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The Effects of Skewed Distributions on the Performance of Variable Sample Size $\bar{X}$ Chart

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Abstract—**The variable sample size (VSS) $\bar{X}$ chart is one of the adaptive control charts that gains prestige in the field of statistical process control during the last decade. Traditionally, the design of the VSS $\bar{X}$ chart is based on the assumption of normally distributed data or measurements. However, in many real-life applications, the normality assumption may be violated. This paper investigates the effects of skewed distributions on the performance of the VSS $\bar{X}$ chart. Two VSS schemes are considered in this paper, i.e. (i) the small sample size (n_S) or (ii) the large sample size (n_L), is predetermined for the first sample (n_1). Monte Carlo simulation is adopted to evaluate the run-length performances of these two VSS $\bar{X}$ schemes for different levels of skewness corresponding to Weibull, lognormal and gamma distributions. The results show that the in-control average run lengths for the VSS $\bar{X}$ chart with $n_1 = n_S$ are closer to the desired value and have a lower false alarm rate compared to that of the VSS $\bar{X}$ chart with $n_1 = n_L$.**

Index Terms—**average run length, skewed distribution, standard deviation of the run length, statistical process control, variable sample size $\bar{X}$ chart**

I. Introduction

Statistical Process Control (SPC), which is a tool of quality control, adopts statistical methods to control and monitor manufacturing and services processes. This will ensure efficient process and high quality products with less scrap being produced. For example, SPC is successfully implemented in short-run job-shop manufacturing environment, semiconductor manufacturing and improving suppliers' process [1]. Control chart is one of the key tools used in SPC.

Variable Sample Size (VSS) chart is one of the adaptive charts that allows sample size n to vary in each sampling interval. Prabhu *et al.* [2] and Costa [3] are the pioneers in proposing the VSS $\bar{X}$ chart. Markov chain approach is employed to study the run-length properties of the VSS $\bar{X}$ chart in both papers. Costa [3] showed that the VSS $\bar{X}$ chart is superior in detecting certain ranges of shifts compared to the Shewhart $\bar{X}$ chart, variable sampling interval (VSI) $\bar{X}$ chart, $\bar{X}$ chart with supplementary run rules, exponentially weighted moving average (EWMA) chart and cumulative sum (CUSUM) chart. The VSS weighted loss function CUSUM chart proposed by Zhang and Wu [4], is favorable to achieve a high capability of detecting process variations. Kooli and Limam [5] claimed that greater cost savings achieved for the VSS *np* chart compared to the static charts. Also, the VSS EWMA and VSS EWMA median charts suggested by Amiri *et al.* [6] and Zhang and Song [7], respectively, outperform the fixed-sample-size EWMA chart. To monitor the Coefficient of Variation (CV), the VSS CV chart proposed by Castagliola *et al.* [8] outperforms the Shewhart CV, synthetic CV, and VSI CV charts for some ranges of shifts in the CV. By means of a linearly covariate error model, Hu *et al.* [9] found that the VSS $\bar{X}$ chart is significantly affected by measurement errors. Recently, Teoh *et al.* [10] optimally designed the VSS $\bar{X}$ chart based on Median Run Length (MRL) and expected MRL.

To date, the VSS $\bar{X}$ chart is designed based on the assumption that the distribution is normal. In many real applications, this assumption may not be true for the processes in reliability engineering, automobiles, semiconductor, cutting tool wear and mechanical [11], [12]. A control chart constructed with normality assumption, produces a higher false alarm rate in such a skewed population. This false alarm rate increases as the skewness increases [13]. This is due to the difference of the variability pattern between the normal and skewed distributions. From the economic point of view, Hsieh and Chen [14] found that the cost saving of a process decreases as the skewness increases. In such a situation, practitioners' confidence in using control charts for process monitoring will decrease. Therefore, the aim of this paper is to investigate the effects of skewed distributions on the performance of the VSS $\bar{X}$ chart. Many researchers, to name a few, Chang and Bai [15], Chen and Cheng [16], Khoo *et al.* [17], Teh and Khoo [18], and Teoh *et al.* [19] have contributed to the area of control charts for skewed populations.

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This paper is organized as follows. Section II demonstrates the operation of the VSS $\bar{X}$ chart and its run-length properties. Section III describes the statistical properties of the Weibull, lognormal and gamma distributions. The in-control and out-of-control performances of the VSS $\bar{X}$ chart under these three skewed distributions are evaluated in Section IV. Concluding remarks are drawn in Section V.

II. THE VARIABLE SAMPLE SIZE $\bar{X}$ CHART

Assume that the quality characteristic X follows an independent normal $N(\mu_0,\sigma_0^2)$ distribution, where μ_0 and σ_0^2 are the in-control mean and variance, respectively. The ith sample statistic is equal to

$$Z_i = \frac{(\bar{X}_i - \mu_0)\sqrt{n_i}}{\sigma_0} \sim N\left(\delta\sqrt{n_i},\, 1\right) \tag{1}$$

where $\bar{X}_i = \sum_{j=1}^{n_i} X_{i,j} / n_i$, for $i = 1, 2, \cdots$, $j = 1, 2, \cdots, n_i$ and $n_i \in \{n_S, n_L\}$. Here, n_S and n_L denote the small and large sample sizes, respectively. Also, δ in (1) refers to the magnitude of mean shifts in multiples of standard deviation units. When the process is in-control (δ = 0), Z_i is a standard normal N (0, 1) distribution.

