The effects of phase noise in COFDM

J. Stott
BBC Research & Development

The reception of a COFDM signal is analyzed here for the case where phase noise has been added to the signal, e.g. by a receiver local oscillator. Two effects are distinguished: common phase error (a rotation of the signal constellation) and inter-carrier interference (similar to additive Gaussian noise).

It is shown that the amounts of these effects can be deduced from the spectrum of the phase noise using a pair of weighting functions. Use of these weighting functions simplifies the process of computation; it also makes it easier to visualize the consequences of any modifications to the phase-noise spectrum. Some illustrations are given of the two phase-noise effects on the constellation of a DVB-T digital television signal, along with some practical observations on receiver implementation.

1. Introduction

The Author first studied the effects of phase noise in COFDM during 1994, while participating in the RACE project "dTTb". The expertise of this and other European projects was eventually pooled under the umbrella of the DVB Consortium, leading to the establishment of a European Telecommunication Standard – the so-called DVB-T Specification [1].

A study of the effects of phase noise in COFDM was originally prompted by some participants' speculation that phase noise would set a limit to the number of carriers that could practicably be used. A better understanding of the detail was clearly needed in order to inform the process of choosing the number of carriers. Therefore, an analysis of the problem was made and two (internal) papers were contributed by the Author to the work of DVB. The first was an analysis of how the effects arose, including equations for use in estimating the extent of the effects. The second added a new viewpoint to quantifying the effects, by introducing the concept of weighting functions. These submissions played a part in raising confidence that a COFDM system for DVB-T could indeed be based on an 8K fast Fourier transform (FFT).

The study showed that phase noise did not present an insuperable barrier to the implementation of an "8K" COFDM system. However, it remains the case that with DVB-T (and, indeed,

Original language: English
Manuscript received: 29/1/98.

any digital transmission system), it is necessary to take careful note of the effects of phase noise when designing receivers – and, of course, transmission equipment.

The purpose of this article is to make the derivations and results [1] of the past study accessible to a wider audience – in a form which it is hoped will be clearer, thanks to the benefits of hind-sight and the intervening experience.

2. How is phase noise introduced – and why do we care?

Practical oscillators suffer from phase noise – a random perturbation of the phase of the steady sinusoidal waveform. Practical modulators and demodulators usually work either at baseband or at a convenient intermediate frequency (IF). As we must transmit our signal at some allocated radio frequency (RF) it follows that in practice we must shift our modulated signal up to RF in the transmitter, and down from RF to IF or baseband in the receiver. To do this we must use practical oscillators, whose phase noise will be imparted to the signal we convey.

Such frequency-shifting oscillators, usually described as local oscillators (LOs), commonly take the form of free-running oscillators whose frequency is then stabilized to the necessary accuracy by means of a phase-locked loop (PLL). The resulting phase-noise spectrum of such an LO is a function of the properties of the free-running oscillator, together with those of the components of the PLL.

The signal which is demodulated in the receiver will have superimposed on it the phase noise of all the LOs in the chain between it and the modulator. The design of those LOs cannot be specified unless we understand the effect that phase noise has on the demodulation process for the type of signal in use. If we get the specification wrong, then reception may be seriously impaired – or the LOs may be needlessly expensive.

This article analyzes the effects of phase noise on the COFDM method of modulation used in DVB-T. It will (i) show that there are two different effects, one of which can be compensated for and (ii) demonstrate how the concept of weighting functions can be applied to predict how much of each effect will arise, given that we know the spectrum of the phase noise.

3. Analysis: where the noise terms come from

3.1. Definition of the problem and the terminology

Let us suppose that there are N points in the DFTs used to generate and demodulate the COFDM [2] signal. Thus there are potentially N carriers, although some at the extremes of the spectrum will be set to zero at the transmitter in order to provide a guard band for easier implementation of analogue filters [3].

1. A brief summary of the results only, without explanation or proof, is given in reference [2].
2. As in the context of DVB-T, which uses COFDM. However, nowhere in this analysis is the presence of coding of any importance, so the results are applicable to uncoded OFDM systems too.
3. DVB-T has “2K” and “8K” modes which imply the use of an FFT with 2048 or 8192 points, while having 1705 or 6817 active carriers respectively.

Let us describe the transmitted signal as [4]:

$$s(t) = \sum_{k=0}^{N-1} S_k e^{j(\omega_0 + k\omega_u)t}$$

where: S_k is the complex amplitude of the k^{th} carrier;

ω_u is the carrier spacing (i.e. $2\pi / T_u$, where T_u is the active symbol period) in rad / s;

ω_0 is the (angular) frequency of the zeroth carrier (at the IF where we process it).

As we have noted, for some values of k (corresponding to the edges of the spectrum), $S_k = 0$.

Consider first an *ideal* receiver which has no LO phase noise. The "ideally-received" signal *r(t)* is affected only by the channel impulse response *h(t)*. So we write:

$$r(t) = \sum_{k=0}^{N-1} H_k S_k e^{j(\omega_0 + k\omega_u)t}$$

where: H_k is the complex frequency response of the channel at the frequency of the k^{th} carrier [5].

We can replace $H_k S_k$ by R_k, the "ideally-received" complex carrier amplitude in the absence of phase noise. Thus:

$$r(t) = \sum_{k=0}^{N-1} R_k e^{j(\omega_0 + k\omega_u)t}$$

3.2. Introduce the phase noise, assuming it is small

In a *real* receiver, the received signal *x(t)* (at IF, remember) is affected by the channel, whose impulse response is *h(t)*, *and* by the phase noise *φ(t)* introduced by the local oscillator(s).

So we write:

$$x(t) = \sum_{k=0}^{N-1} e^{j\varphi(t)} H_k S_k \, e^{j(\omega_0 + k\omega_u)t}$$

$$= \sum_{k=0}^{N-1} e^{j\varphi(t)} R_k \, e^{j(\omega_0 + k\omega_u)t}$$

---

4. For simplicity and clarity of presentation, it is assumed that the transmitted signal is continuous, without being divided into symbols. Provided that a guard interval of sufficient length (i.e. so that $h(t) = 0$ for $t > \Delta$) is used, the operation of the receiver will be just the same when, as in practice, the transmitted data changes from symbol to symbol.

5. Again, we can do this so long as the guard interval is sufficiently long. Strictly, the H_k *also* include a phase slope which is actually the consequence of any timing error rather than the channel itself.

At the receiver we correlate this signal *x(t)* with each of the possible carrier waveforms to determine the demodulated carrier amplitudes, e.g. for the l th carrier:

$$X_l = \frac{1}{T_u}\int_0^{T_u} x(t)e^{-j(\omega_0 + l\omega_u)t}dt$$

$$= \frac{1}{T_u}\int_0^{T_u} e^{j\varphi(t)}\sum_{k=0}^{N-1} R_k e^{j(k-l)\omega_u t}dt$$

This equation describes the process used in a hypothetical receiver with individual correlator / integrate-and-dump demodulators for each carrier. Practical receivers use the FFT implementation of a DFT, thereby replacing the integration operation by the accumulation of many discrete samples [6], and combining all the "demodulators" into one efficient mathematical operation.

The previous equation is difficult to progress further for the completely general case. However, one way to simplify it is to note that we want the receiver to work without all the margin of the COFDM signal being taken up by the LO phase noise. So we assume that the angle $\varphi(t)$ is always small and make the *small-angle approximation:*

$$e^{j\varphi(t)} \approx 1 + j\varphi(t)$$

This can be deduced from a simple sketch; it is equivalent to taking just the first two terms in the expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Keeping careful track of which processes are linear, we can then write the following expression for the output of the demodulator of the l th carrier:

$$X_l = \frac{1}{T_u}\int_0^{T_u}(1 + j\varphi(t))\sum_{k=0}^{N-1} R_k e^{j(k-l)\omega_u t}dt$$

$$= \frac{1}{T_u}\sum_{k=0}^{N-1} R_k\int_0^{T_u} e^{j(k-l)\omega_u t}dt$$

$$+ \frac{1}{T_u}\sum_{k=0}^{N-1} R_k\int_0^{T_u} j\varphi(t)e^{j(k-l)\omega_u t}dt$$

$$= R_l + Y_l \text{, say.}$$

The first term is the "ideally-received" value R_l that we hoped to recover, while the second term Y_l in turn represents N contributions which result from the presence of the phase noise $\varphi(t)$.

