THE EFFECT OF TENSOR CONDUCTIVITY ON CONTINUUM MAGNETOGASDYNAMIC FLOWS*

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1. Introduction. In recent years much work, both theoretical and experimental, has been done in the exploration of magnetogasdynamic or plasmadynamic phenomena, their engineering applications and their astrophysical implications. A good deal of this work, however, has included the approximation that the electric current always flows in the direction of the electric field vector. In reality, charged particles in a magnetic field do not move in straight lines, but rather tend to drift in a direction perpendicular to both the electric and magnetic fields, giving rise to the so-called Hall current or Hall effect. Because of the Hall effect, Ohm's law is modified in such a way that the electrical conductivity becomes a tensor quantity. The usual approximation of scalar conductivity is valid only when the collision frequency of the particles is so large that the particles have negligible time to drift across the magnetic field lines between collisions, i.e. when the cyclotron frequency of the particles is much less than their collision frequency. For cases where this condition is not met, the Hall effect must be taken into consideration.

This paper extends our earlier results [1] to account for this phenomenon. We consider a dense plasma with a plasma frequency much greater than the cyclotron frequency of the particles present which, in turn, is much larger than the inverse of the characteristic flow time (i.e. the ratio of the free stream velocity to the characteristic length of the obstacle). In addition, the cyclotron frequency for the electrons is assumed to be of the same order of magnitude as the collision frequency between electrons and ions. We begin Sec. 2 by stating the equations pertinent to this problem. After linearization and nondimensionalization, we proceed to find the basic equation governing the perturbation quantities. A normal-mode analysis is applied to this equation to determine all the inherent field modes for the two-dimensional and axisymmetric flow problems. We then express the general form of the solutions as an arbitrary combination of all the field modes. Finally, Fourier synthesis is employed to treat simple slender bodies. In Sec. 3, we restrict our analysis to the case of large magnetic Reynolds number. This approximation allows us to simplify the mode solutions and their coefficients, and renders the Fourier integrals tractable. The simplified integrands of these Fourier integrals are identical in form to those of the scalar conductivity case which were studied earlier. Therefore, in Sec. 4, we are able to make qualitative predictions about Hall current effects without having to evaluate the integrals themselves. Finally, our conclusions are given in Sec. 5.

2. Formulation. To investigate the influence of the Hall effect upon the steady motion of a dense, compressible, fully ionized, quasi-neutral gas, the pertinent equations

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are the usual continuum magnetogasdynamic equations with a generalized Ohm's law:

$$\begin{aligned} &\nabla\cdot(\rho\bar{q}) = 0, \\ &\rho\bar{q}\cdot\nabla\bar{q} + \nabla p = \bar{J}\times\bar{B}, \\ &\rho T\bar{q}\cdot\nabla S = \bar{J}\cdot(\bar{E} + \bar{q}\times\bar{B}), \\ &p = \rho RT, \\ &\nabla\cdot\bar{B} = 0, \\ &\nabla\times\bar{B} = \mu\bar{J}, \\ &\nabla\times\bar{E} = 0 \end{aligned} \tag{1}$$

and

$$\bar{J} = \sigma(\bar{E} + \bar{q}\times\bar{B} + n^{-1}e^{-1}\nabla p_e) - \omega_e\tau\,|\bar{B}|^{-1}\,\bar{J}\times\bar{B}.$$

Here S, p, ρ, R, T, $\bar{q}$, $\bar{B}$, $\bar{J}$ and $\bar{E}$ have their usual meaning and μ, n, e, ω_e, τ, σ and p_e are respectively the magnetic permitivity, electron number density, electron charge, electron cyclotron frequency, collision time between ions and electrons, electrical conductivity and electron gas pressure. The only difference between this set of equations and that for scalar conductivity is in Ohm's law. This set now includes the effect of the electron pressure gradient on current density in the plasma and also the Hall current, which describes how the electrons, in addition to moving parallel to the effective electric field, also drift across the magnetic field lines in a direction perpendicular to both electric and magnetic fields.

