values of La_i would tend to increase proportionally to L, because of the factor $S(\omega/L)$ which scales with the depth of penetration. A relative sound-speed difference $\delta c/c$ varying as $(1-r/R)^{-a}(a<\frac{1}{2})$ beneath the upper turning point of the modes would cause the frequency perturbation to vary as L^{2a} (Gough & Thompson 1988b).

The even component of the splitting reported by Duvall *et al.* (1986) does indeed show some evidence of an increase with L. By performing a logarithmic regression of the values of the reported splittings between sectoral and zonal modes, Gough & Thompson found a best fit of $L^{0.2}$ for the L dependence of the data. Moreover, the m/L dependence suggested that the cause was somewhat concentrated in equatorial regions. One can perform a similar analysis on the data of Brown & Morrow (1987). In this case, La_2 looks rather flat, suggesting a cause less concentrated towards the surface. Generally speaking, the values of La_2 reported by Brown & Morrow are only a third those of Duvall *et al.*

In addition to the asymptotic variation (equation 8.6) with frequency of the splitting coefficients, it will be interesting to examine unaveraged data for the cyclic variation with frequency associated with a thin perturbing layer. Such behaviour may be detectable if there is a thin but sufficiently strong layer of magnetic field at the base of the convection zone, forming the seat of the solar cycle (e.g. Speigel & Weiss 1980).

Barring fortuitous cancellations, one can use the magnitude of the observed coefficients to put an upper bound on the strength of any large-scale magnetic field in the outer radiative zone. The values of the La_2 determined by Brown & Morrow (1987) are nowhere greater than about 300 nHz. If one postulates a magnetic origin for the even component of the splitting one expects

$$La_{2k}/\nu \sim \langle\!\langle v_A^2/c^2 \rangle\!\rangle. \tag{8.7}$$

From this or from the numerical results of Section 6, one infers that the present data put an upper bound of about $10^7 G$ on any large-scale field except perhaps in the core, for which we do not yet have sufficient data. Certainly we may expect to put more stringent detailed constraints on the internal field using data to be gathered in the not too distant future by networks of ground-based observing stations and by spacecraft.

Meanwhile, how do uncertainties in our knowledge of the internal rotation and magnetic field affect the accuracy of inversions to obtain the spherically symmetric component of the Sun's structure? In this regard it is not just the coefficient I_0, which is equal to the mean frequency shift of the multiplet, which is important. The frequency of a mode of given azimuthal order m depends in general on all the coefficients I_λ, as generally all the functions $Q_{\lambda lm}$ will be non-zero. At present, however, this is not a problem (except perhaps for the core), as the uncertainty in the absolute frequencies is of the order of a few hundred nanohertz, as are the typical values of the La_{2k} (for $k\neq 0$); and the results of this study suggest that the values of I_0 should be no bigger than this. The uncertainty in the absolute frequencies is inherently much larger than that in the frequency splittings, so even when the observational accuracy is great enough for rotational and magnetic effects to have to be taken into account in spherically symmetric inversions, we may hope in turn to know their effect sufficiently accurately to make due allowance.

Finally, it is tempting to speculate how the differences in the reported values of the splitting coefficients reflect temporal changes of solar origin (*cf.* Gough 1988; Kuhn 1988a, b). We can certainly expect in the future to be able to measure the splitting sufficiently accurately to monitor changes over the solar cycle. It would be of great interest if one were able to detect cycle variations in the internal flow, for example. Changes in the m dependence would also be very interesting and might enable us to say whether, for example, the observed drift of sunspots towards the equator during the cycle reflects the migration of a deep-seated field or whether it is a phenomenon that occurs only in the upper convection zone.

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APPENDIX

In this appendix, an outer boundary condition to be used when solving the adiabatic oscillation equation in a slowly rotating star is derived. The boundary condition is also applicable if the star has a buried magnetic field provided that the magnetic effects are negligible compared with the first-order effects of rotation.

The boundary condition, which is of the form $d\xi/dr-\kappa\xi=0$, where ξ is the radial component of the displacement vector $\boldsymbol{\xi}$, is derived by requiring the interior solution to match on to the causal solution in a plane-parallel atmosphere above the outer boundary. The radius at which the boundary condition is applied will be denoted as $r=R$: this point may be higher in the atmosphere than the photosphere.