6. This discrete accumulation should, strictly speaking, be represented by very slightly different equations. However, the end results are for all practical purposes the same.

3.3. Identify two types of phase-noise contribution

So, for each received carrier we have an additive error Y_l which is the sum of N terms. For the purposes of insight, it is interesting to examine the different effects of its constituent parts.

3.3.1. The "common" phase-error part

Consider the single contribution to the summation for which $k = l$. In this case the exponential factor cancels and, noting $\varphi(t)$ is real, we have a term (call it P_l) of the form:

$$P_l = \frac{R_l}{T_u}\int_0^{T_u} j\varphi(t)dt = jR_l \times (\text{some real number})$$

This is the "phase" noise that we expect. It is proportional to the "received carrier" R_l and is perpendicular to it on the phasor (Argand) diagram: i.e. it corresponds to rotation of the signal constellation by an angle φ_0 given by:

$$\varphi_0 = \frac{1}{T_u}\int_0^{T_u} \varphi(t)dt$$

Interestingly, *all* carriers are rotated by the same angle simultaneously. This means that, if the rotation in a given symbol can be measured using some carriers which bear reference information, it is then possible to correct the remaining carriers in the symbol. This is known as *common phase-error correction* and its use was foreseen in the DVB-T Specification which provides some "continual-pilot" carriers for this purpose.

3.3.2. The "thermal-noise-like" part

So far we have accounted for the "ideally-received" carrier value R_l and an expected phase-error term P_l. What of the other $(N - 1)$ contributions from the summation in the final equation of *Section 3.2*, arising when $k \neq l$? Let us write:

$$A_l = \frac{1}{T_u}\sum_{k=0}^{l-1} R_k\int_0^{T_u} j\varphi(t)e^{j(k-l)\omega_u t}dt + \frac{1}{T_u}\sum_{k=l+1}^{N-1} R_k\int_0^{T_u} j\varphi(t)e^{j(k-l)\omega_u t}dt$$

This is somewhat cumbersome, so a slightly shorthand notation is proposed for greater clarity, the total range of the summation being carefully remembered hereafter:

$$A_l = \frac{1}{T_u}\sum_{k \neq l} R_k\int_0^{T_u} j\varphi(t)e^{j(k-l)\omega_u t}dt$$

Examining the integrand we can note that $\varphi(t)e^{j(k-l)\omega_u t}$ represents a frequency-shifted version of $\varphi(t)$, the frequency shift being $(k-l)f_u$.

We can interpret A_l as follows [7]. It is the sum of $(N-1)$ contributions. Each is weighted by the (complex) "received" amplitude R_k of one of the other carriers and derives mostly from those components of $\varphi(t)$ around the frequency $-(k-l)f_u$. In *Section 5*, we shall develop a physical interpretation of this.

Since the R_k are complex, so too are the A_l. So this noise term is *not* pure phase noise, it is more like additive thermal noise. It represents the so-called effect of "loss of orthogonality".

3.3.3. The relationship between the results for different carriers

We have already defined φ_0. More generally we can define:

$$\varphi_m = \frac{1}{T_u}\int_0^{T_u} \varphi(t)e^{jm\omega_u t}dt$$

We can then present the results for all carriers simultaneously as a matrix equation, $\mathbf{Y} = j\Phi\mathbf{R}$, short for:

$$\begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_{N-2} \\ Y_{N-1} \end{bmatrix} = j \begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \cdots & \varphi_{N-2} & \varphi_{N-1} \\ \varphi_{-1} & \varphi_0 & \varphi_1 & \cdots & \varphi_{N-3} & \varphi_{N-2} \\ \varphi_{-2} & \varphi_{-1} & \varphi_0 & \cdots & \varphi_{N-4} & \varphi_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{-(N-2)} & \varphi_{-(N-3)} & \varphi_{-(N-4)} & \cdots & \varphi_0 & \varphi_1 \\ \varphi_{-(N-1)} & \varphi_{-(N-2)} & \varphi_{-(N-3)} & \cdots & \varphi_{-1} & \varphi_0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ \vdots \\ R_{N-2} \\ R_{N-1} \end{bmatrix}$$

If the φ_m decrease as m increases, then we have a diagonal stripe (centred on the leading diagonal) of significant entries. In this case the middle carriers will suffer somewhat more than those at the edges. If φ_0 were the *only* significant entry, then the common phase-error term would predominate.

This matrix also makes it very clear why the "thermal-noise-like" component is also described as *inter-carrier interference* (ICI).

7. There is an analogy here with the "reciprocal mixing" problem familiar to designers of HF communications receivers with synthesized LOs. The HF band contains very many strong broadcast signals. If it is required to receive a weak signal at a frequency adjacent to a congested broadcast band, there is little protection to be obtained from front-end selectivity. Once the obvious problem of front-end and mixer intermodulation is reduced sufficiently (e.g. by using a high-power mixer), there remains the problem of reciprocal mixing whereby high-frequency phase-noise sidebands of the LO are mixed back on top of the weak wanted signal by the strong unwanted signals.

3.3.4. The consequences if φ(t) is periodic

Suppose that $\varphi(t)$ is itself periodic. Consider it first as just a single harmonic of the carrier spacing, e.g. $\varphi(t) = Qe^{jq\omega_u t}$. We then have:

$$\varphi_m = \frac{1}{T_u}\int_0^{T_u} Qe^{j(q+m)\omega_u t}dt$$

It is of interest to note that the φ_m are *all* identically zero except for φ_{-q}.

If $\varphi(t)$ is periodic, with a repetition rate which is a multiple of the carrier spacing, then it will contain one or more such harmonics q_1, q_2, ... In this case most of the φ_m will be zero except for φ_{-q_1}, φ_{-q_2}, ...

We can infer from this that if $\varphi(t)$ is periodic in this way, then the ICI is not general in nature (all carriers crosstalking to all, to some degree) but only occurs between carriers that are q_1, q_2, ... slots apart.

3.4. *Quantifying the* φ_m

3.4.1. Quantifying φ_0

We need to quantify the result of the integral:

$$\varphi_0 = \frac{1}{T_u}\int_0^{T_u} \varphi(t)dt$$

We can think of this as the output of an integrate-and-dump process which is fed by a "waveform" $\varphi(t)$. Now this integrate-and-dump process is itself equivalent to feeding the "waveform" $\varphi(t)$ through a hypothetical filter whose impulse response is zero except for a rectangle of height $1/T_u$ and duration T_u, and then sampling the output. The hypothetical filter with rectangular impulse response is sometimes called a "top-hat" filter, from the visual appearance of the impulse response. The corresponding frequency response $H(f)$ is given by a standard Fourier-transform result, namely $H(f) = \mathrm{sinc}\,(f/f_u)$, where the "sinc" function is defined by $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$.

Since the demodulator has in any case to cope with arbitrary fixed phase shifts (as might be introduced by the channel), we can confine our attention purely to the random phase variations introduced by the LOs. In effect we make $\varphi(t)$ a zero-mean random variable by definition. It follows that the values of φ_0 evaluated in different symbols will also have zero mean.

The values of φ_0 are, in effect, samples of a zero-mean noise-like "signal" of spectral "power" density $\mathrm{sinc}^2(f/f_u)|\Phi(f)|^2$. The variance, σ_0^2 of φ_0 is therefore given by:

$$\sigma_0^2 = \int_{-\infty}^{\infty} \mathrm{sinc}^2(f/f_u)|\Phi(f)|^2 df$$

3.4.2. Quantifying φ_m in general

More generally we shall need to quantify the result of the integral:

$$\varphi_m = \frac{1}{T_u}\int_0^{T_u} \varphi(t)e^{jm\omega_u t}dt$$

Unlike φ_0, the quantity φ_m can be complex, so we shall need a special treatment.