The nondimensional version of Eqs. (1), linearized to first order in perturbed quantities, may be expressed for two-dimensional or axisymmetric flows as the single equation

$$\begin{aligned} P\phi = \Bigg\{&\left[R_m^{-1}\left(\Delta + \frac{\partial^2}{\partial x^2}\right) + (m^{-2} - 1)\frac{\partial}{\partial x}\right] \\ &\cdot\left\{\left[R_m^{-1}\left(\Delta + \frac{\partial^2}{\partial x^2}\right) + (m^{-2} - 1)\frac{\partial}{\partial x}\right]\left(\Delta + \beta^2\frac{\partial^2}{\partial x^2}\right) - M^2m^{-2}\Delta\frac{\partial}{\partial x}\right\} \\ &+ \omega_e^2\tau^2R_m^{-2}\left(\Delta + \frac{\partial^2}{\partial x^2}\right)\left(\Delta + \beta^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2}{\partial x^2}\Bigg\}\phi = 0, \end{aligned} \tag{2}$$

where

$$\beta^2 = 1 - M^2, \qquad \Delta = y^{-i}\frac{\partial}{\partial y}\left(y^i\frac{\partial}{\partial y}\right) - jy^{-2i}$$

and $j = 0$ or 1 for two-dimensional or axisymmetric flow. Here P is a linear, sixth-order partial differential operator, ϕ is one of the perturbations in the velocity or magnetic field components, M, m are the freestream Mach and Alfvén numbers and Rm is the magnetic Reynolds number based on a typical length. If we define the total velocity and magnetic field vectors in a three-dimensional orthogonal coordinate system, which may be either the system (x, y, z) or $(x, r, -\theta)$, to be $\bar{q} = U[1 + u, v, w]$ and $\bar{H} = H_\infty[1 + h_x, h_y, h_z]$ then all the perturbation quantities satisfy Eq. (2). This equation was first obtained by Sears and Resler [2]. We are concerned here with how the solutions to this equation for large magnetic Reynolds numbers are modified by the inclusion of the Hall effect [i.e. we are considering $\omega_e = O(\tau^{-1})$]. In particular, we will be concerned with the flow over a finite slender body.

When $\omega_e\tau$ vanishes, i.e. when the scalar conductivity approximation is valid, Eq. (2) degenerates into the fourth-order partial differential equation

$$\left\{\left[R_m^{-1}\left(\Delta + \frac{\partial^2}{\partial x^2}\right) + (m^{-2} - 1)\frac{\partial}{\partial x}\right]\left(\Delta + \beta^2\frac{\partial^2}{\partial x^2} - M^2m^{-2}\Delta\frac{\partial}{\partial x}\right\}\phi = 0.$$

A detailed quantitative description of the solutions to this equation is to be found in Tang and Seebass [1]. The other limit for which qualitative predictions of the behavior of the flow exist is that of incompressible flow over an infinite sinusoidal wall [2].

To proceed with our problem, we follow the procedure employed in our analysis of the scalar conductivity case. That is, we shall first solve Eq. (2) for the y-component of the perturbation velocity $v(x, y)$. The other two components of the perturbation velocity can then be obtained from the continuity equation and Ohm's law. Having determined the flow field, the magnetic field can easily be determined from the curl of the linearized momentum equation, Ampere's law and the selenoidal requirement on the magnetic field.

Applying a normal mode analysis to Eq. (2), we assume that $v(x, y)$ is proportional to $\exp(i\lambda x - ky)$ in the two-dimensional case and to $K_1(ky)\exp(i\lambda x)$, a modified Bessel function of the first kind, in the axisymmetric case. Then $v(x, y)$ satisfies $Pv = 0$ provided that the ratio, $r = k\lambda^{-1}$, satisfies the dispersion relation

$$\begin{aligned} r^6 &+ [2iN(m^{-2} - 1 - \tfrac{1}{2}M^2m^{-2}) - (2 + \beta^2 + \omega_e^2\tau^2)]r^4 + \{[1 + 2\beta^2 + (1 + \beta^2)\omega_e^2\tau^2] \\ &- 2iN[(1 + \beta^2)(m^{-2} - 1) - \tfrac{1}{2}M^2m^{-2}] - N^2(m^{-2} - 1)(\beta^2m^{-2} - 1)\}r^2 \\ &+ [2i\beta^2N(m^{-2} - 1) + N^2(m^{-2} - 1)^2\beta^2 - \beta^2(1 + \omega_e^2\tau^2)] = 0, \end{aligned}$$