In the rotating frame $\mathscr{S}$, the adiabatic oscillation equation is

$$\mathscr{L}\boldsymbol{\xi}+\omega^2\rho\boldsymbol{\xi}=-2i\varepsilon\omega\rho(\boldsymbol{\Omega}_c\times\boldsymbol{\xi}+\mathbf{v}\cdot\nabla\boldsymbol{\xi}), \tag{A1}$$

where the notation is the same as in Section 2 of the main text.

The zero-order, non-rotating case (in which the right-hand side of the above equation is zero) has spheroidal-mode solutions of the form

$$\boldsymbol{\xi}=\boldsymbol{\xi}_0=\left[\xi_0(r)\,Y_l^m,\ \eta_0(r)\frac{\partial Y_l^m}{\partial\theta},\ \frac{im\eta_0(r)}{\sin\theta}\,Y_l^m\right];\ \omega=\omega_0. \tag{A2}$$

Then equation (A1) yields two scalar first-order differential equations for ξ_0 and χ_0, where $\nabla\cdot\boldsymbol{\xi}_0\equiv\chi_0(r)Y_l^m$:

$$\frac{1}{r^2}\frac{d}{dr}(r^2\xi_0)-\frac{gk^2}{\omega_0^2}\,\xi_0-\left(1-\frac{k^2c^2}{\omega_0^2}\right)\chi_0=0, \tag{A3a}$$

$$\frac{d\chi_0}{dr}+\frac{\omega_0^2}{c^2}\left(1-\frac{g^2k^2}{\omega_0^4}+\frac{2g}{r\omega_0^2}\right)\xi_0-\left(\frac{\gamma g}{c^2}-\frac{k^2g}{\omega_0^2}\right)\chi_0=0 \tag{A3b}$$

[where $k^2\equiv l(l+1)/r^2$].

The first-order effect of rotation on the eigenfunction is to perturb it to $\boldsymbol{\xi}_0+\varepsilon\boldsymbol{\xi}_1$; the eigenfrequency becomes $\omega_0+\varepsilon\omega_1$. As discussed in Section 3, $\boldsymbol{\xi}_1=\boldsymbol{\xi}_S+\boldsymbol{\xi}_T$, where $\boldsymbol{\xi}_S$ is spheroidal and $\boldsymbol{\xi}_T$ is toroidal. The latter has zero radial component and divergence, so that $\mathscr{L}\boldsymbol{\xi}_T=0$ (to this order). Thus equation (A1) yields

$$\mathscr{L}\boldsymbol{\xi}_S+\omega_0^2\rho\boldsymbol{\xi}_S=-2i\omega_0\rho(\boldsymbol{\Omega}_c\times\boldsymbol{\xi}_0+\mathbf{v}\cdot\nabla\boldsymbol{\xi}_0)-2\omega_0\omega_1\rho\boldsymbol{\xi}_0-\omega_0^2\rho\boldsymbol{\xi}_T. \tag{A4}$$

For rotation that is a function only of radius, $\boldsymbol{\xi}_S$ has the form $\boldsymbol{\xi}_S=[\xi_S(r)Y_l^m,\ \eta_S(r)\,\partial Y_l^m/\partial\theta,\ im\eta_S(r)Y_l^m/\sin\theta]$. As shown in Section 3, $\boldsymbol{\xi}_T$ can be determined algebraically by evaluating the vertical component of the curl of equation (A4). Then (*cf.* equations 3.38, 3.39) from equation (A4) one obtains two scalar equations for ξ_S and χ_S (in an obvious notation):

$$\frac{1}{r^2}\frac{d}{dr}(r^2\xi_S)-\frac{gk^2}{\omega_0^2}\,\xi_S-\left(1-\frac{k^2c^2}{\omega_0^2}\right)\chi_S=-2k^2r\eta_0\frac{\tilde{\omega}_1}{\omega_0}-\frac{2}{r}(\xi_0+\eta_0)\frac{m\Omega}{\omega_0}, \tag{A5a}$$

$$\frac{d\chi_S}{dr}+\frac{\omega_0^2}{c^2}\left(1-\frac{g^2k^2}{\omega_0^4}+\frac{2g}{r\omega_0^2}\right)\xi_S-\left(\frac{\gamma g}{c^2}-\frac{k^2g}{\omega_0^2}\right)\chi_S=\frac{-2g}{rc^2}\left\{\left(k^2r^2\eta_0+\frac{r\omega_0^2}{g}\,\xi_0\right)\frac{\tilde{\omega}_1}{\omega_0}+\left[\xi_0+\left(1+\frac{r\omega_0^2}{g}\right)\eta_0\right]\frac{m\Omega}{\omega_0}\right\}. \tag{A5b}$$

