THE EFFECTS OF COLLISIONS ON SATURN'S RINGS

Harold Jeffreys

(Received 1947 February 15)

Summary

If Saturn's rings were several particles thick the damping effect of collisions on the inclinations and eccentricities would reduce the rings in less than a year to a state where the particles were piled one on another. Dissipation by friction and impact would not cease at this stage. Its later effects would extend the ring inwards and outwards in its own plane, until it was nowhere more than one particle thick and the spacing was just enough for collisions to be avoided. The time needed to attain this state is estimated to be of the order of 10^6 years for particles of diameter 1 cm. and less for larger ones. Some comments are made on Maxwell's and Goldsbrough's criteria for stability.

The outstanding observational facts about Saturn's rings are as follows. (1) Spectroscopic evidence shows that the velocity at any point is nearly that of a particle in a circular orbit at the same distance. (2) The reflecting power is high, that of Ring B exceeding that of the planet and that of Ring A approaching it. (3) Nevertheless the rings are not quite opaque, stars having occasionally been seen through them. (4) The reflecting power is appreciably lower when Saturn is at quadrature than at opposition, even when allowance is made for the difference of distance. (5) The ring is extremely thin.

Maxwell showed that a set of satellites moving in one circle about the planet would be stable, and that all the other suggested types of constitution that he considered would be unstable. Since his essay his results have usually been quoted as the chief evidence for the meteoric constitution, but they are not quite decisive, because in considering a liquid or gaseous ring he supposed it to be in uniform rotation like a rigid body. Previous work of Laplace and Kowalewsky had shown that a ring in such a state of motion would have a thickness comparable with its width, and in view of our present knowledge of the thickness of the ring this hypothesis scarcely merits further consideration. But a fluid ring could be arbitrarily thin if we abandon the hypothesis of rigid-body rotation, which in itself would suggest a very high viscosity.* The stability of a fluid ring with variable rotation has not, I think, been discussed. The best way of presenting the case for a meteoric constitution at present, I think, is to rely on the observational data (1), (4) and (5) above. (1) and (5) limit us to the meteoric theory or to a gas or liquid with the velocities mainly controlled by gravity and not viscosity. A gas can be excluded, since the distribution of density normal to the plane would satisfy the usual laws for a gas. But in the small normal field that must exist in a mass 13 km. thick at most the density could not build up sufficiently to give great scattering of light. A liquid, again, would have a smooth surface and give regular reflection. Images of the ball of Saturn and of stars would be formed in the ring, and could not have escaped observation. Accordingly, quite apart from the mechanical arguments, which are incomplete, I think that optical considerations alone are

* For an analogous problem *cf.* Jeffreys, *The Earth*, 1929, pp. 49–52, Cambridge.

enough to show that the meteoric theory is the only tenable one. Further, the meteoric theory explains (4), as has been shown by H. Seeliger.*

Nevertheless the meteoric theory needs considerable mechanical investigation before it can be regarded as complete. Maxwell's essay was confined to a set of particles in a single circle, and recent investigations of perturbations have considered only disturbances from such a circle. Attractions between particles at different distances are ignored. But in any case, since the masses of the satellites are small, and that of the ring also small, all mutual gravitation is of the second order of small quantities. But in a system of freely moving solid bodies we may expect collisions to be frequent and to give discontinuous changes of momentum of the first order of small quantities. A preliminary treatment of these has been given †, but further investigation is needed.