where

$$N = R_m\lambda^{-1}.$$

Since the above equation is bicubic in form, there are three independent roots that satisfy the uniform undisturbed boundary condition as $y \to \infty$. Therefore, we may express v as the sum of the product of arbitrary functions of λ with the mode solutions. The arbitrary functions of λ are to be determined from the boundary conditions at the interface. For a finite body the solution $v(x, y)$ must consist of contributions from all wave numbers, λ. Thus the general solution for a finite body has the form

$$v(x, y) = \int_{-\infty}^{\infty}\sum_{i=1}^{3} V_i(\lambda)\left\{\begin{matrix}\exp(-\lambda r_i y) \\ K_1(\lambda r_i y)\end{matrix}\right\}e^{i\lambda x}\,d\lambda.$$

From the inviscid boundary condition on the velocity, we find

$$\sum_{i=1}^{3} V_i(\lambda)\left\{\begin{matrix}1 \\ \lambda^{-1}r_i^{-1}\end{matrix}\right\} = (2\pi)^{-1}\int_{-\infty}^{\infty}\left\{\begin{matrix}v(s, 0) \\ (2\pi)^{-1}S'(s)\end{matrix}\right\}e^{-i\lambda s}\,ds, \tag{3}$$

where $v(s, 0)$ is the slope of the body on the x-axis, and $S'(s)$ is the slope of the cross-sectional area of the body. Other relations between the V's must be supplied by the boundary conditions on the electromagnetic quantities. Only the boundary conditions on the magnetic field vector need be considered here. That is, we need only require that the magnetic induction be continuous across the interface. We shall assume that the permeability of the body is the same as that of the fluid, and impose this continuity on the magnetic field vector $\bar{H}$. If we require that the body be an insulator, the above boundary condition coupled with the requirement that all components of $\bar{H}$ be harmonic inside the body leads to two more linear relationships between the V's. For a slender body, these relations reduce to

$$\begin{Bmatrix} 1 \\ (\lambda r_2)^{-1} \end{Bmatrix} V_2(\lambda) = -Q(\lambda) \frac{r_3^2 - \beta^2}{r_2^2 - \beta^2} \begin{Bmatrix} 1 \\ (\lambda r_3)^{-1} \end{Bmatrix} V_3(\lambda) \tag{4}$$

and

$$\begin{Bmatrix} 1 \\ (\lambda r_1)^{-1} \end{Bmatrix} V_1(\lambda) = \frac{r_1^2 - 1}{r_1^2 - \beta^2} \left[\frac{r_3^2 - \beta^2}{r_2^2 - 1} Q(\lambda) - \frac{r_3^2 - \beta^2}{r_3^2 - 1} \right] \begin{Bmatrix} 1 \\ (\lambda r_3)^{-1} \end{Bmatrix} V_3(\lambda), \tag{5}$$

where

$$Q(\lambda) = \left[\frac{r_1^2 - \alpha^2}{r_3^2 - \alpha^2} - \frac{r_1^2 - 1}{r_3^2 - 1} \right] \Big/ \left[\frac{r_1^2 - \alpha^2}{r_2^2 - \alpha^2} - \frac{r_1^2 - 1}{r_2^2 - 1} \right] \quad \text{and} \quad \alpha^2 = 1 + iR_m \lambda^{-1}(1 - m^{-2}).$$

Combining Eqs. (3)–(5), we find

$$\begin{Bmatrix} 1 \\ (\lambda r_3)^{-1} \end{Bmatrix} V_3(\lambda) = \left\{ 1 - \frac{(r_3^2 - \beta^2)(r_1^2 - 1)}{(r_3^2 - 1)(r_1^2 - \beta^2)} \right.$$
$$\left. + (r_3^2 - \beta^2) \left[\frac{r_1^2 - 1}{(r_2^2 - 1)(r_1^2 - \beta^2)} - \frac{1}{r_2^2 - \beta^2} \right] Q(\lambda) \right\}^{-1} (2\pi)^{-1} \int_{-\infty}^{\infty} \begin{Bmatrix} v(s, 0) \\ (2\pi)^{-1} S'(s) \end{Bmatrix} e^{-i\lambda s} \, ds. \tag{6}$$