We can see that if we treat the exponential term as frequency-shifting *φ(t)*, then the same sort of "noise-power" analysis as applied for the result for φ_0 gives us:

$$E\{\varphi_m \varphi_m^*\} = \int_{-\infty}^{\infty} \mathrm{sinc}^2(f/f_u)|\Phi(f-mf_u)|^2 df$$

where *E{.}* is the expected-value operator and $*$ denotes conjugation.

Assuming that φ_m has zero mean, we can for example write $E\{\varphi_m\varphi_m^*\}$ as:

$$\begin{aligned} E\{\varphi_m \varphi_m^*\} &= \mathrm{var}(\mathrm{Re}(\varphi_m)) + \mathrm{var}(\mathrm{Im}(\varphi_m)) \\ &= \sigma_m^2 \text{ , say,} \end{aligned}$$

so that:

$$\sigma_m^2 = \int_{-\infty}^{\infty} \mathrm{sinc}^2(f/f_u)|\Phi(f-mf_u)|^2 df$$

4. How to quantify the added "noise" of ICI

4.1. The general problem

This is a problem in statistics. Many variables and statistical processes are involved.

Consider first the radiated complex amplitudes S_k of the carriers. These are the sum of two approximately random variables, being the real and imaginary parts. Each has a discrete uniform distribution, taking one of q equiprobable values when the modulation is q^2-QAM. Note that the S_k are not strictly independent since it is the job of the channel coding to introduce redundancy. If there is no time interleaving, then the S_k of the same symbol must be related to some degree.

The sampled (complex) frequency response of the channel H_k will obey some distribution depending on the path. Their *magnitudes* might, for example, tend towards a Rayleigh distribution [8]. The phases may be partly random [9].

8. Note that if the delay spread of the channel is finite, then there will be a degree of correlation between the H_k of nearby carriers.

9. As stated before, there may be a phase slope, resulting from a timing error, superimposed on whatever the channel gives us. However, if the phases resulting from the channel have a degree of randomness, then adding a phase slope will not remove that randomness.

The “received carriers” $R_k = H_k S_k$ are thus already the multiplicative combination of two random processes.

We then get a random result φ_m from integrating a frequency-shifted version of $\varphi(t)$. This result multiplies R_k in turn. Clearly, the spectrum $\Phi(f)$ of the phase noise matters.

Finally, we add up $(N-1)$ of these.

4.2. How to proceed to a general result for a flat channel

Let us simplify the problem by assuming that the channel is flat so that the H_k are all unity [10].

We can for example write:

$$\mathrm{var}(\mathrm{Re}(S_k)) = \mathrm{var}(\mathrm{Im}(S_k)) = \sigma_s^2 \text{ , say,}$$

where σ_s^2, the variance of one component of S_k, is assumed to be the same for all the carriers [11]. The mean “power” [12] per carrier is $2\sigma_s^2$. This has units of $(volts)^2$. We assume each component of S_k has zero mean.

We evaluate and sum $(N-1)$ terms, each term being a product of the form $R_k\varphi_m = S_k\varphi_m$ (given that the channel is assumed flat). Note that both φ_m and S_k are complex.

We can deduce that the real and imaginary parts of one of these $(N-1)$ terms each have variance $\sigma_s^2\sigma_m^2$. In order to do this we must: (i) account properly for the product of two complex numbers, in terms of their real and imaginary components, and (ii) note that the variance of the product of two (real) zero-mean random variables is the product of their variances [13].

Now consider the sum:

$$A_l = j\sum_{k \neq l} R_k\varphi_{k-l} = j\sum_{k \neq l} S_k\varphi_{k-l}$$

We assume that each of the $(N-1)$ terms is an independent random result [14] so that the variance of the real part of the sum is simply the sum of the variances of the real parts of the $(N-1)$ terms, i.e.:

$$\mathrm{var}(\mathrm{Re}(A_l)) = \sum_{k \neq l} \sigma_s^2\,\sigma_{k-l}^2 = \sigma_s^2 \sum_{k \neq l} \sigma_{k-l}^2$$

10. Strictly, we should say that the *magnitudes* are unity since, as noted, there may be a phase slope resulting from a timing error.
11. This assumes that all the carriers are modulated in a similar way. This is not strictly true for DVB-T. Most of the carriers are similarly modulated with data using a form of QAM, but a few “continual pilot” carriers always carry reference information, while others carry some “scattered pilot” reference information. A few “TPS” carriers are modulated using BPSK. The pilot information (continual or scattered) is transmitted with approximately 2.5 dB greater power than used for QAM data
12. In a 1-ohm system, as conventionally discussed, and ignoring the DVB-T fine detail noted above.
13. This result is given in Reference [3].
14. This will be true if either the S_k are independent or the φ_m are independent. Perhaps neither are completely independent but, since S_k and φ_m appear unconnected (unless perhaps the LO used some form of data-dependent loop), it seems reasonable that the product terms are independent.

The same is true for the imaginary component, so the thermal-noise-like "power" affecting one carrier is $2\sigma_s^2\ \sigma_m^2 \sum_{k \neq l} \sigma_{k-l}^2$ while the "power" of that carrier is $2\sigma_s^2$.

So, finally, the signal-to-noise power ratio is simply given by:

$$\frac{S}{N} = \frac{1}{\sum_{k \neq l} \varphi_{k-l}^2}$$

Remember:

The summation is over $(N-1)$ terms excluding φ_0 ;

The specific range of the summation differs from carrier to carrier (see the matrix equation of *Section 3.3.3* for clarification). We may expect the carriers in the middle of the band to suffer the greatest effect.

The channel is assumed to be flat;

All carriers are assumed to be similarly modulated with independent data.

Since we have a large number of carriers, we may apply the Central Limit Theorem to confirm that the "noise" will tend to have Gaussian characteristics [15].

4.3. How to evaluate the result for a particular noise spectrum

We now need to evaluate $\sum_{k \neq l} \sigma_{k-l}^2$ for any particular noise spectrum $\Phi(f)$.

Section 3.4.2 showed that:

$$\sigma_m^2 = \int_{-\infty}^{\infty} \operatorname{sinc}^2(f/f_u)\left|\Phi(f-mf_u)\right|^2 df$$

The limits of $\pm\infty$ are somewhat academic since, even if $\Phi(f)$ is not band-limited at source, it will become so in passing through the IF filter of any practical receiver. Since the integrand is always positive, it follows that the above result may therefore be an over-estimate.

In principle, for any known $\Phi(f)$ we may now determine the effective signal-to-noise ratio using the formula:

$$\frac{S}{N} = \frac{1}{\sum_{k \neq l} \sigma_{k-l}^2}$$

$$= \frac{1}{\sum_{k \neq l} \int_{-\infty}^{\infty} \operatorname{sinc}^2(f/f_u)\left|\Phi(f-\overline{k-l}f_u)\right|^2 df}$$

15. It will however have a strictly limited amplitude, unlike Gaussian noise which (with small probability) can take an arbitrarily large value.

Such heroic integration may be useful if you are sure that $\Phi(f)$ is imposed and immutable and you simply want to know the result. However, as written, it does not give much insight into how to modify a given LO in order to obtain a better result (or, if the performance is better than needed, how to make a cheaper one whose performance is good enough). The next section introduces a concept of *weighting functions* which gives this insight while also simplifying the calculation of ICI.

5. Derivation of weighting functions

5.1. A weighting function for common phase error

Section 3.4.1. showed that the variance of the common phase error φ_0 is given by:

$$\sigma_0^2 = \int_{-\infty}^{\infty} \mathrm{sinc}^2(f/f_u)|\Phi(f)|^2 df$$

in which the "power" spectrum of the phase noise $|\Phi(f)|^2$ is multiplied by the factor $\mathrm{sinc}^2(f/f_u)$ before being integrated. The latter factor is thus, in effect, a weighting function, so we can define:

$$W_{CPE}(f) = \mathrm{sinc}^2(f/f_u)$$

Using it, we can then write:

$$\sigma_0^2 = \int_{-\infty}^{\infty} W_{CPE}(f)|\Phi(f)|^2 df$$

Note that the weighting function and the "power" spectrum should be treated as double-sided, i.e. the integral is from $-\infty$ to $+\infty$. Alternatively, since both the power spectrum $|\Phi(f)|^2$ and the weighting function $W_{CPE}(f)$ are symmetrical about zero frequency, we could write instead:

$$\sigma_0^2 = 2\int_{0}^{\infty} W_{CPE}(f)|\Phi(f)|^2 df$$

The weighting function is illustrated in *Fig.1* (in linear form) and *Fig.2* (in dB). It is easy to see that only the low-frequency part of the phase-noise spectrum is likely to have much impact on the common phase error unless, unusually, the phase-noise "power" spectrum rises sharply with increasing frequency.