The essential point is that the high opacity of the rings shows that on an average most rays of light striking the ring at angles up to 27° meet a particle on their way. But every particle of the ring must cross the mean plane of the ring twice in each revolution. Consequently we may expect it to undergo at least two collisions on the way. The point is that the opacity and the frequency of collisions depend on the same function of the number and size of the particles, namely the total surface of the particles per unit area in the plane of the ring; and if the departures of the particles from steady motion in circles can be treated as random the opacity shows that collisions will have a dominating effect. But collisions between solids are essentially non-conservative; at each collision the relative velocity is reduced by imperfection of restitution and by friction, usually by a fraction in the neighbourhood of $\frac{1}{2}$. Hence relative velocities between neighbouring particles will be rapidly annulled, with a time of relaxation not more than the orbital period, about a day. If for instance the particles were originally in an anchor ring, an average normal to the plane of revolution intersecting several particles, the velocities normal to this plane would be practically annihilated in a year. Further, the radial velocities corresponding to the orbital eccentricities would be removed at the same rate. But there is little to alter the mean motions. Hence the state reached in a year would be a peculiar one. The ring would be thin, but at any distance several particles would be piled one on another, in permanent contact. Such a state could last with little change for a long time, because the particles would acquire such rotations that there would be little difference of velocity at the points of contact. Nevertheless dissipation would not be altogether abolished. Detailed treatment is beyond the present resources of statistical mechanics, for even in the absence of dissipation the problem would be that of a fluid with the molecular spacing comparable with the molecular dimensions. But motion in and out, or up and down, would persist, though its actual amount would be only that needed to maintain rolling. Interchange between motions in different directions remains possible, since the lines joining centres of bodies in contact might be in any direction in relation to the mean plane of revolution. We have in fact a case of kinetic theory where there is no independent agitation; such agitation as there is would be parasitic on the general motion.

The outstanding cause of further dissipation would be the differences in orbital period. For simplicity take the particles to be spheres of radius a, and

* *Abh. Bayer. Akad.*, **18**, 1–72, 1893. For full discussion *cf.* E. Schönberg, *Handb. Astrophys.*, **2**, part I, 130–170, 1929.

† Jeffreys, *M.N.*, **77**, 89–92, 1916.

consider two sets moving in circles of radii r, $r+b$, where $b<2a$. In free motion the ratio of the periods is $1-\frac{3}{2}b/r$. Hence if the spacing in longitude is also b, a particle on the inner circle will encounter on an average $\frac{3}{2}$ particles in a revolution and share momentum with them. If v is the orbital velocity at distance r, m the mass of a particle, the outward transfer of angular momentum at a collision is of order $-mbr\,dv/dr$, the numerical factor being less than 1 but probably more than 0·1. Hence a particle transfers outwards, per unit time, an angular momentum of order

$$-\frac{3mbv}{4\pi}\frac{dv}{dr}.$$

As the particle occupies an area $2ab$, this is equivalent to a tangential stress

$$-\lambda\frac{mv}{ar}\frac{dv}{dr},$$

where λ will probably be between 0·01 and 0·1. Using the relation

$$v^2=\mu/r$$

we have for the tangential stress

$$-\lambda m\mu/2ar^3,$$

and for the total rate of outward transfer of angular momentum in a belt where the normal section is c,

$$\pi\lambda\mu cm/ar^2.$$

The total angular momentum in a ring, if a, b, c are constant, is

$$\tfrac{1}{3}\mu^{\frac{1}{2}}\frac{cm}{ab}[r^{\frac{3}{2}}].$$

Comparing these results, we see that the time that would be needed to transfer the whole of the angular momentum from the inner to the outer half of the mass would be of order

$$\frac{2}{3\pi\lambda}\frac{r^{\frac{1}{2}}[r^{\frac{3}{2}}]}{nb},$$

where n is the mean motion of a particle at the mean distance of the rings. If we take them as having their present extent, and $b=1$ cm., this quantity is of the order of 10^6 years. In the early state that we are considering, when the values of r would differ by much less, the time would be shorter. It would also be shortened by taking b larger, as we should have to do if we are to maintain the hypothesis that the particles were produced by tidal disruption of a solid body. The conclusion to draw is therefore that though the evolution in this state would be slow compared with the damping out of inclinations and eccentricities, it would still be rapid on a cosmogonical scale.

The nature of the changes is clear; on account of the steady outward transfer of angular momentum the outer parts of the ring would be driven outwards and the inner parts inwards, so that the ring would become thinner. The process would stop when the relation $b<2a$ is no longer satisfied. That is, we start from the state of close packing in three dimensions, and arrive at one where the particles are just widely enough spaced to avoid collisions altogether. They follow each other around in circles, the circles being spaced at intervals slightly greater than the diameters of the particles; and the ring is nowhere more than one particle thick.

The result is, I think, consistent with Seeliger's conclusion that a distance between particles decidedly more than their diameters is needed to explain the reduction of albedo near quadrature. It cannot be many times the diameters, since the high albedo even when the rings are open to their fullest extent shows that a straight line from the Earth to the ring must usually intersect at least one particle. If we were in a position to see the ring normally a much smaller fraction of the area would appear occupied than from our actual viewpoint.