Fig. 1 shows the schematic representation of the VSS $\bar{X}$ chart's operation. In Fig. 1, W (>0) and $K(\geq W)$ are the warning and control limits, respectively. There are three main regions for the VSS $\bar{X}$ chart, i.e. the out-of-control region $(I_{\text{ooc}} \in \{(-\infty,\ -K) \cup (K,\ \infty)\})$, the warning region $(I_L \in \{[-K,\ -W) \cup (W,\ K]\})$, and the central region $(I_S \in [-W,\ W])$.

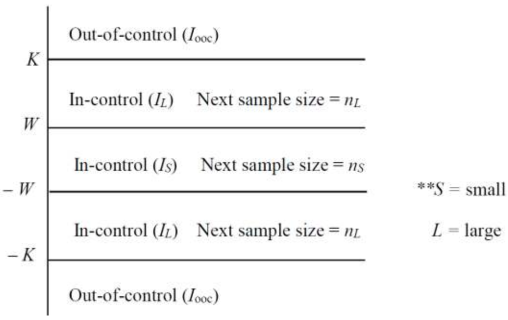

Fig. 1. Schematic representative of the VSS $\bar{X}$ chart's operation.

By referring to Fig. 1, the VSS $\bar{X}$ chart can be implemented as follows:

(i) Take a sample of size n_i.

(ii) Compute the ith sample statistic as in (1).

(iii) Declare the process as in-control if $Z_i \in I_S$; thus $n_{i+1} = n_S$ is the size for the next sample.

(iv) Declare the process as in-control if $Z_i \in I_L$; thus $n_{i+1} = n_L$ is the size for the next sample.

(v) Declare the process as out-of-control if $Z_i \in I_{\text{ooc}}$, then investigate and eliminate potential assignable cause(s).

The Markov chain approach is used to characterize the run-length properties of the VSS $\bar{X}$ chart when the underlying distribution is normal. The matrix of transient probabilities $\mathbf{Q}$ for the VSS $\bar{X}$ chart is [3]

$$\mathbf{Q} = \begin{pmatrix} p_S(n_S) & p_L(n_S) \\ p_S(n_L) & p_L(n_L) \end{pmatrix} \tag{2}$$

where the probabilities $p_S(n_i)$ and $p_L(n_i)$ are equal to

$$p_S(n_i) = \Phi\left(W - \delta\sqrt{n_i}\right) - \Phi\left(-W - \delta\sqrt{n_i}\right) \tag{3}$$

and

$$p_L(n_i) = \Phi\left(-W - \delta\sqrt{n_i}\right) - \Phi\left(-K - \delta\sqrt{n_i}\right) + \Phi\left(K - \delta\sqrt{n_i}\right) - \Phi\left(W - \delta\sqrt{n_i}\right) \tag{4}$$

respectively. Here, $\Phi(\cdot)$ in (3) and (4), is the cumulative distribution function of N (0, 1).

The average run length (ARL) and the standard deviation of the run length (SDRL) can be obtained as

$$\text{ARL} = \mathbf{q}^T\left(\mathbf{I} - \mathbf{Q}\right)^{-1}\mathbf{1} \tag{5}$$

and

$$\text{SDRL} = \sqrt{2\mathbf{q}^T\left(\mathbf{I} - \mathbf{Q}\right)^{-2}\mathbf{Q}\mathbf{1} - \text{ARL}^2 + \text{ARL}} \tag{6}$$

respectively, where $\mathbf{q}$ is the initial probability vector, $\mathbf{I}$ is the identity matrix and $\mathbf{1}$ is the column vector of ones. If $n_1 = n_S$, then $\mathbf{q} = (1,\ 0)^T$; while if $n_1 = n_L$, then $\mathbf{q} = (0,\ 1)^T$. Here, n_1 is the size of the first sample. The average sample size (ASS) of the VSS $\bar{X}$ chart corresponds to a process functioning over an infinite horizon. Castagliola *et al.* [8] showed that the ASS is equal to

$$\text{ASS} = (n_S,\ n_L,\ n_1)\mathbf{R}^{-1}\begin{pmatrix} \mathbf{q} \\ 0 \end{pmatrix} \tag{7}$$

where $n_1 \in \{n_S,\ n_L\}$, matrix $\mathbf{R}$ for $n_1 = n_S$ or $n_1 = n_L$ can be obtained as

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ p_L(n_S) & p_L(n_L) - 1 & 0 \\ 1 - p_S(n_S) - p_L(n_S) & 1 - p_S(n_L) - p_L(n_L) & -1 \end{pmatrix} \tag{8}$$

or

$$\mathbf{R} = \begin{pmatrix} p_S(n_S) - 1 & p_S(n_L) & 0 \\ 1 & 1 & 1 \\ 1 - p_S(n_S) - p_L(n_S) & 1 - p_S(n_L) - p_L(n_L) & -1 \end{pmatrix} \tag{9}$$

respectively.