5.2. A weighting function for ICI

Consider the equation for the signal-to-ICI-noise ratio derived in *Section 4.3.*, re-written in reciprocal form:

$$\left.\frac{N}{S}\right|_{l\text{ th carrier}} = \sum_{k \neq l} \int_{-\infty}^{\infty} \mathrm{sinc}^2(f/f_u)\left|\Phi(f - \overline{k-l} f_u)\right|^2 df$$

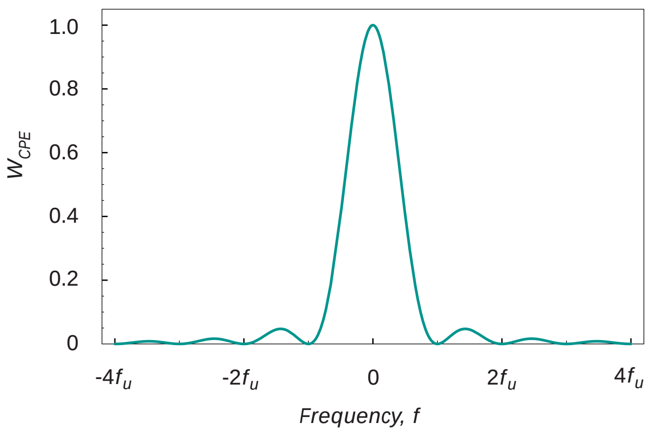


Figure 1
The weighting function for the common phase error, W_{CPE} *(f)*.

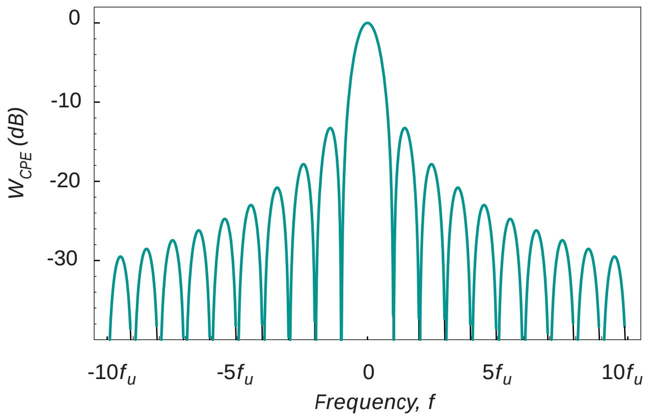


Figure 2
The weighting function for the common phase error, W_{CPE}*(f)*, in dB.

This can be considered to reflect the physical form of what is happening. The demodulator for the l th carrier in effect has a $\mathrm{sinc}^2(f/f_u)$ power frequency response (which is considered, for the purposes of explaining this equation, to be centred on zero frequency – a kind of homodyne receiver). It receives $(N-1)$ frequency-shifted [16] versions of the phase-noise spectrum, each modulated on one of the other $(N-1)$ carriers [17].

This physical view is illustrated in *Fig.3*, in which the spectra of the carriers are sketched separately so that the phase-noise sidebands can be seen.

If we perform the substitution $v = f - \overline{k-l} f_u$ then we can reformulate the equation as:

$$\left.\frac{N}{S}\right|_{l\text{ th carrier}} = \sum_{k \neq l} \int_{-\infty}^{\infty} \mathrm{sinc}^2(v/f_u + \overline{k-l}) |\Phi(v)|^2 dv$$

$$= \int_{-\infty}^{\infty} \left(\sum_{k \neq l} \mathrm{sinc}^2(v/f_u + \overline{k-l}) \right) |\Phi(v)|^2 dv$$

$$= \int_{-\infty}^{\infty} W_{ICI\;l\text{ th carrier}}(f) |\Phi(f)|^2 \, df$$

(substituting back to a more familiar variable)

where we have defined a weighting function:

$$W_{ICI\;l\text{ th carrier}}(f) = \sum_{k \neq l} \mathrm{sinc}^2(f/f_u + \overline{k-l})$$

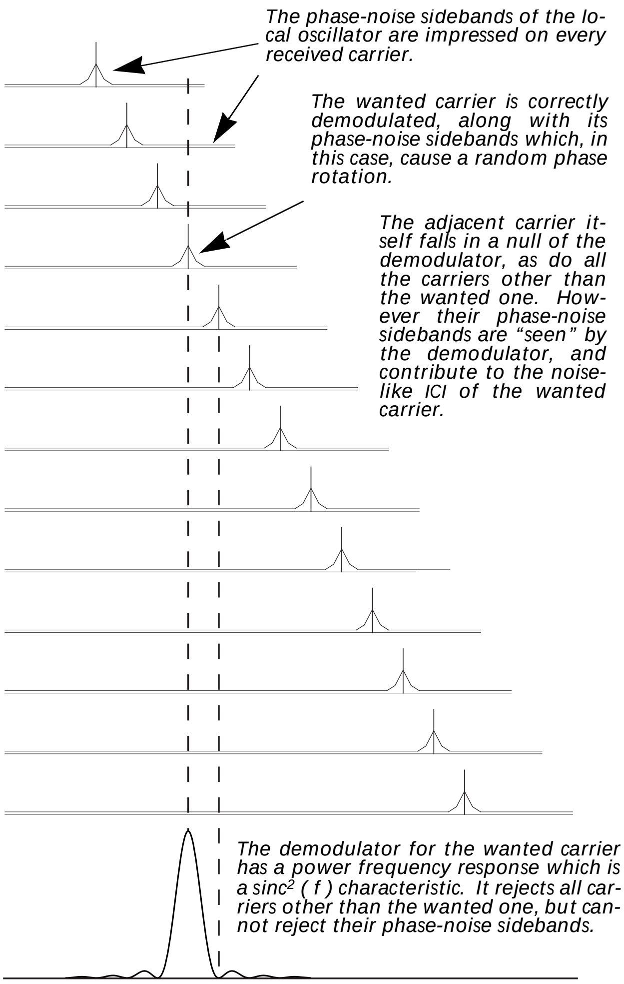


Figure 3
The occurrence of ICI – how the demodulation of one of the carriers in an OFDM ensemble is affected by the phase-noise sidebands of the others.

16. It also receives one unshifted version – but this causes the common phase error, rather than contributing to the noise-like ICI.

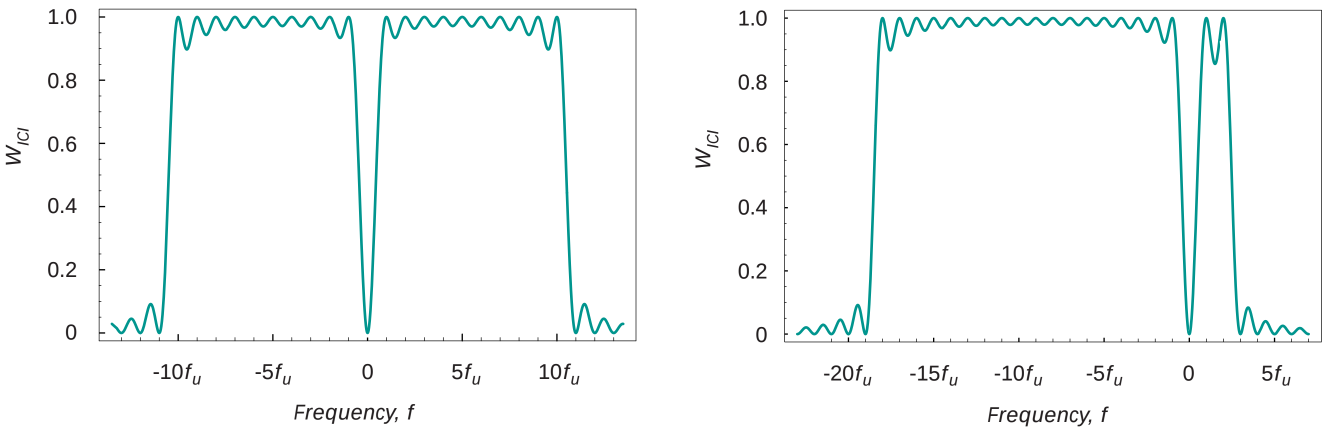


Figure 4
The ICI weighting function for the middle one of 21 carriers *(left trace)* **and the 3rd of 21 carriers** *(right trace)***.**

Note that this too is a two-sided weighting function. We shall see later that this is important as it is not symmetrical. Be reminded that the summation is over all values of the index k corresponding to an active carrier, *except* the wanted carrier l.