Some suggestions arise from these results that may be relevant to the question of the sizes of the particles and to the stability. H. Struve * has inferred from the failure to detect secular perturbations due to the rings that the total mass is not more than 1/27000 of that of Saturn, say 2×10^{25} g. Supposing the whole area of the rings covered to a thickness $2a$ by matter of density 1, this gives $a < 1{\cdot}8 \times 10^4$ cm. This is less than we have inferred from the hypothesis of tidal disruption of a solid, and of course also less than the maximum thickness inferred from observation, which is of the order of 10 km.; and we seem to be driven to the additional hypothesis that the bodies produced in this way were broken up further by collisions in the early stage or to think of some different explanation altogether, such as that the rings were formed by direct condensation from the gaseous state as in the formation of snowflakes. Allowance for the fact that the whole surface need not be covered would not bridge the gap. The collisions in the later stage would be quite gentle, since the relative velocity of neighbouring particles would be of the order of their diameter in a day—much less than the velocity of fall of snowflakes.

According to Maxwell's theory, a set of particles moving in a circle would be stable only if $mp^3 < 2{\cdot}3M$, where p is the number of particles and M is the mass of Saturn, $5{\cdot}6 \times 10^{29}$ g. With spacing equal to the diameters we have

$$\frac{mp^3}{M} = \left(2\pi \, . \frac{1{\cdot}4 \times 10^{10}}{2a}\right)^3 \frac{4}{3} \pi a^3 / 5{\cdot}6 \times 10^{29}$$

which is very roughly 6. The result is independent of a; but if we allow for the spacing being greater than $2a$ Maxwell's criterion will be satisfied by the particles in each ring separately. It would not, however, have been satisfied in the earlier stages. But the stability in any case needs re-examination, on account of the effects of the attractions between particles in different circles.

If there was a division in the ring while the particles were still in contact, the tendency of the transfer of angular momentum would always be to fill it up. Consequently it is likely that the divisions in the ring have been formed since collisions became rare.

Goldsbrough has recently published † a detailed discussion of the perturbations of a circle of satellites by an independent satellite, and concludes that for certain ranges of distance the perturbations would produce instability; he finds that these ranges, for perturbations by Mimas, show interesting correspondences with the boundaries of the rings, Cassini's division, and Encke's division. The theory is very intricate and it seems hypercritical to suggest that it is not intricate enough, since it neglects the attractions between particles at different distances and the reaction on Mimas, whose mass is at any rate much less than the maximum possible mass of the ring. Goldsbrough works in terms of a parameter νT_s, which also

* H. Struve, *Publ. Obs. cent. Nicolas*, **11**, 228, 1898.
† *Phil. Trans.* A, **239**, 183–216, 1946.

appears in Maxwell's stability condition. Cassini's division, in particular, corresponds to the inequalities

$$2 \leqslant \Omega \leqslant \frac{2}{1-\frac{7}{4}\nu T_s}$$

and ω/ω', the ratio of the mean motions of the particles and Mimas, is given by

$$\frac{\omega}{\omega'} = \frac{\Omega}{\Omega - 1}.$$

From the width of the division he derives a value of νT_s, which is consistent with Maxwell's criterion. His argument is then that a circle of particles not satisfying this condition and moving in the division would be perturbed until some of its members collided with particles in the main rings; the circle would then be broken up. But it seems to me that this only shows that for each departure of Ω from 2 there is a critical value of νT_s, below which the circle would be safe. The division would not be sharp; the brightness would fall off continuously towards $\Omega = 2$, but would vanish only at this value.

On Goldsbrough's theory the inner edge of Ring B and the outer edge of Ring A correspond to instabilities at $\Omega = \frac{3}{2}$ and $\Omega = 3$. It is natural to take the former as an indication that particles near the inner edge of Ring B would overshoot the danger zone and join the crape ring; but if so there will be a systematic loss of angular momentum which, as far as I can see, would be compensated by a gain by Mimas, which would thus be driven further off. But if we accept this explanation for the crape ring we should expect another crape ring outside Ring A. This is not definitely confirmed, though some observers have suggested it.

St. John's College,
Cambridge :
1947 *February* 14.