With $V_1(\lambda)$, $V_2(\lambda)$ and $V_3(\lambda)$ determined, we can easily express each of the perturbations in terms of the $V_i(\lambda)$'s:

$$i\beta^2 u(x, y) = \int_{-\infty}^{\infty} \sum_{i=1}^{3} r_i(\lambda) V_i(\lambda) \begin{Bmatrix} \exp(-\lambda r_i y) \\ K_0(\lambda r_i y) \end{Bmatrix} e^{i\lambda x} \, d\lambda;$$

$$v(x, y) = \int_{-\infty}^{\infty} \sum_{i=1}^{3} V_i(\lambda) \begin{Bmatrix} \exp(-\lambda r_i y) \\ K_1(\lambda r_i y) \end{Bmatrix} e^{i\lambda x} \, d\lambda;$$

$$\beta^2 w(x, y) = \int_{-\infty}^{\infty} \sum_{i=1}^{3} \omega_e \tau \left[\frac{r_i^2 - \beta^2}{r_i^2 - \alpha^2} \right] V_i(\lambda) \begin{Bmatrix} \exp(-\lambda r_i y) \\ K_1(\lambda r_i y) \end{Bmatrix} e^{i\lambda x} \, d\lambda; \tag{7}$$

$$i\beta^2 m^{-2} h_x(x, y) = \int_{-\infty}^{\infty} \sum_{i=1}^{3} \left[\frac{r_i^2 - \beta^2}{r_i^2 - 1} \right] r_i V_i(\lambda) \begin{Bmatrix} \exp(-\lambda r_i y) \\ K_0(\lambda r_i y) \end{Bmatrix} e^{i\lambda x} \, d\lambda;$$

$$\beta^2 m^{-2} h_y(x, y) = \int_{-\infty}^{\infty} \sum_{i=1}^{3} \left[\frac{r_i^2 - \beta^2}{r_i^2 - 1} \right] V_i(\lambda) \begin{Bmatrix} \exp(-\lambda r_i y) \\ K_1(\lambda r_i y) \end{Bmatrix} e^{i\lambda x} \, d\lambda;$$

$$h_z(x, y) = m^2 w(x, y).$$

Here $K_0(z)$ is also a modified Bessel function of the first kind. Finally, by requiring that the ions and electrons have the same temperature, so that we can simply replace p_e

by one-half the total gas pressure p in Ohm's law, we obtain for the electric field

$$E_x(x, y) = m^2R_m^{-1}\left[\tfrac{1}{2}\omega_e\tau \frac{\partial}{\partial x} u(x, y) - \frac{\partial}{\partial y} w(x, y)\right],$$

$$E_y(x, y) = \left[(1 + \tfrac{1}{2}\omega_e^2\tau^2)m^2R_m^{-1} \frac{\partial}{\partial x} + (1 - m^2)\right]w(x, y) + \tfrac{1}{2}\omega_e\tau\left[m^2R_m^{-1} \frac{\partial}{\partial x} - 1\right]v(x, y) + \tfrac{1}{2}\omega_e\tau h_y(x, y), \tag{8}$$

and

$$E_z(x, y) = 0.$$

If one is interested in determining the distribution of surface charge density on the obstacle, it can be obtained by taking the difference of the normal components of the electric field across the interface. In our case, the electric field inside the obstacle satisfies Laplace's equation, and its amplitude can be determined by matching the tangential components of the electric field across the interface.

Equations (4)–(8) complete our description of linearized magnetogasdynamic flow, including the effects of tensor conductivity, past a slender two-dimensional or axisymmetric body. As in our previous analysis for the scalar conductivity case, if we let ϵ be representative of the body slope and thickness, the important requirement that our linearization be valid again leads to the condition $(m^{-2}R_m)^{1/2}\epsilon = o(1)$. In addition, for a symmetric body with no current flowing inside, all the cyclic constants vanish for this problem.