5.3. The form of this ICI weighting function

For a small number of carriers it is feasible to plot the weighting function directly, as shown in the arbitrary example of *Fig. 4*.

In each case we see that the weighting function takes (give or take some ripple) the form of a block of unit height but with a notch at zero frequency. This gives us a valuable clue, which we develop in the next section. Meanwhile we can note that all of the phase-noise spectrum, within a block of total width roughly Nf_u (i.e. the system bandwidth) contributes more or less equally to the ICI, *excepting* the part near zero frequency.

5.4. A simplified version of the ICI weighting function

The resemblance of the ICI weighting function to a block, with a notch cut out of it, leads us to the following development where the range of summation has been spelt out in full and, as discussed already, N is taken as the number of *active* carriers [18]:

$$\underset{l\text{ th carrier}}{W_{ICI}(f)} = \sum_{k=0,\,k\neq l}^{N-1} \operatorname{sinc}^2(f/f_u + \overline{k-l})$$

$$= \left(\sum_{k=0}^{N-1} \operatorname{sinc}^2(f/f_u + \overline{k-l})\right) - \operatorname{sinc}^2(f/f_u)$$

17. Remember as always that only *active* carriers contribute to ICI, so N in this case should be interpreted as the number of active carriers.

18. Strictly we therefore renumber the carrier indices so that k is zero for the first active carrier, and $(N-1)$ for the last.

The summation is now taken over all the active carriers and has a well-known shape. It is the power spectrum of an OFDM signal [19] which we know to be a close approximation to a rectangular block of width equal to Nf_u (remember N represents just the active carriers). In this case, the block is frequency-inverted and frequency-shifted so that the wanted carrier is at zero-frequency. *Fig. 5* shows the calculated block for all the sinc^2 functions in a block of 21.

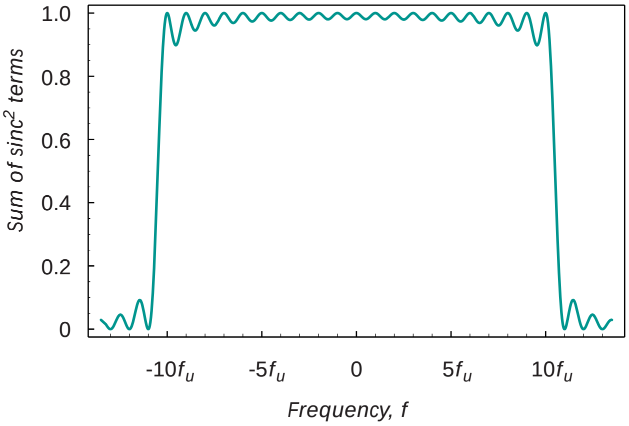


Figure 5
A group of adjacent sinc² functions (21 in this case) approximates to a block.

If we replace this slightly "wiggly" block by an idealized one, we obtain the idealized, simplified version of the weighting function $W_{ICI,\,l}(f)$ illustrated in *Fig. 6*.

5.5. A one-sided version of the ICI weighting function

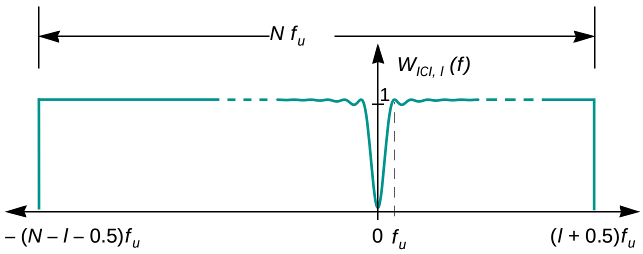


Figure 6
A simplified weighting function based on a rectangle, minus a single sinc² function.

The non-symmetrical nature of $W_{ICI,\,l}(f)$ is slightly disorientating. We are unaccustomed to spectra which are not symmetrical (that's because those of real signals are always symmetrical [20] in appearance). $W_{ICI,\,l}(f)$ is simply a mathematical convenience which we use in the calculation:

$$\left.\frac{N}{S}\right|_{l\text{ th carrier}} = \int_{-\infty}^{\infty} W_{ICI,\,l}(f)|\Phi(f)|^2 df$$

The power spectrum $|\Phi(f)|^2$ *is* symmetrical, so we can recast the problem to do the integral over positive frequencies only, as follows:

$$\left.\frac{N}{S}\right|_{l\text{ th carrier}} = \int_{0}^{\infty} 2W'_{ICI,\,l}(f)|\Phi(f)|^2 df$$

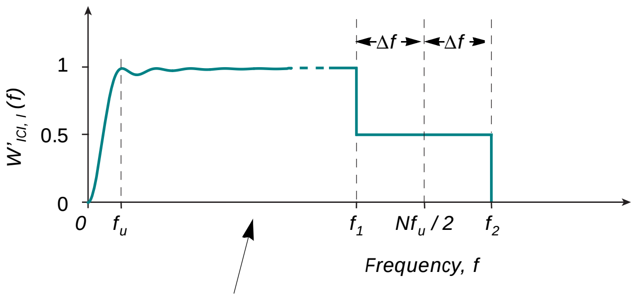
$W'_{ICI,\,l}(f)$ is now a *one*-sided weighting function made from $W_{ICI,\,l}(f)$ by folding it over the y axis and adding. The factor "2" has been slipped in so that the weighting function (in general) has a maximum amplitude of 1, as before (see *Fig. 7*).

19. Strictly, the power spectrum of an OFDM signal with no guard interval. The spectrum when there is a guard interval (as in DVB-T) has regular ripples throughout the occupied frequency band. In contrast, those visible in *Fig. 5* become insignificant (except right at the band edges) when the number of carriers increases to the thousands used in DVB-T.

20. As long as you only look at the magnitudes. The real part of the spectrum of a real signal is symmetrical while the imaginary part is antisymmetrical, i.e. $H(-f) = H^*(f)$.

The weighting function shown in *Fig. 7* is for the general case, excluding edge carriers (i.e. $l = 0$ or $l = N - 1$). Two special cases are of particular interest: a middle carrier [21] or an edge one. These are shown in *Figs. 8* and *9* respectively.

It is easy to see that if the phase-noise spectrum is flat, all carriers will suffer roughly equally for ICI. For practical spectra, with more power at lower frequencies than at high ones, the middle carrier is the worst case, with the edge-most carrier then being up to 3 dB better off. For most practical purposes it will be sufficient to take the “middle-carrier” weighting function (shown in *Fig. 8*) as representative of the performance of the whole COFDM signal.



$$f_1 = \mathrm{Min}\left\{\left(l+\frac{1}{2}\right)f_u,\ \left(N-l-\frac{1}{2}\right)f_u\right\}$$

$$f_2 = \mathrm{Max}\left\{\left(l+\frac{1}{2}\right)f_u,\ \left(N-l-\frac{1}{2}\right)f_u\right\}$$

$$\Delta f = \left|\frac{(N-1)}{2} - l\right| f_u$$

Figure 7
A simplified one-sided weighting function for ICI into the l^{th} carrier.