III. STATISTICAL PROPERTIES OF THE WEIBULL, LOGNORMAL AND GAMMA DISTRIBUTIONS

Since Weibull, lognormal and gamma distributions provide a wide variety of shapes from approximately symmetric to highly skewed [20], they are considered in this study. Let $\lambda > 0$ and $\beta > 0$ be the scale and shape parameters, respectively, for the Weibull distribution. Prabhakar et al. [21] demonstrated that the skewness (γ) for the Weibull distribution is equal to

$$\gamma = \frac{\Gamma\left(1+\frac{3}{\beta}\right) - 3\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{2}{\beta}\right) + 2\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^3}{\left[\Gamma\left(1+\frac{2}{\beta}\right) - \left\{\Gamma\left(1+\frac{1}{\beta}\right)\right\}^2\right]^{3/2}} \quad (10)$$

where $\Gamma(\cdot)$ is the gamma function. For convenience, we choose $\lambda = 1$ throughout this paper. This is because the skewness in (10) only depends on β. The in-control mean ($\mu_{W,0}$) and standard deviation ($\sigma_{W,0}$) for Weibull distribution are acquired as [17]

$$\mu_{W,0} = \Gamma(1+1/\beta) \quad (11)$$

and

$$\sigma_{W,0} = \sqrt{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2} \quad (12)$$

respectively.

Let θ and σ_{LN} be the location and scale parameters, respectively, for the lognormal distribution. Blackwood [22] showed that the skewness (γ) for the lognormal distribution is obtained as

$$\gamma = \left(e^{\sigma_{\text{LN}}^2} + 2\right)\sqrt{e^{\sigma_{\text{LN}}^2} - 1} \quad (13)$$

Since the skewness in (13) does not involve θ, $\theta = 0$ is set throughout this paper. The $\mu_{W,0}$ and $\sigma_{W,0}$ for lognormal distribution are acquired as [17]

$$\mu_{W,0} = e^{\frac{1}{2}\sigma_{\text{LN}}^2} \quad (14)$$

and

$$\sigma_{W,0} = \sqrt{e^{\sigma_{\text{LN}}^2}\left(e^{\sigma_{\text{LN}}^2} - 1\right)} \quad (15)$$

respectively.

Let α be the shape parameter for the gamma distribution. By referring to Bowman and Shenton [23], the skewness (γ) for the gamma distribution is

$$\gamma = \frac{2}{\sqrt{\alpha}} \quad (16)$$

The $\mu_{W,0}$ and $\sigma_{W,0}$ for gamma distribution are acquired as [17]

$$\mu_{W,0} = \alpha \quad (17)$$

and

$$\sigma_{W,0} = \sqrt{\alpha} \quad (18)$$

respectively.

IV. Performance Studies

Table I, Table II, and Table III investigate the in-control ARL (ARL_0) and in-control SDRL (SDRL_0) performances of the VSS $\bar{X}$ chart with $n_1 = n_S$ and $n_1 = n_L$, when the underlying distributions are Weibull, lognormal, and gamma, respectively. The chart's parameters (n_S, n_L, W, K) of the VSS $\bar{X}$ chart which are optimized under normal distribution, are applied here to simulate the ARLs and SDRLs of the VSS $\bar{X}$ chart under Weibull, lognormal, and gamma distributions. Monte-Carlo simulation programs written in Statistical Analysis System (SAS) software are employed to obtain these ARLs and SDRLs.

TABLE I: THE (ARL_0, SDRL_0) VALUES OF THE VSS $\bar{X}$ CHART WHEN THE UNDERLYING DISTRIBUTION IS WEIBULL

β	γ	$n_1 = n_S$	$n_1 = n_L$
		(ARL_0, SDRL_0)	(ARL_0, SDRL_0)
3.6024	0.0	(266.48, 265.30)	(320.67, 317.52)
2.2156	0.5	(247.26, 244.74)	(223.86, 221.86)
1.5639	1.0	(194.89, 193.19)	(131.29, 130.61)
1.2111	1.5	(148.49, 148.62)	(91.44, 90.48)
1.0000	2.0	(116.42, 114.93)	(71.35, 70.40)
0.8632	2.5	(95.93, 95.88)	(59.57, 58.75)
0.7686	3.0	(82.63, 81.72)	(53.13, 52.24)

