The efficiency of fair division*

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Abstract

In this paper we study the impact of fairness on the efficiency of allocations. We consider three different notions of fairness, namely proportionality, envy-freeness, and equitability for allocations of divisible and indivisible goods and chores. We present a series of results on the price of fairness under the three different notions that quantify the efficiency loss in fair allocations compared to optimal ones. Most of our bounds are either exact or tight within constant factors. Our study is of an optimistic nature and aims to identify the potential of fairness in allocations.

1 Introduction

Fair division (or fair allocation) dates back to the ancient times and has found applications such as border settlement in international disputes, greenhouse gas emissions reduction, allocation of mineral riches in the ocean bed, inheritance, divorces, etc. In the era of the Internet, it appears regularly in distributed resource allocation and cost sharing in communication networks.

We consider allocation problems in which a set of *goods* or *chores* has to be allocated among several agents. Fairness is an apparent desirable property in these situations and means that each agent gets a *fair share*. Depending on what the term "fair share" means, different notions of fairness can be defined. An orthogonal issue is *efficiency* that refers to the total happiness of the agents. An important notion that captures the minimum efficiency requirement from an allocation is that of *Pareto-efficiency*; an allocation is Pareto-efficient if there is no other allocation that is strictly better for at least one agent and is at least as good for all the others.

Problem statement. We consider two different allocation scenarios, depending on whether the items to be allocated are goods or chores. In both cases, we distinguish between *divisible* and *indivisible* items.

The problem of allocating divisible goods is better known as *cake-cutting*. In instances of cake-cutting, the term *cake* is used as a synonym of the whole set of goods to be allocated. Each agent has a *utility function* on each piece of the cake corresponding to the happiness of the agent if she gets the particular piece; this function is non-negative, non-atomic, and additive. We assume that the utility of each agent for the whole cake is 1 (normalized utilities). Divisibility means that the cake can be cut in arbitrarily small pieces which can then be allocated to the agents. In instances with indivisible goods, the utility function of an agent is defined over sets of items; again, utilities are non-negative and additive and the utility of each agent for the whole set of items is 1. Each item cannot be cut in pieces and has

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to be allocated as a whole to some agent. Given an allocation, the utility of an agent is simply the sum of her utilities over the (pieces of) items she receives. An allocation with n agents is proportional if the utility of each agent is at least $1/n$. It is envy-free if the utility of an agent is not smaller than the utility she would have when exchanging the (pieces of) items she gets with the items of any other agent. It is equitable if the utilities of all agents are equal. An allocation is *optimal* if it maximizes the total utility of all agents, i.e., each (piece of) item is allocated to the agent that values it the most (ties are broken arbitrarily). Here, through the assumption on normalized utilities, we have adopted a definition of efficiency (and optimality) that does not discriminate between agents.

In instances with divisible chores, each agent has a *disutility function* for each piece of the cake which denotes the regret of the agent when she gets the particular piece. Again, the disutility functions are non-negative, non-atomic, and additive and the disutility of an agent for the whole cake is 1. The case of indivisible chores is defined accordingly; indivisibility implies that an item cannot be cut into pieces and has to be allocated as a whole to some agent. Given an allocation, the disutility of an agent is simply the sum of her disutilities over the (pieces of) items she receives. An allocation with n agents is proportional if the disutility of each agent is at most $1/n$. It is envy-free if the disutility of an agent is not larger than the disutility she would have when exchanging the (pieces of) items she gets with the items of any other agent. It is equitable if the disutilities of all agents are equal. An allocation is optimal if it minimizes the total disutility of all agents, i.e., each (piece of) item is allocated to the agent that values it the least (ties are broken arbitrarily).

Note that envy-freeness implies proportionality. Furthermore, instances with divisible items always have proportional, envy-free, or equitable allocations. It is not hard to see that this is not always the case for instances with indivisible items. Furthermore, in any case, there are instances in which no optimal allocation is fair.

Models similar to ours have been considered in the literature; the focus has been on the design of protocols for achieving proportionality, envy-freeness, and equitability or on the design of approximation algorithms in settings where fulfilling the fairness objective exactly is impossible. However, the related literature seems to have neglected the issue of efficiency. Although several attempts have been made to characterize fair division protocols in terms of Pareto-efficiency [9], the corresponding results are almost always negative. Most of the existing protocols do not even provide Pareto-efficient solutions and this seems to be due to the limited amount of information they use for the utility functions of the agents. Recall that in the case of divisible goods and chores, complete information about the utility or disutility functions of the agents may not be compactly representable. Furthermore, recall that Pareto-efficiency is a minimum efficiency requirement; indeed, the total utility of the agents in a Pareto-efficient allocation may be very far from the maximum possible total utility.