6. Application of the weighting functions

The weighting functions are of great help in visualizing the impact of local-oscillator phase noise on the operation of an OFDM system. Beyond promoting general understanding, they have two very real applications. The first is to help system designers appreciate what kind of constraints the oscillator phase noise imposes on the choice of OFDM parameters. (The analysis presented in this article was originally produced for just this purpose: to inform the process of choosing the parameters for DVB-T.) The second is to assess whether particular phase-noise spectra are satisfactory for a particular application – and if not, to give some pointers to the designer as to where attention should be focused in trying to improve the performance.

6.1. The implications of changing the FFT size

This very question arose when the DVB-T Specification was being written.

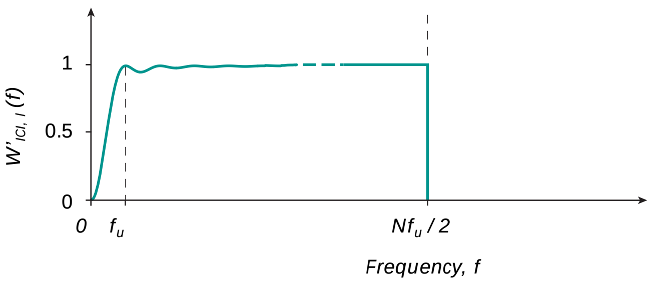


Figure 8
The simplified one-sided weighting function for ICI into the middle carrier.

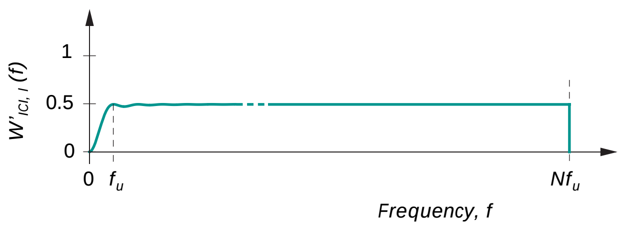


Figure 9
The simplified one-sided weighting function for ICI into an edge carrier.

21. Strictly, a middle carrier only exists where (as in DVB-T) the total number of active carriers is odd!

6.1.1. The effect on ICI

The ICI weighting function explains clearly what happens to the amount of ICI if the number of carriers is changed while adjusting their spacing to keep the bandwidth constant – corresponding to changing the FFT size with a constant clock frequency. In this case the weighting function remains mostly unchanged. The only change is in the region near zero frequency since, as N increases, f_u decreases and thus the width of the notch from 0 to f_u decreases too.

If the phase-noise spectrum is broadly flat in this region and some way beyond it, the ICI will increase only slowly as N increases. Only if the amplitude of the phase-noise spectrum falls quickly with increasing f in this region will the ICI increase very significantly.

The exact *shape* of the phase-noise spectrum, and the level of its lower-frequency part relative to its *noise floor,* are thus crucial in determining exactly what will happen. No general rule of proportionality (such as "an x dB increase in ICI for each doubling of the FFT size") can be deduced. That this should be the case is easily confirmed, since it is obvious that for the case of a flat phase-noise spectrum the amount of ICI is virtually independent of the FFT size.

6.1.2. The effect on CPE

In contrast, CPE arises predominantly from the low-frequency components of the phase-noise spectrum, up to f_u or so. Thus, if the phase-noise spectrum is broad, CPE actually *decreases* as N increases and thus f_u decreases. If the phase-noise spectrum is predominantly narrow, CPE would remain roughly constant as N varies. Of course, since all carriers suffer the same CPE during any given symbol, it is in any case possible to correct for CPE by measuring it, for example by using some carriers devoted to reference information. The so-called *continual pilots* of DVB-T can be used for this purpose.

6.2. Applications to the computation of CPE and ICI

If we have measurements of a particular oscillator, or just a hypothetical mathematical model of one, then we can apply the weighting functions to evaluate how much CPE and ICI will arise. The Author has done this in two ways.

One method – particularly well-suited where *measured* data are available – is to use a computer spreadsheet to perform the integrations numerically. Measuring the phase noise over the whole spectrum is not always easy in practice – especially in the critical lower-frequency region. Some care is needed to ensure that sufficient measurements have been made in this frequency range to make the calculation valid. The Author's spreadsheet performs upper and lower bound calculations of the amounts of CPE and ICI, given the data supplied. If these bounds differ significantly, data about more spot frequencies must be supplied.

The other method uses a tool such as *Mathematica* [22] to perform the integration. This is particularly useful when investigating hypothetical mathematical models of oscillators. (The spreadsheet can also be used for this, but it is less easy to accommodate a range of arbitrary model types.)

22. Wolfram Research, Inc.

In either case it is easy (and very quick) to observe the effects of varying the *system parameters* (the number of carriers and their spacing) for a given spectrum or, for model-based spectrum descriptions, of varying the *model parameters* (e.g. the slopes or level of the noise floor).

In practice we can measure the amount of ICI indirectly by measuring the loss of noise margin that it causes in an experimental receiver. If the spectrum of the phase noise is also measured, we can then predict the amount of ICI. Good agreement has been obtained, within the limits of experimental error of both measurements, for some prototype tuners, thus giving some experimental support for the theory expounded in this article.

6.3. What determines the phase-noise spectrum?

The phase noise that matters is that present at the input to the receiver FFT. It arises from all the frequency-shifting oscillators before this point in the path from the IFFT in the modulator. Of course, in a broadcast system, the transmission chain should normally be specified so that the local oscillator(s) in the receiver are the dominant cause.

The spectrum is not, however, necessarily that of the local oscillator by itself. The LO in a typical receiver has to be tunable over a wide range of frequencies (e.g. the UHF band for DVB-T). Free-running oscillators of such tuning range will normally be incorporated in a PLL in order to provide easy channel selection and improved frequency-setting accuracy [23]. The PLL locks the oscillator frequency to that of a fixed reference oscillator (often a crystal oscillator) in an integer ratio, say m:n. In the process the phase-noise spectrum of the tunable oscillator is changed. At lower frequencies, within and just above the bandwidth of the PLL, the resulting spectrum is determined by the reference-oscillator noise and imperfections (usually noise-like) in the phase comparator and divider chain. The dominant effect is usually to “cap” the phase noise at a plateau in this region.

At very low frequencies, the noise of the PLL reference oscillator will cause the phase noise to increase once again as frequency decreases. Fortunately we do not normally need to worry about this because another receiver mechanism comes to our aid. Even a PLL tuner does not satisfy the requirements of DVB-T for frequency stability and accuracy [4] so a system of automatic frequency control (AFC) must be provided. Various methods are possible. Measurements are made of the signal, either before or after the FFT in the receiver, from which an AFC signal is derived. This signal then controls one of the analogue LOs or, alternatively, a digital frequency shifter prior to the FFT. Whatever the method, the AFC loop will tend to serve as a high-pass filter, suppressing or reducing the tuning error and also reducing the very-low-frequency phase noise within its bandwidth [24].

7. Practical illustrations

We can breathe life into the rather dull equations derived in this article by observing the behaviour of a real receiver, the DVB-T demodulator built by BBC R&D. Fortunately, as it has

23. Indeed, PLLs are virtually universal even in conventional (analogue) television receivers whose tolerance to mis-tuning is greater than DVB-T.

24. The AFC bandwidth is likely to be rather lower than that of the PLL. Incidentally, with some receiver configurations, it is not appropriate to consider the PLL and AFC loop as separate systems since they interact. This interaction should be taken into account in system design to ensure stability, as well as a desirable characteristic for phase noise.

been designed as an experimental tool, it is possible to explore some of the internal workings. In particular, a constellation display (the plot on an Argand diagram of the complex values X_l output by the demodulators) can be provided using an oscilloscope. In the examples we shall show, the constellations for all the carriers are superimposed. In every case, 64-QAM modulation is used with the "2K" (1705 carriers) mode of DVB-T.

By inserting up- and down-frequency conversions between the modulator and demodulator, we can introduce phase noise in a controlled way by modulating the phase of any of the conversion oscillators. Obviously, in this case the oscillators must have sufficiently low phase noise themselves for it to be negligible. This is partially verified by checking for a "clean" constellation with no phase modulation intentionally added (see *Fig. 10*). A more thorough test would be to measure the modem performance with and without the frequency conversions present.