TABLE II: THE (ARL_0, SDRL_0) VALUES OF THE VSS $\bar{X}$ CHART WHEN THE UNDERLYING DISTRIBUTION IS LOGNORMAL

σ_{LN}	γ	$n_1 = n_S$	$n_1 = n_L$
		(ARL_0, SDRL_0)	(ARL_0, SDRL_0)
0.0003	0.0	(249.40, 247.85)	(248.18, 248.77)
0.1641	0.5	(227.82, 226.90)	(188.72, 185.90)
0.3143	1.0	(181.38, 181.49)	(123.74, 122.11)
0.4435	1.5	(140.42, 140.33)	(91.14, 90.43)
0.5514	2.0	(112.69, 111.79)	(75.64, 74.43)
0.6409	2.5	(95.05, 94.65)	(66.67, 65.63)
0.7156	3.0	(84.38, 84.07)	(61.02, 59.91)

TABLE III: THE (ARL_0, SDRL_0) VALUES OF THE VSS $\bar{X}$ CHART WHEN THE UNDERLYING DISTRIBUTION IS GAMMA

α	γ	$n_1 = n_S$	$n_1 = n_L$
		(ARL_0, SDRL_0)	(ARL_0, SDRL_0)
40000	0.0	(249.54, 250.22)	(250.82, 249.23)
16.0000	0.5	(231.83, 230.23)	(194.49, 193.14)
4.0000	1.0	(189.74, 188.42)	(125.47, 124.58)
1.7778	1.5	(147.26, 146.33)	(91.23, 90.31)
1.0000	2.0	(116.42, 114.93)	(71.35, 70.40)
0.6400	2.5	(95.69, 95.04)	(58.36, 57.20)
0.4444	3.0	(82.60, 81.67)	(51.33, 50.23)

By means of the formulae shown in Section II, the chart's parameters (n_S, n_L, W, K) are obtained by minimizing the out-of-control ARL (ARL_1) at the desired mean shift $\delta_{\text{opt}} = 1.0$, subject to the desired ARL_0=250 and desired in-control ASS = 10. From the optimization programs written in ScicosLab, we obtain (n_S, n_L, W, K) =(9, 22, 1.744, 2.878) and (2, 31, 1.096, 2.878) for the VSS $\bar{X}$ chart with $n_1 = n_S$ and $n_1 = n_L$, respectively. The corresponding (ARL_0, SDRL_0)=(250.0, 249.50) are obtained for both the VSS schemes under normal distribution. These chart's parameters (n_S, n_L, W, K) are used to compute all the values of in-control and out-of-control ARL and SDRL displayed in Table I to Table VIII. For example, by using (n_S, n_L, W, K)=(9, 22, 1.744, 2.878) for the VSS $\bar{X}$ chart with $n_1 = n_S$, we obtain (ARL_0, SDRL_0) = (247.26, 244.74) and (95.93, 95.88), respectively, for $\gamma \in \{0.5, 2.5\}$ when the underlying distribution is Weibull (see Table I). Similarly, when γ = 0.5 and the underlying distribution is lognormal, (ARL_1, SDRL_1)=(31.58, 30.76) and (1.41, 0.58) (see Table VII) are acquired for the VSS $\bar{X}$ chart with $n_1 = n_S$ for $\delta \in$ {0.25, 1.00}, respectively. These values of ARL_1 and $SDRL_1$ are also computed by using the chart's parameters (n_S, n_L, W, K) = (9, 22, 1.744, 2.878).

The skewness $\gamma \in \{0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}$ are considered in Table I to Table III. For a given γ, the parameters β, σ_{LN}, and α shown in Table I to Table III, respectively, are uniquely determined by means of the Mathematica software. The distribution is approximately symmetric when γ=0. The skewness $\gamma \in \{0.5, 1.0\}$ represent low levels of skewness; $\gamma \in \{1.5, 2.0\}$ represent moderate levels of skewness; while $\gamma \in \{2.5, 3.0\}$ represent high levels of skewness. A shift in the process mean for a skewed distribution is obtained as $\mu_{W,1} = \mu_{W,0} + \delta\sigma_{W,0}$. For an in-control process, δ=0; while for an out-of-control process, $\delta \neq 0$.

From Table I to Table III, when γ=0, the (ARL_0, $SDRL_0$) values for the three skewed distribution are generally close to the desired (ARL_0, $SDRL_0$) = (250.0, 249.50) under normal distribution. This is expected as the Weibull, lognormal, and gamma distributions are approximately symmetric when γ=0. As γ increases, the (ARL_0, $SDRL_0$) values under Weibull, lognormal, and gamma distributions, deviate significantly from that of those under normal distribution. For example, as mentioned in previous paragraph, the desired (ARL_0, $SDRL_0$) values for the VSS $\bar{X}$ chart with n_1=n_S is (250.0, 249.50) when the underlying distribution is normal. For Weibull distribution, these values decrease to (ARL_0, $SDRL_0$) = (247.26, 244.74) when γ=0.5 (see Table I) and further decrease to (ARL_0, $SDRL_0$) = (116.42, 114.93) when γ=2.0 (see Table I). For lognormal distribution, these values decrease to (ARL_0, $SDRL_0$)=(227.82, 226.90) when γ=0.5 (see Table II) and reduce significantly to (ARL_0, $SDRL_0$)=(112.69, 111.79) when γ=2.0 (see Table II). While for gamma distribution, these values decrease to (ARL_0, $SDRL_0$)=(231.83, 230.23) when γ=0.5 (see Table III) and further reduce to (ARL_0, $SDRL_0$)=(116.42, 114.93) when γ=2.0 (see Table III). This single example shows that the (ARL_0, $SDRL_0$) values under skewed distributions decrease as γ increases. From this single example, we observe that the performance of the VSS $\bar{X}$ chart under lognormal distribution is the worst among all the three skewed distributions. Low values of ARL_0 indicate that the false alarm rate for the VSS $\bar{X}$ chart under a skewed distribution is high. This false alarm rate increases as γ increases. This is an unfavorable performance.