Here

$$\tilde{\omega}_1\equiv\omega_1+m[\Omega_c-\Omega(R)] \tag{A6}$$

where Ω is the total angular velocity in the inertial frame which we have approximated with a constant: $\Omega(R)$; also $\varepsilon\omega_1$ is the frequency perturbation in frame $\mathscr{S}$ and is given by the right-hand side of equation (2.35). Hence in the notation of that equation $\tilde{\omega}_1=m[«\Omega»-\Omega(R)]$.

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Specializing now to an isothermal plane-parallel atmosphere with constant γ, g and c, the zero-order equations (A3) become

$$\frac{d}{dz}\xi_0-\frac{gk^2}{\omega_0^2}\xi_0-\left(1-\frac{k^2c^2}{\omega_0^2}\right)\chi_0=0, \tag{A7a}$$

$$\frac{d}{dz}\chi_0+\frac{\omega_0^2}{c^2}\left(1-\frac{g^2k^2}{\omega_0^4}\right)\xi_0-\left(\frac{\gamma g}{c^2}-\frac{k^2g}{\omega_0^2}\right)\chi_0=0, \tag{A7b}$$

where $z\equiv r-R$ is the height above $r=R$. These have solutions ξ_0, χ_0 proportional to $\exp(\kappa_0 z)$, where κ_0 is given by

$$\det\begin{pmatrix}\kappa_0-gk^2/\omega_0^2 & k^2c^2/\omega_0^2-1\\ \omega_0^2/c^2-g^2k^2/\omega_0^2c^2 & \kappa_0-\gamma g/c^2+k^2g/\omega_0^2\end{pmatrix}=0; \tag{A8}$$

that is,

$$\kappa_0=\frac{1}{2H}\pm\left[\frac{1}{4H}-\frac{\omega^2}{c^2}+k^2(1-N^2/\omega^2)\right]^{1/2},$$

where $H\equiv c^2/\gamma g$ is the (density) scale height and $N\equiv(\gamma-1)^{1/2}g/c$ is the Brunt–Väisälä frequency.

For five-minute solar p modes κ_0 is real so the modes are evanescent in the atmosphere. For a physically acceptable solution with energy density declining with height, $\rho\boldsymbol{\xi}^2$ must decrease as $z\to\infty$: because in the isothermal atmosphere the density is proportional to $\exp(-z/H)$, this requires that

$$\kappa_0=\frac{1}{2H}-\left[\frac{1}{4H}-\frac{\omega^2}{c^2}+k^2(1-N^2/\omega^2)\right]^{1/2}. \tag{A9}$$

The vertical displacement and Lagrangian pressure perturbation must be continuous at $r=R$. On the assumption that γ, g and c are also continuous, χ and [from equation (A3)] $d\xi/dr$ are continuous also. Thus at $r=R-$ a zero-order boundary condition on the interior solution is

$$\frac{d\xi}{dr}-\kappa_0\xi=0. \tag{A10}$$

The first-order correction to this boundary condition can similarly be found from equations (A5), provided r and Ω are treated as constant in $r>R$. Then in the atmosphere

$$\frac{d}{dz}\xi_S-\frac{gk^2}{\omega_0^2}\xi_S-\left(1-\frac{k^2c^2}{\omega_0^2}\right)\chi_S=Fe^{\kappa_0 z}, \tag{A11a}$$

$$\frac{d}{dz}\chi_S+\frac{\omega_0^2}{c^2}\left(1-\frac{g^2k^2}{\omega_0^4}\right)\xi_S-\left(\frac{\gamma g}{c^2}-\frac{k^2g}{\omega_0^2}\right)\chi_S=Ge^{\kappa_0 z}, \tag{A11b}$$

where F and G are constants which depend on the zero-order eigensolution and ω_1; the latter is found in the manner described in Section 2. Hence for a physical solution, and normalizing so that $\xi_S=0$ at $r=R$,