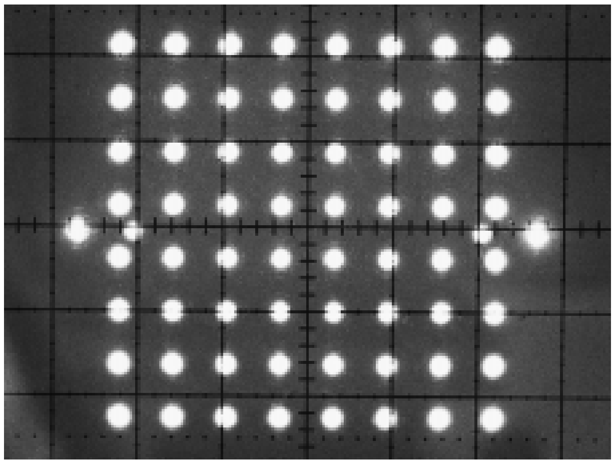


Figure 10
The clean demodulated 64-QAM constellation which is obtained when no phase modulation is applied to any frequency-conversion stage.

The weighting functions tell us that in principle it should be possible (in the laboratory!) to illustrate the CPE and ICI effects almost independently by choosing to inject phase noise with particular spectral content. In particular, if the phase noise is predominantly low frequency, then there should be little ICI and mostly CPE. Unfortunately, it can be difficult to demonstrate this directly. Although CPE will be caused in this situation, other receiver circuits will reduce its amplitude:

as noted already, AFC action reduces low-frequency phase noise falling within the AFC loop bandwidth (thus reducing both CPE and the small amount of ICI that LF phase noise causes);

low-frequency CPE effects which remain are further reduced by the action of the channel equalizer (which tracks the effects of arbitrary gain and phase responses of the channel at each carrier frequency).

It can thus be difficult to show CPE purely in isolation.

This might suggest that CPE correction is redundant, but this is not necessarily so. The AFC bandwidth is small, of the order of a few tens of Hz, while the channel equalizer for DVB-T fundamentally cannot have a temporal bandwidth exceeding one-quarter of the OFDM symbol rate [25] (and in practice it is somewhat less). As the CPE weighting function tells us that the phase-noise spectrum up to about the symbol rate is responsible for the majority of CPE effects, it follows that the spectrum between say 0.15 – 1.1 times symbol rate will cause CPE which would not be reduced significantly by AFC or channel-equalizer action, but could be corrected by a CPE-correction circuit. Whether such a circuit is worth having thus depends on how much phase noise is present in this part of the spectrum with the local oscillator in use. These frequency components of phase noise also cause some ICI – the more so as frequency increases within this range – potentially confusing any demonstration.

25. Strictly, this assumes that the equalizer is simultaneously required to deal with the channel delay spread permitted by the greatest guard-interval fraction of DVB-T (i.e. one quarter). It is theoretically possible to have an equalizer with a greater temporal bandwidth if the need to deal with channel delay spread (and hence frequency selectivity) is removed, but this would not be a practical proposition.

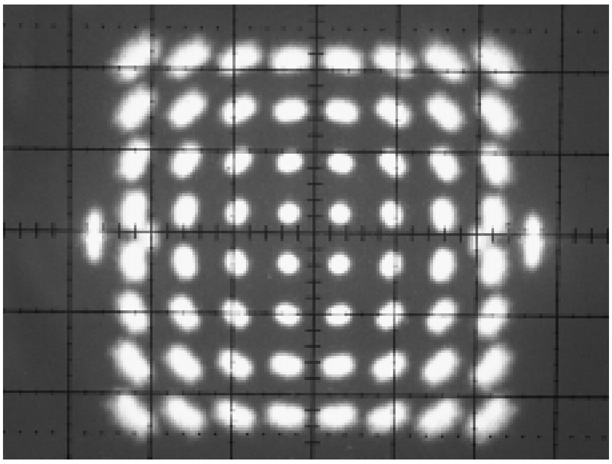


Figure 11
The 64-QAM constellation when 600 Hz phase modulation is applied, showing the rotation of CPE is dominant.

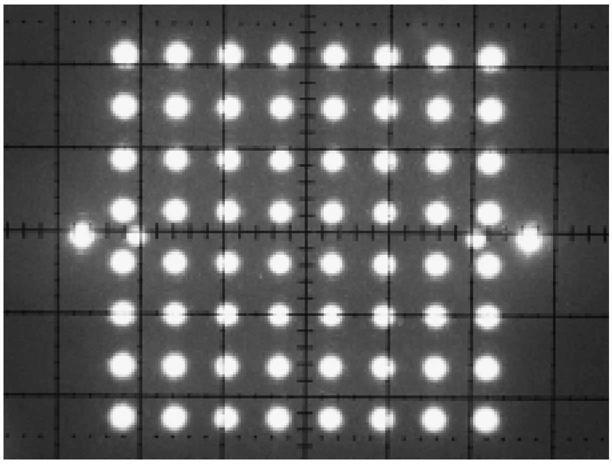


Figure 12
The constellation with 600 Hz phase modulation (as in *Fig. 11*) but with CPE correction enabled, thereby cancelling the rotation.

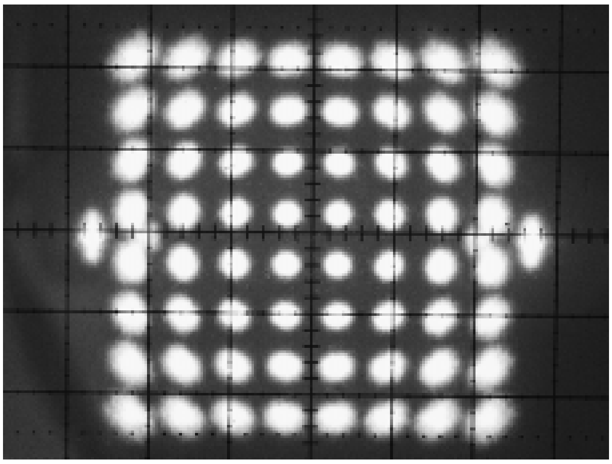


Figure 13
The constellation when 2 kHz phase modulation is applied, again showing the rotation of CPE.

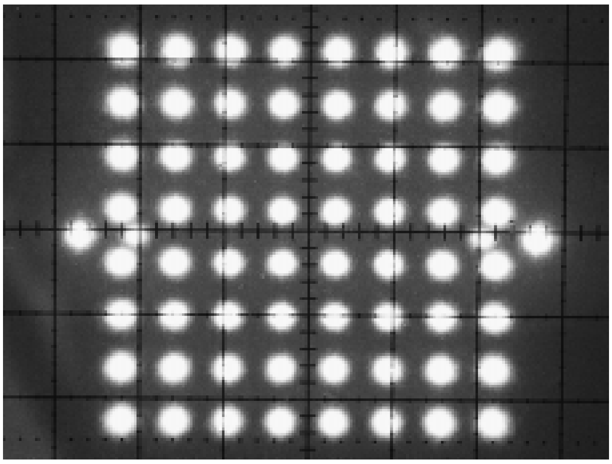


Figure 14
The constellation with 2 kHz phase modulation (as in *Fig. 13*) but with CPE correction enabled, showing that some "fuzziness" remains.

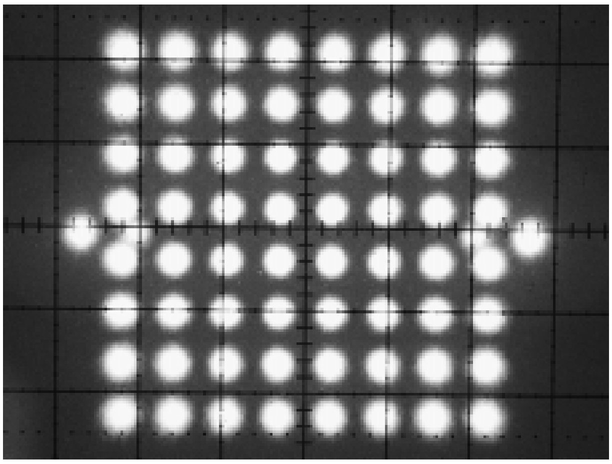


Figure 15
The constellation with 5 kHz phase modulation and CPE correction off, showing the "fuzziness" of the ICI but negligible rotation from CPE.