Teoh *et al*. [10] revealed that the VSS $\bar{X}$ chart with $n_1 = n_L$ outperforms the VSS $\bar{X}$ chart with $n_1 = n_S$ when the underlying distribution is normal. However, when the distribution is skewed, it is obvious that the (ARL_0, $SDRL_0$) values for the VSS $\bar{X}$ chart with $n_1 = n_L$ are lower than those of the VSS $\bar{X}$ chart with $n_1 = n_S$ (see Table I to Table III). This suggests that the in-control performance for the VSS $\bar{X}$ chart with $n_1 = n_L$ is worse than that of the VSS $\bar{X}$ chart with $n_1 = n_S$. For example, (ARL_0, $SDRL_0$) = (231.83, 230.23) (see Table III) for the VSS $\bar{X}$ chart with $n_1 = n_S$ under gamma distribution, are closer to the desired (ARL_0, $SDRL_0$) = (250.0, 249.50) under normal distribution, as opposed to (ARL_0, $SDRL_0$) = (194.49, 193.14) (see Table III) for the VSS $\bar{X}$ chart with $n_1 = n_L$.

Table IV and Table VIII present the out-of-control ARL (ARL_1) and out-of-control SDRL ($SDRL_1$) values for the VSS $\bar{X}$ chart with $n_1 = n_S$ and $n_1 = n_L$, when the underlying distributions are normal, Weibull, and lognormal. Due to space constraints, the results for gamma distribution are excluded from this paper. However, these results can be obtained from the corresponding author. The trend and conclusion for the results obtained when the underlying distribution is gamma, are quite similar with that of the Weibull distribution. In Table V to Table VIII, we consider $\gamma \in$ {0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0} and $\delta \in$ {0.25, 0.50, 0.75, 1.00, 1.50, 2.00}.

From Table V and Table VII, when γ=0, the (ARL_1, $SDRL_1$) values for the Weibull and lognormal distribution are generally close to the (ARL_1, $SDRL_1$) under normal distribution (see Table IV). When γ increases, the difference between the (ARL_1, $SDRL_1$) values under Weibull distribution (see Tables V and VI) or lognormal distribution (see Tables VII and VIII) and that of the normal distribution (see Table IV) increases, especially for small and moderate δ (≤ 0.75).

TABLE IV: THE (ARL_1, $SDRL_1$) VALUES OF THE VSS $\bar{X}$ CHART WHEN THE UNDERLYING DISTRIBUTION IS NORMAL

δ	$n_1 = n_S$	$n_1 = n_L$
	(ARL_1, $SDRL_1$)	(ARL_1, $SDRL_1$)
0.25	(46.95, 46.21)	(29.34, 30.31)
0.50	(6.23, 5.09)	(2.41, 2.16)
0.75	(2.31, 1.24)	(1.11, 0.36)
1.00	(1.52, 0.63)	(1.00, 0.06)
1.50	(1.05, 0.22)	(1.00, 0.00)
2.00	(1.00, 0.03)	(1.00, 0.00)

TABLE V: THE (ARL_1, $SDRL_1$) VALUES OF THE VSS $\bar{X}$ CHART FOR $\gamma \in \{0.0, 0.5, 1.0\}$ WHEN THE UNDERLYING DISTRIBUTION IS WEIBULL

β	γ	δ	$n_1 = n_S$	$n_1 = n_L$
			(ARL_1, $SDRL_1$)	(ARL_1, $SDRL_1$)
3.6024	0.0	0.25	(47.63, 46.53)	(29.67, 30.70)
		0.50	(6.21, 5.04)	(2.40, 2.15)
		0.75	(2.30, 1.23)	(1.11, 0.36)
		1.00	(1.51, 0.63)	(1.00, 0.06)
		1.50	(1.05, 0.22)	(1.00, 0.00)
		2.00	(1.00, 0.03)	(1.00, 0.00)
2.2156	0.5	0.25	(40.52, 39.62)	(27.26, 28.13)
		0.50	(6.25, 5.10)	(2.43, 2.20)
		0.75	(2.32, 1.25)	(1.11, 0.35)
		1.00	(1.52, 0.63)	(1.00, 0.05)
		1.50	(1.05, 0.21)	(1.00, 0.00)
		2.00	(1.00, 0.01)	(1.00, 0.00)
1.5639	1.0	0.25	(35.55, 34.69)	(25.52, 26.28)
		0.50	(6.32, 5.22)	(2.46, 2.27)
		0.75	(2.34, 1.26)	(1.10, 0.34)
		1.00	(1.54, 0.62)	(1.00, 0.04)
		1.50	(1.04, 0.20)	(1.00, 0.00)
		2.00	(1.00, 0.00)	(1.00, 0.00)