$$\left.\begin{aligned}\xi_S&=Az\exp(\kappa_0 z)\\ \chi_S&=Bz\exp(\kappa_0 z)+C\exp(\kappa_0 z)\end{aligned}\right\}, \tag{A12}$$

where A and B are related as ξ_0 and χ_0:

$$(\kappa_0-gk^2/\omega_0^2)A-(1-k^2c^2/\omega_0^2)B=0; \tag{A13}$$

and, from equations (A11),

$$\left.\begin{aligned}A-(1-k^2c^2/\omega_0^2)C&=F\\ B+\kappa_0C-\left(\frac{\gamma g}{c^2}-\frac{k^2g}{\omega_0^2}\right)C&=G.\end{aligned}\right\} \tag{A14}$$

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Eliminating B and C between equations (A13) and (A14) gives

$$A=\left(2\kappa_0-\frac{\gamma g}{c^2}\right)^{-1}\left[\left(\kappa_0-\frac{\gamma g}{c^2}+\frac{k^2 g}{\omega_0^2}\right)F+\left(1-\frac{k^2c^2}{\omega_0^2}\right)G\right]. \tag{A15}$$

From equation (A7a) and the definition of χ_0 one obtains

$$\eta_0=\frac{-c^2}{R\omega_0^2}\left(\kappa_0-\frac{g}{c^2}\right)\left(1-\frac{k^2c^2}{\omega_0^2}\right)^{-1}\xi_0 \tag{A16}$$

so that

$$F\equiv 2\frac{\tilde{\omega}_1}{\omega_0}\left[\frac{k^2c^2}{\omega_0^2}\left(1-\frac{k^2c^2}{\omega_0^2}\right)^{-1}\left(\kappa_0-\frac{g}{c^2}\right)\right]\xi_0(R)+\frac{2}{R}\frac{m\Omega}{\omega_0}\left[\frac{c^2}{R\omega_0^2}\left(\kappa_0-\frac{g}{c^2}\right)\left(1-\frac{k^2c^2}{\omega_0^2}\right)^{-1}-1\right]\xi_0(R), \tag{A17}$$

$$G\equiv 2\frac{\tilde{\omega}_1}{\omega_0}\left[\frac{k^2g}{\omega_0^2}\left(1-\frac{k^2c^2}{\omega_0^2}\right)^{-1}\left(\kappa_0-\frac{g}{c^2}\right)-\frac{\omega_0^2}{c^2}\right]\xi_0(R)+\frac{2}{R}\frac{m\Omega}{\omega_0}\left[\left(1+\frac{g}{R\omega_0^2}\right)\left(\kappa_0-\frac{g}{c^2}\right)\left(1-\frac{k^2c^2}{\omega_0^2}\right)^{-1}-\frac{g}{c^2}\right]\xi_0(R). \tag{A18}$$

Hence

$$\frac{A}{\xi_0(R)}=\left(\kappa_0-\frac{\gamma g}{2c^2}\right)^{-1}\left(1-\frac{k^2c^2}{\omega_0^2}\right)^{-1}\left[\left\{\frac{k^2c^2}{\omega_0^2}\left(\kappa_0-\frac{g}{c^2}\right)\left[\kappa_0-\frac{(\gamma-1)g}{c^2}\right]-\frac{\omega_0^2}{c^2}\left(1-\frac{k^2c^2}{\omega_0^2}\right)^2\right\}\frac{\tilde{\omega}_1}{\omega_0}\right.$$
$$\left.+\left\{\frac{c^2}{R^2\omega_0^2}\left(\kappa_0-\frac{g}{c^2}\right)\left[\kappa_0-\frac{(\gamma-1)g}{c^2}\right]-\frac{(\gamma-2)g}{R^2c^2}\left(1-\frac{k^2c^2}{\omega_0^2}\right)\right\}\frac{m\Omega}{\omega_0}\right]. \tag{A19}$$

As before, one thus obtains a boundary condition of the form $d\xi/dr-\kappa\xi=0$ at $r=R-$. Because, at $r=R$, $\xi=\xi_0$ and $d\xi/dr=\kappa_0\xi_0+\varepsilon A$ one thus has

$$\frac{d\xi}{dr}-(\kappa_0+\varepsilon\kappa_1)\xi=0, \tag{A20}$$

where $\kappa_1\equiv A/\xi_0(R)$ is given by equation (A19).

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