3. Large magnetic Reynolds number. Our next step is to solve the dispersion relation for its roots. While in principle we can find the exact forms for the roots, the evaluation of the integrals in Eq. (7) would be formidable, judging from the complicated functional dependence of the roots and the V_i's upon λ. Consequently, to obtain an analytic description of the flow field it will be necessary for us to approximate the integrands in Eq. (7). In this regard, the quantity of primary importance in the dispersion relation is $N = R_m\lambda^{-1}$.* We shall limit our investigation to the case of large magnetic Reynolds number, i.e. to large N.

With this in mind we shall assume that $r^2 = \sum_{-\infty}^{\infty} \alpha_n N^n$. If we substitute this form for r^2 into the dispersion relation and equate coefficients of like powers of N, we obtain

$$r_1 = \pm[iR_m\lambda^{-1}m^{-2}(m^2 + M^2 - 1)]^{1/2} + O(N^{-1/2}),$$
$$r_2 = \pm[iR_m\lambda^{-1}m^{-2}(m^2 - 1)]^{1/2} + O(N^{-1/2}) \tag{9}$$

and

$$r_3 = \pm i(1 - c^2)^{1/2}[1 + (i/2)\lambda R_m^{-1}(1 - m^2)^{-1}c^4] + O(N^{-2}),$$

where

$$c^2 = m^2M^2(M^2 + m^2 - 1)^{-1}.$$

*Note: This parameter differs from that of our earlier analysis for the scalar conductivity case by a factor M^{-2}. While our former analysis is valid for the incompressible case, the normal mode analysis used here fails in this limit.

Furthermore, if we substitute Eqs. (9) into Eqs. (4)–(6), we find

$$V_1(\lambda) = \begin{Bmatrix} 1 \\ \lambda r_1 \end{Bmatrix} \left[\frac{M^2 - 1}{M^2 + m^2 - 1} + O(N^{-1}) \right] (2\pi)^{-1} \int_{-\infty}^{\infty} \begin{Bmatrix} v(s, 0) \\ (2\pi)^{-1} S'(s) \end{Bmatrix} e^{-i\lambda s} \, ds$$

$$V_2(\lambda) = \begin{Bmatrix} 1 \\ \lambda r_2 \end{Bmatrix} \left[\frac{m^2 \omega_e^2 \tau^2 (M^2 - 1)}{i M^4 R_m \lambda^{-1}} + O(N^{-2}) \right] (2\pi)^{-1} \int_{-\infty}^{\infty} \begin{Bmatrix} v(s, 0) \\ (2\pi)^{-1} S'(s) \end{Bmatrix} e^{-i\lambda s} \, ds \tag{10}$$

and

$$V_3(\lambda) = \begin{Bmatrix} 1 \\ \lambda r_3 \end{Bmatrix} \left[\frac{m^2}{M^2 + m^2 - 1} + O(N^{-1}) \right] (2\pi)^{-1} \int_{-\infty}^{\infty} \begin{Bmatrix} v(s, 0) \\ (2\pi)^{-1} S'(s) \end{Bmatrix} e^{-i\lambda s} \, ds.$$

A comparison of these results with our former analysis [1] indicates that r_1 and r_3 are respectively the usual parabolic wake and either the hyperbolic or elliptic modes of the scalar conductivity solution, and that r_2 is the new addition due to the effect of tensor conductivity. From the form of r_2 in Eqs. (9), we see that it is identical to r_1 in the limit of $M \to 0$. Thus, we would expect the qualitative behavior of this new mode to be that of a parabolic wake, which we shall call the 'Hall wake'. Notice also that, to the order of accuracy of our analysis, the scalar conductivity modes are completely independent of the presence of the Hall effect: $\omega_e \tau$ appears neither in the roots nor in the coefficients. On the other hand, the coefficient of the new component, V_2, is proportional to $\omega_e^2 \tau^2$. This guarantees that in the limit of $\omega_e \tau \to 0$ we will recover the scalar conductivity results. Furthermore, from the form of the roots, Eqs. (9), and the coefficients, Eqs. (10), we can expect that the contribution of the Hall mode to $v(x, y)$ is of $O(R_m^{-1})$ relative to the scalar components.