TABLE VI: THE (ARL_1, $SDRL_1$) VALUES OF THE VSS $\bar{X}$ CHART FOR $\gamma \in \{1.5, 2.0, 2.5, 3.0\}$ WHEN THE UNDERLYING DISTRIBUTION IS WEIBULL

β	γ	δ	$n_1 = n_S$	$n_1 = n_L$
			(ARL_1, $SDRL_1$)	(ARL_1, $SDRL_1$)
1.2111	1.5	0.25	(32.07, 31.52)	(24.66, 25.40)
		0.50	(6.39, 5.30)	(2.49, 2.33)
		0.75	(2.37, 1.26)	(1.10, 0.34)
		1.00	(1.54, 0.61)	(1.00, 0.03)
		1.50	(1.03, 0.18)	(1.00, 0.00)
		2.00	(1.00, 0.00)	(1.00, 0.00)
1.0000	2.0	0.25	(29.47, 28.72)	(24.02, 25.00)
		0.50	(6.49, 5.43)	(2.52, 2.35)
		0.75	(2.39, 1.28)	(1.10, 0.32)
		1.00	(1.55, 0.60)	(1.00, 0.02)
		1.50	(1.03, 0.16)	(1.00, 0.00)
		2.00	(1.00, 0.00)	(1.00, 0.00)
0.8632	2.5	0.25	(27.63, 26.96)	(23.28, 24.12)
		0.50	(6.58, 5.56)	(2.54, 2.38)
		0.75	(2.42, 1.29)	(1.09, 0.32)
		1.00	(1.55, 0.59)	(1.00, 0.01)
		1.50	(1.02, 0.13)	(1.00, 0.00)
		2.00	(1.00, 0.00)	(1.00, 0.00)
0.7686	3.0	0.25	(26.26, 25.76)	(22.94, 23.57)
		0.50	(6.72, 5.64)	(2.55, 2.40)
		0.75	(2.44, 1.29)	(1.09, 0.31)
		1.00	(1.56, 0.58)	(1.00, 0.01)
		1.50	(1.01, 0.10)	(1.00, 0.00)
		2.00	(1.00, 0.00)	(1.00, 0.00)

TABLE VII: THE (ARL_1, $SDRL_1$) VALUES OF THE VSS $\bar{X}$ CHART FOR $\gamma \in \{0.0, 0.5, 1.0\}$ WHEN THE UNDERLYING DISTRIBUTION IS LOGNORMAL

σ_{LN}	γ	δ	$n_1 = n_S$	$n_1 = n_L$
			(ARL_1, $SDRL_1$)	(ARL_1, $SDRL_1$)
0.0003	0.0	0.25	(46.48, 45.78)	(29.13, 30.09)
		0.50	(6.22, 5.08)	(2.40, 2.15)
		0.75	(2.30, 1.24)	(1.11, 0.37)
		1.00	(1.52, 0.63)	(1.00, 0.06)
		1.50	(1.05, 0.22)	(1.00, 0.00)
		2.00	(1.00, 0.03)	(1.00, 0.00)
0.1641	0.5	0.25	(31.58, 30.76)	(21.40, 22.27)
		0.50	(5.03, 3.99)	(2.13, 1.85)
		0.75	(2.07, 1.09)	(1.08, 0.31)
		1.00	(1.41, 0.58)	(1.00, 0.05)
		1.50	(1.03, 0.17)	(1.00, 0.00)
		2.00	(1.00, 0.02)	(1.00, 0.00)
0.3143	1.0	0.25	(22.49, 21.74)	(15.99, 16.65)
		0.50	(4.08, 3.10)	(1.82, 1.48)
		0.75	(1.84, 0.94)	(1.05, 0.24)
		1.00	(1.30, 0.51)	(1.00, 0.03)
		1.50	(1.01, 0.12)	(1.00, 0.00)
		2.00	(1.00, 0.01)	(1.00, 0.00)

TABLE VIII: THE (ARL_1, $SDRL_1$) VALUES OF THE VSS $\bar{X}$ CHART FOR $\gamma \in \{1.5, 2.0, 2.5, 3.0\}$ WHEN THE UNDERLYING DISTRIBUTION IS LOGNORMAL