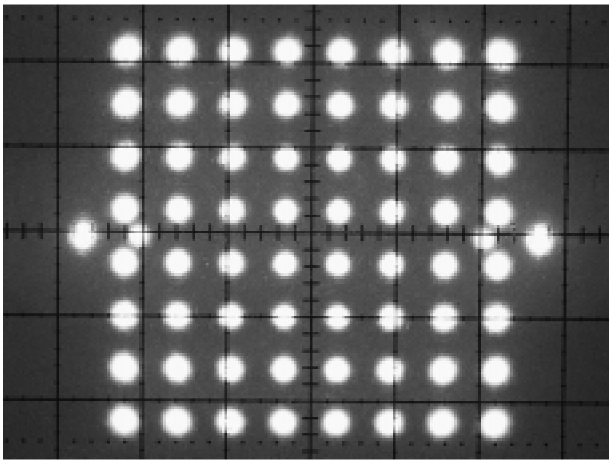


Figure 16
The constellation when using a prototype example of a domestic tuner. In this particular case the CPE is negligible but some ICI is visible.

With all this borne in mind we now examine some examples. The effect of injecting low-frequency phase disturbance (in the form of a sinusoidal phase modulation at 600 Hz, with a deviation of 0.06 rad) is shown in *Fig. 11*. It is clear that the constellation is rotated about its nominal position.

Fig. 12 shows the same conditions as *Fig. 11*, except that the receiver's common-phase-error correction circuitry has been enabled. This shows clearly that CPE can be removed.

If the phase-modulation frequency is increased to 2 kHz, with the same deviation, more ICI is caused as well. *Figs. 13* and *14* show the constellation without and with CPE correction, *Fig. 14* showing that some ICI remains after the CPE has been corrected.

Fig. 15 shows the result when the phase-modulation frequency is increased once more to 5 kHz, with the CPE correction off. In this case, there is essentially no CPE, so enabling the CPE correction makes no difference and is not illustrated. The characteristic noise-like fuzziness of the ICI is clear.

Finally, *Fig. 16* shows the constellation obtained from a fairly good prototype example of a "domestic" UHF-to-IF tuner used in conjunction with the BBC demodulator. In this case the level of phase noise is sufficient to cause visible fuzziness from ICI [26], but the phase-noise spectrum is such that there is very little CPE, and thus enabling CPE correction (not shown) makes no *visible* difference. A small difference in performance when correction is enabled can just be *measured* in some circumstances. However, other prototype tuners have been tested where CPE correction was visibly worth having, so each case must be treated on its merits.

8. Conclusions

Analysis of the effects of phase noise on an OFDM signal, such as is used in the DVB-T system for digital terrestrial television, shows that two effects can be distinguished:

> A *common phase error* (CPE) arises simultaneously on all carriers – i.e. the signal constellation within a given symbol is rotated by the same angle for all carriers. This effect can be corrected by using "pilots" (reference information) within the same symbol, as provided within DVB-T.

> There is also a "thermal-noise-like" addition (i.e. a blurring rather than rotation of the constellation), which is different for all carriers. This effect amounts to a form of *inter-carrier interference* (ICI). It can be interpreted as a loss of orthogonality.

The CPE and ICI effects can each be quantified by simply applying *weighting functions* to the phase-noise spectrum. This aids both visualization and computation.

For CPE, the weighting function is $W_{CPE}(f) = \mathrm{sinc}^2(f/f_u)$.

This means that only the low-frequency part of the phase-noise spectrum (roughly from zero-frequency up to the carrier spacing) is normally of importance for CPE as long as the spectrum of the phase noise does not exhibit an (uncommon) *increase* in power with frequency.

26. Note that the fuzziness of the constellation in this example is indeed caused by the phase noise; the RF signal-to-noise ratio was high enough that pure thermal noise was negligible.

The weighting function for ICI is essentially the complement of that for CPE. Thus, all parts of the phase-noise spectrum *except* the lowest frequencies contribute with equal force to ICI. The "weight" of the very low frequencies is smaller, although their impact may sometimes still dominate by virtue of their usually-greater amplitude. This simple insight was not previously obvious.

The weighting functions can be used to assess the extent to which the phase noise of particular oscillators would degrade the performance of a COFDM system. More importantly, should the noise of a given oscillator be excessive they guide the designer to those parts of its phase-noise spectrum most in need of improvement.

The question "*What happens when the number of carriers in a COFDM system is changed, while the system bandwidth is held constant*" can be answered:

> The common phase error decreases as the carrier spacing decreases, if the phase-noise spectrum is broad, or remains constant if the phase-noise spectrum is predominantly narrow.
>
> Inter-carrier interference (more important for digital terrestrial television since, unlike CPE, it cannot be corrected for) remains roughly constant as the carrier spacing changes, provided that the LO phase-noise spectrum is predominantly broad. Only if the spectrum is predominantly narrow, with a broadband noise floor whose level is low in comparison, does the "noise-like" ICI get significantly worse as the carrier spacing decreases.

Practical oscillators often have spectra which lie between these extremes. Their behaviour must be evaluated for any particular case. In a DVB-T receiver, proper account should be taken of the way in which the phase-noise spectrum is shaped by the action of PLL and AFC loops.

Acknowledgements

The Author gratefully acknowledges the assistance of his colleagues in BBC Research and Development in preparing this document, and offers his special thanks to Adrian Robinson for providing useful mathematical hints, and John Salter for setting up the experimental arrangement which yielded the illustrations shown in *Section 7*.

Jonathan Stott *studied Engineering and Electrical Sciences at Churchill College, Cambridge University, graduating with Distinction in 1972. He then joined the BBC Research Department (now BBC R&D), where he is a Project Manager in the Spectrum Planning Group. Most of his career, from his very first assignment, has been taken up with applying digital techniques to broadcasting, primarily to television signals.*

In recent years Mr Stott has been deeply involved with the development and introduction of digital terrestrial television, starting with participation in the European RACE dTTb project. This led to his becoming a member of the Task Force on System Comparison which, under the leadership of Lis Grete Møller from Denmark, drew up the DVB-T specification for modulation and coding of digital terrestrial television. He currently leads the theoretical and simulation work within the BBC R&D team that is at the forefront of digital television developments in Europe.

Abbreviations

64-QAM	64-state quadrature amplitude modulation	**ETS**	European Telecommunication Standard
AFC	Automatic frequency control	**FFT**	Fast Fourier transform
BPSK	Binary phase-shift keying	**ICI**	Inter-carrier interference
COFDM	Coded orthogonal frequency division multiplex	**IFFT**	Inverse fast Fourier transform
CPE	Common phase error	**LO**	Local oscillator
DFT	Discrete Fourier transform	**OFDM**	Orthogonal frequency division multiplex
dTTb	digital Terrestrial Television broadcasting	**PLL**	Phase-locked loop
DVB	Digital Video Broadcasting	**QAM**	Quadrature amplitude modulation
DVB-T	Digital Video Broadcasting – Terrestrial	**RACE**	R&D in Advanced Communications Technologies in Europe
		TPS	Transmission-parameter signalling

Bibliography

[1] ETS 300 744 (1997): **Digital broadcasting systems for television, sound and data services; framing structure, channel coding and modulation for digital terrestrial television.**
ETSI, 1997.
http://www.etsi.fr

[2] J.H. STOTT: **The DVB terrestrial (DVB-T) specification and its implementation in a practical modem.**
Proceedings of the 1996 International Broadcasting Convention, IEE Conference Publication No. 428, pp 255-260, September 1996.

[3] R.V. HOGG, A.T. CRAIG: **Introduction to mathematical statistics**
Macmillan, New York, 1978.

[4] J.H. STOTT: **The effects of frequency errors in OFDM**
BBC Research and Development Report No. RD 1995/15.
http://www.bbc.co.uk/rd/pubs/reports/1995_15.html