With the simplifications afforded by Eqs. (9) and (10), we can formally combine Eqs. (4)–(7) to obtain the asymptotic representations for all the perturbation quantities:

$$i\beta^2 \begin{bmatrix} u(x, y) \\ m^{-2} h_x(x, y) \end{bmatrix} = (2\pi)^{-1} \iint_{-\infty}^{\infty} \sum_{i=1}^{3} \begin{bmatrix} 1 \\ \gamma_i \end{bmatrix} \begin{Bmatrix} v(s, 0) r_i \exp(-\lambda r_i y) \\ (2\pi)^{-1} S'(s) \lambda r_i^2 K_0(\lambda r_i y) \end{Bmatrix} V^{(i)} e^{i\lambda(x-s)} \, ds \, d\lambda,$$

and

$$\begin{bmatrix} v(x, y) \\ -\beta^2 (\omega_e \tau)^{-1} w(x, y) \\ \beta^2 m^{-2} h_y(x, y) \\ -\beta^2 (\omega_e \tau)^{-1} m^{-2} h_z(x, y) \end{bmatrix} = (2\pi)^{-1} \iint_{-\infty}^{\infty} \sum_{i=1}^{3} \begin{bmatrix} 1 \\ \delta_i \\ \gamma_i \\ \delta_i \end{bmatrix} \begin{Bmatrix} v(s, 0) \exp(-\lambda r_i y) \\ (2\pi)^{-1} S'(s) \lambda r_i K_1(\lambda r_i y) \end{Bmatrix} V^{(i)} e^{i\lambda(x-s)} \, ds \, d\lambda,$$

where the r_i's are given by Eqs. (9) and where

$$V^{(1)} = \frac{M^2 - 1}{M^2 + m^2 - 1} + O(N^{-1}), \qquad \gamma_1 = 1 + O(N^{-1}),$$

$$V^{(2)} = \frac{m^2 \omega_e^2 \tau^2 (M^2 - 1)}{i M^4 R_m \lambda^{-1}} + O(N^{-2}), \qquad \gamma_2 = 1 + O(N^{-1}),$$

$$V^{(3)} = \frac{m^2}{M^2 + m^2 - 1} + O(N^{-1}); \qquad \gamma_3 = c^{-2}(c^2 - M^2) + O(N^{-1});$$

$$\delta_1 = M^{-2}(M^2 + m^2 - 1) + O(N^{-1}),$$

$$\delta_2 = -iR_m M^2 \lambda^{-1} m^{-2} \omega_e^{-2} \tau^{-2} + O(N^0),$$

$$\delta_3 = -\frac{m^2(M^2 - c^2)}{iR_m \lambda^{-1}(m^2 - 1)} + O(N^{-2}).$$

We might point out here that the large $N(= R_m\lambda^{-1})$ approximation, when used in the context of Fourier synthesis, requires the special assumption that the higher harmonics of the mode solutions are not important. It is necessary, therefore, to give an indication of how rapidly the integrand occurring in the Fourier transforms should fall off for large wave numbers. A detailed investigation of the errors introduced by large wave numbers has been carried out in references [1] and [5]. It was found that, with the possible exception of circular regions of diameter $O(R_m^{-1})$ centered at the leading and trailing edges, these errors are negligible.

4. Qualitative conclusions. The close correspondence of the above solutions with those for scalar conductivity immediately leads us to the following qualitative observations:

(1) Because the charged particles now drift in the direction perpendicular to both the electric and magnetic fields, our problem is no longer strictly a two-dimensional one. The Hall effect introduces a perturbation current density in the y-direction, which interacts with the unperturbed magnetic field to cause a Lorentz force in the z-direction. Since we have assumed that there can be no pressure gradient in the z-direction to offset this force, a cross flow component results. This in turn causes other perturbations. Altogether six new components, w, h_z, E_x, E_y, J_x and J_y, are excited by the Hall effect, in addition to those present in the scalar conductivity case.

(2) Tensor conductivity also introduces an entirely new field mode in the form of a wake, which we have termed the 'Hall wake'. Thus all the perturbation solutions of this problem are made up of the superposition of three modes: the scalar wake mode, the Hall wake mode, and either the elliptic or hyperbolic mode.

(3) To the order of accuracy of our analysis, the scalar components (u, v, h_x, h_y, E_z, J_z) for the scalar wake and the elliptic and the hyperbolic modes are unaffected by the inclusion of Hall effect, and are therefore identical to those obtained for the scalar conductivity case.