σ_{LN}	γ	δ	$n_1 = n_S$	$n_1 = n_L$
			(ARL_1, $SDRL_1$)	(ARL_1, $SDRL_1$)
0.4435	1.5	0.25	(16.77, 16.12)	(12.16, 12.79)
		0.50	(3.33, 2.40)	(1.57, 1.15)
		0.75	(1.65, 0.79)	(1.02, 0.16)
		1.00	(1.21, 0.43)	(1.00, 0.01)
		1.50	(1.00, 0.07)	(1.00, 0.00)
		2.00	(1.00, 0.00)	(1.00, 0.00)
0.5514	2.0	0.25	(13.19, 12.40)	(9.67, 10.28)
		0.50	(2.81, 1.91)	(1.38, 0.87)
		0.75	(1.49, 0.68)	(1.01, 0.10)
		1.00	(1.13, 0.34)	(1.00, 0.00)
		1.50	(1.00, 0.04)	(1.00, 0.00)
		2.00	(1.00, 0.00)	(1.00, 0.00)
0.6409	2.5	0.25	(10.79, 10.09)	(7.86, 8.58)
		0.50	(2.44, 1.56)	(1.24, 0.64)
		0.75	(1.37, 0.58)	(1.00, 0.06)
		1.00	(1.08, 0.27)	(1.00, 0.00)
		1.50	(1.00, 0.02)	(1.00, 0.00)
		2.00	(1.00, 0.00)	(1.00, 0.00)
0.7156	3.0	0.25	(9.12, 8.38)	(6.52, 7.20)
		0.50	(2.17, 0.30)	(1.15, 0.47)
		0.75	(1.28, 0.50)	(1.00, 0.03)
		1.00	(1.04, 0.21)	(1.00, 0.00)
		1.50	(1.00, 0.01)	(1.00, 0.00)
		2.00	(1.00, 0.00)	(1.00, 0.00)

For example, when δ =0.25 and n_1=n_L, we obtain (ARL_1, $SDRL_1$)=(29.34, 30.31) for normal distribution (see Table IV). For Weibull distribution, these values decrease to (ARL_1, $SDRL_1$)=(25.52, 26.28) when γ=1.0 (see Table V), as opposed to (ARL_1, $SDRL_1$)=(22.94, 23.57) when γ=3.0 (see Table VI). While for lognormal distribution, these values decrease significantly to (ARL_1, $SDRL_1$) = (15.99, 16.65) when γ=1.0 (see Table VII) and further reduce to (ARL_1, $SDRL_1$)=(6.52, 7.20) when γ=3.0 (see Table VII). This indicates that the performance of the VSS $\bar{X}$ chart under lognormal distribution is worse than that under the Weibull distribution. This is because the (ARL_1, $SDRL_1$) values for lognormal distribution deviate greatly from that of the normal distribution. Also, from this example, it is clear that for small and moderate shifts, the performance of the VSS $\bar{X}$ chart under skewed distributions is remarkably different from that of the normal distribution when γ increases.

For large δ ($\geq$ 1.00), the (ARL_1, $SDRL_1$) values for normal, Weibull, and lognormal distributions are quite similar. This indicates that the run-length performances of the VSS $\bar{X}$ chart are less impacted by skewed distributions when the shifts are large. For instant, when δ =1.50 and n_1=n_S, we obtain (ARL_1, $SDRL_1$)=(1.05, 0.22) for normal distribution (see Table IV). For Weibull distribution, these values are (ARL_1, $SDRL_1$)=(1.05, 0.21) when γ=0.5 (see Table V) and (ARL_1, $SDRL_1$)=(1.02, 0.13) when γ=2.5 (see Table VI). While for lognormal distribution, these values are (ARL_1, $SDRL_1$)=(1.03, 0.17) when γ=0.5 (see Table VII) and (ARL_1, $SDRL_1$)=(1.00, 0.02) when γ=2.5 (see Table VII). From this example, regardless the changes of γ, we obtain similar ARL_1 values for normal and skewed distributions. For variation, the $SDRL_1$ values for lognormal distribution and large γ, tend to be low.

V. CONCLUSIONS

The results in this paper reveal that the in-control and out-of-control performances of the VSS $\bar{X}$ chart with $n_1 = n_S$ and $n_1 = n_L$, are remarkably influenced by skewed distributions, especially when the process is in-control or slightly out-of-control. Though the out-of-control performance for the VSS $\bar{X}$ chart with $n_1 = n_L$ is better than that of the chart with $n_1 = n_S$, the poorer in-control performance of the chart with $n_1 = n_L$ put it into the dilemma of favoring this chart. High false alarm rate of a control chart will cause practitioners to conclude that the SPC is a failure if they keep on encountering false alarms with non-existence assignable cause(s). In such a case, the VSS $\bar{X}$ chart with $n_1 = n_S$ is favorable compared to the chart with $n_1 = n_L$ as the (ARL_0, $SDRL_0$) are closer to the desired values.

Also, as γ increases, the VSS $\bar{X}$ chart's performance is tremendously undesirable. Therefore, the chart's parameters (n_S, n_L, W, K) specially designed for the normal distribution are not suitable to be used for skewed distributions. Future research need to be conducted to propose new formulae and chart's parameters for the VSS $\bar{X}$ chart under skewed distributions.

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REFERENCES

[1] D. C. Montgomery, *Statistical Quality Control: A Modern Introduction*, 8th ed. New York: John Wiley & Sons, 2018.

[2] S. S. Prabhu, G. C. Runger, and J. B. Keats, "An adaptive sample size $\bar{X}$ control scheme," *Int. Journal of Production Research*, vol. 31, pp. 2895-2909, 1993.

[3] A. F. B. Costa, "$\bar{X}$ chart with variable sample size," *Journal of Quality Technology*, vol. 26, pp. 155-163, 1994.

[4] S. Zhang and Z. Wu, "A CUSUM scheme with variable sample sizes for monitoring process shifts," *Int. Journal of Advanced Manufacturing Technology*, vol. 33, no. 9-10, pp. 977-987, 2007.

[5] I. Kooli and M. Limam, "Economic design of an attribute *np* control chart using a variable sample size," *Sequential Analysis: Design Methods and Applications*, vol. 30, pp. 145-159, 2011.

[6] A. Amiri, A. Nedaie, and M. Alikhani, "A new adaptive variable sample size approach in EWMA control chart," *Communications in Statistics - Simulation and Computation*, vol. 43, no. 4, pp. 804-812, 2014.

[7] L. Zhang and X. Song, "EWMA median control chart with variable sampling size," *Information Technology Journal*, vol. 13, no. 14, pp. 2369-2373, 2014.

[8] P. Castagliola, A. Achouri, H. Taleb, G. Celano, and S. Psarakis, "Monitoring the coefficient of variation using a variable sample size control chart," *Int. Journal of Advanced Manufacturing Technology*, vol. 80, no. 9-12, pp. 1561-1576, 2015.

[9] X. L. Hu, P. Castagliola, J. Sun, and M. B. C. Khoo, "The performance of variable sample size $\bar{X}$ chart with measurement errors," *Quality and Reliability Engineering International*, vol. 32, no. 3, pp. 969-983, 2016.

[10] W. L. Teoh, J. K. Chong, M. B. C. Khoo, P. Castagliola, and W.C. Yeong, "Optimal designs of the variable sample size $\bar{X}$ chart based on median run length and expected median run length," *Quality and Reliability Engineering International*, vol. 33, no. 1, pp. 121-134, 2017.

[11] K. H. Ong, S. Y. Teh, M. B. C. Khoo, W. L. Teoh, and K. L. Soh, "Comparing the effects of skewed distributions on the *S*-chart and S^2-EWMA chart based on the median run length criterion," *Jurnal Teknologi (Sciences & Engineering)*, vol. 78, no. 6, pp. 53-58, 2016.

[12] W. L. Teoh, W. C. Yeong, M. B. C. Khoo, and S. Y. Teh, "The performance of the double sampling $\bar{X}$ chart with estimated parameters for skewed distributions," *Academic Journal of Science*, vol. 5, pp. 237-252, 2016.

[13] Y. S. Chang and D. S. Bai, "Control charts for positively-skewed populations with weighted standard deviations," *Quality and Reliability Engineering International*, vol. 17, pp. 397-406, 2001.

[14] K. L. Hsieh and Y. K. Chen, "An economic design of the VSSI $\bar{X}$ control charts for the means of positively skewed distributions," *African Journal of Business Management*, vol. 7, pp. 526-535, 2013.

[15] Y. S. Chang and D. S. Bai, "Median control charts for skewed populations," *Asia Pacific Management Review*, vol. 6, pp. 267-277, 2001.

[16] H. F. Chen and Y. Y. Cheng, "Non normality effects on the economic-statistical design of $\bar{X}$ charts with Weibull in-control time," *European Journal of Operational Research*, vol. 176, pp. 986-998, 2007.

[17] M. B. C. Khoo, Z. Wu, and A. M. A. Atta, "A synthetic control chart for monitoring the process mean of skewed populations based on the weighted variance method," *Int. Journal of Reliability, Quality and Safety Engineering*, vol. 15, pp. 217-245, 2008.

[18] S. Y. Teh and M. B. C. Khoo, "A study on the effects of skewed distributions on the performances of Max-EWMA and Max-GWMA charts," *Risk and Decision Analysis*, vol. 3, pp. 255-262, 2012.

[19] W. L. Teoh, W. C. Yeong, M. B. C. Khoo, and S. Y. Teh, "The performance of the double sampling $\bar{X}$ chart with estimated parameters for skewed distributions," *Academic Journal of Science*, vol. 5, pp. 237-252, 2016.

[20] K. Derya and H. Canan, "Control charts for skewed distributions: Weibull, gamma, and lognormal," *Metodoloski Zvezki*, vol. 9, pp. 95-106, 2012.

[21] D. N. Prabhakar, M. Xie, and R. Jiang, *Weibull Models*, New York: John Wiley & Sons, 2004.

[22] L. G. Blackwood, "The lognormal distribution, environmental data, and radiological monitoring," *Environmental Monitoring and Assessment*, vol. 21, pp. 193-210, 1992.

[23] K. O. Bowman and L. R. Shenton, "Gamma distribution," in *International Encyclopedia of Statistical Science*, M. Lovric, Ed. New York: Springer Berlin Heidelberg, 2014, pp. 573-575.

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