(4) All the new components (w, h_z, E_x, E_y, J_x, J_y) for the scalar wake and the elliptic and hyperbolic modes, as well as all the perturbation components for the Hall wake, are proportional to $(\omega_e\tau)^n$ where n is positive. Thus, in the limit of vanishing $\omega_e\tau$ all the Hall effects disappear and we recover the scalar conductivity solutions of Tang and Seebass [1] in the two-dimensional case and the mode solutions of Lear [3] in the axisymmetric case. If we also let $M \to 0$, we would recover the incompressible scalar conductivity solutions of Lary [4].

(5) An order of magnitude analysis shows that the Hall effects are important only for the wake modes. That is, the magnitudes of components w, h_z, E_x, E_y, J_x and J_y in the elliptic and hyperbolic modes are much smaller than those of the scalar conductivity components; while in the tensor or scalar wake modes, the tensor conductivity components dominate or are of the same order of magnitude as the scalar conductivity components.

(6) In spite of the mathematical similarities that exist between the two wake com-

ponents, a closer look reveals some difference in their characteristics. The width of the scalar wake grows as $(R_m^{-1}x\,|m^2(M^2+m^2-1)^{-1}|)^{1/2}$ in the x-direction, that of the Hall wake grows as $(R_m^{-1}x\,|m^2(m^2-1)^{-1}|)^{1/2}$. Thus, the Hall wake can be thicker or thinner than the scalar wake, depending on the initial conditions of our flow problem. The Hall wake is forward-facing in sub-Alfvénic flow and rearward-facing in super-Alfvénic flow, whereas the transition from forward- to rearward-facing occurs at $m^2 + M^2 = 1$ for the scalar wake. In the scalar wake, the cross flow (two-dimensional) or spinning component (axisymmetric), w, is of the same order of magnitude as v; in the Hall wake, w is much larger than u and v. Furthermore, w is of the same order of magnitude in both wakes but with opposite signs. At the surface of our body, the Hall wake contribution is equal to but opposite in sign from the scalar wake contribution, and their sum vanishes. This satisfies the current-free boundary condition for the insulator which implies that h_z = constant inside the body. However, because $\oint h_z\, d_z$ must also vanish, $h_z = 0$ on the boundary. Then from Eq. (7), which gives $h_z = m^2 w$, we conclude that $w = 0$ on the boundary. Finally, in contrast to the scalar wake, the difference in the orders of magnitude of the perturbation velocities in the Hall wake indicates that the compressibility effects which are important in the scalar wake are negligible in the Hall wake.

To test the validity of the above qualitative observations, we may go back to the governing equation. The existence of the Hall wake mode can be demonstrated if one applies the boundary layer approximation to Eq. (2). This is possible since the analytic forms of the roots in Eq. (9) suggest that the width of the wake is proportional to $(xR_m^{-1})^{1/2}$. Thus, in the wake regions, the length that characterizes the changes in the y-direction is $(xR_m^{-1})^{1/2}$; the characteristic length for variations in the x-direction is simply x. But this means that the variation in y is $O(R_m^{1/2})$ relative to the variation in x in the wake regions. However, by normalizing the x and y coordinates separately with respect to their own characteristic lengths, we can effectively make variations in $\hat{x}$ and $\hat{y}$ (of the new normalized coordinate system) comparable. This procedure is equivalent to a stretching of the y-coordinate. If we apply this normalization procedure to Eq. (2), the relative importance of each term in the governing equation can be estimated from the order of magnitude of the coefficient preceding it. By retaining only the largest terms in orders of R_m the following equation results for the two-dimensional case:

$$\mathfrak{L}_1\mathfrak{L}_2\mathfrak{L}_3\phi \equiv \left[\frac{\partial^2}{\partial\hat{y}^2} + (m^{-2} - 1)\frac{\partial}{\partial\hat{x}}\right]\left[\frac{\partial^2}{\partial\hat{y}^2} + m^{-2}(1 - m^2 - M^2)\frac{\partial}{\partial\hat{x}}\right]\frac{\partial^2}{\partial\hat{y}^2}\phi = 0,$$

where now $(\partial/\partial\hat{x})\phi = O((\partial/\partial\hat{y})\phi)$ and $\mathfrak{L}_3$ is spurious because of boundary conditions at infinity. In this equation we immediately recognize the two bracketed operators as merely parabolic diffusion operators. The solution to $\mathfrak{L}_2\phi = 0$ gives the usual scalar wake component, while the solution to $\mathfrak{L}_1\phi = 0$ is the new addition, the Hall wake component. The transition of each wake component from forward- to rearward-facing is governed by the change of sign of the coefficient multiplying the x-derivative. Thus, except in the incompressible limit, the transition lines are different for the two wake components. In addition, since $\mathfrak{L}_1$ is independent of M, the compressibility effects which play an important role in the scalar wake mode are negligible here. The main result of this new mode is the excitation of the cross-flow component.

For the flow region outside the wakes we expect the characteristic lengths to be of the same order of magnitude in both the x and y directions. Thus, no re-normalization is necessary here. If we again retain only the largest terms in Eq. (2), we obtain in the two-dimensional case:

$$\left[\frac{(1 - M^2)(1 - m^2)}{1 - m^2 - M^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\phi = 0.$$

This equation describes the scalar conductivity magnetoacoustic component, and confirms our observation that Hall effects are important only in the wake regions.

The above results are summarized in Figure 1, which includes the qualitative effects of magnetic Reynolds number and tensor conductivity upon flow fields in different regions of the Taniuti-Resler-Imai diagram. The detailed solution for a specific biconvex profile has been carried out to confirm the qualitative predictions made here [6].

5. Conclusion. We have succeeded in determining the effects of tensor conductivity upon aligned-fields magnetogasdynamic flow when the electron cylotron frequency is of the same order of magnitude as the collision frequency. Analytical solutions for the flow fields about simple profiles can easily be obtained for all regions of the Taniuti-Resler-Imai diagram. These solutions exist everywhere provided $R_m^{1/2}\epsilon = o(1)$, are valid when $R_m^{-1} = o(1)$, and satisfy the appropriate boundary conditions on the body and at infinity.

The major effect of tensor conductivity is to introduce a new mode in the form of a wake which we have called the 'Hall wake'. The structure of the Hall wake is quite different from that of the scalar wake. This wake may have significant implications with regard to the interpretation of probe measurements made for the purpose of plasma diagnostics.

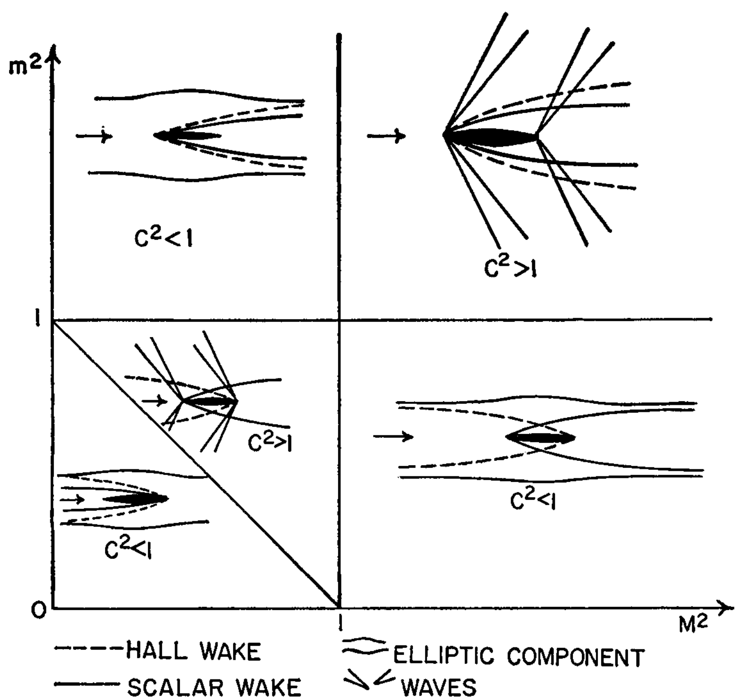

FIG. 1. Sketch of the flow for various regions of the Taniuti-Resler-Imai diagram.

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