Instead, in the current paper we are interested in quantifying the decrease of efficiency due to fairness (*price of fairness*). Our study has an optimistic nature and aims to identify the efficiency loss in the most efficient fair allocation. We believe that such a study is well-motivated since the knowledge of tight bounds on the price of fairness may detect whether a fair allocation can be improved. In many settings, complete information about the utility functions of the agents is available (e.g., in a divorce) and computing an efficient and fair allocation may not be infeasible. Fair allocations can be thought of as counterparts of equilibria in strategic games; hence, our work is similar in spirit to the line of research that studies the price of stability in games [1].

In order to capture the price of fairness, we define the price of proportionality, envy-freeness, and equitability. Given an instance for the allocation of goods, its price of proportionality (respectively, envy-freeness, equitability) is defined as the ratio of the total utility of the agents in the optimal allocation for the instance over the total utility of the agents in the best proportional (respectively, envy-free, equitable) allocation for the instance. Similarly, if we consider an instance for the allocation of chores, its price of proportionality (respectively, envy-freeness, equitability) is defined as the ratio of the total disutility of the agents in the best proportional (respectively, envy-free, equitable) allocation for the instance over the total disutility of the agents in the optimal allocation for the instance. The price of proportionality (respectively, envy-freeness, equitability) of a class of instances is then the maximum price of proportionality (respectively, envy-freeness, equitability) over all instances of the class. We remark that, in the case of indivisible items, we consider only instances for which proportional, envy-free, and equitable allocations, respectively, do exist.

Related work. Research on fair division originated in the 1940s with a focus on protocols for achieving fairness objectives in cake-cutting [27] (i.e., for divisible goods). Since then, the problem of achieving a proportional allocation with the minimum number of operations has received much attention and is now well-understood [16, 17, 9, 25, 31]. The problem of achieving envy-freeness has been proven to be much more challenging [23, 8, 28]; in fact, under the most common computational model of cut and evaluation queries [25], no algorithm with bounded running time is known for more than 3 agents. Very recently, envy-freeness was proved to be a harder property to achieve than proportionality [24, 29, 15]. Better solutions exist for different computational models (e.g., moving knife algorithms [10, 26]). Equitability seems to be a harder goal; the objective that all agents must get the same utility is computationally costly to achieve.

Optimization problems with objectives related to fairness have been studied in the recent literature. Lipton et al. [22] studied envy minimization with indivisible goods (where envy-freeness may not be guaranteed). Among other results, they showed how to compute allocations with bounded envy in polynomial time. They also present algorithms that compute allocations that approximate the minimum *envy-ratio*; the envy ratio of an agent p for an agent q is the utility of agent p for the items allocated to agent q over p's utility for the items allocated to her. Complexity considerations about envy-freeness for indivisible goods and more general non-additive utilities are presented in [7]. The papers [13, 14] study the problem of achieving envy-free and efficient allocations in distributed settings and when the allocation of items is accompanied by monetary side payments (in this case, envy-freeness is always a feasible goal [30]).

Another fairness objective that has been extensively considered recently for indivisible goods is *max-min fairness*. Here, the objective is to compute an allocation in which the utility of the least happy agent is maximized. The problem was studied by Bezáková and Dani [6] and Golovin [18] who obtained approximation algorithms that provably return a solution that is always a factor of $O(n)$ within the optimal value. The problem was popularized by Bansal and Sviridenko [4] as the *Santa Claus problem*, where Santa Claus aims to distribute presents to the kids so as to maximize the happiness of the least happy kid. Subsequently, Asadpour and Saberi [2] presented an $O(\sqrt{n}\log^3 n)$-approximation algorithm for this problem. See [5] and [12] for some very recent related results.

Fair division with chores is discussed in [9]. A related optimization problem is scheduling on unrelated machines [21] where the objective is to compute an allocation that minimizes the disutility of the most unhappy agent. Different notions of fairness have also been studied for other scheduling and resource allocation problems [11, 20, 19].

Overview of results. In this paper we provide upper and lower bounds on the price of proportionality, envy-freeness, and equitability in fair division with divisible and indivisible goods and chores. Our work reveals an almost complete picture. In all subcases except the price of envy-freeness with divisible goods and chores, our bounds are either exact or tight within a small constant factor.

Table 1 summarizes our results for fair division of goods. For divisible goods, the price of proportionality is very close to 1 (i.e., $8-4\sqrt{3}\approx 1.072$) for two agents and $\Theta(\sqrt{n})$ in general. The upper bound for two agents follows by a detailed analysis that takes into account the properties of the best proportional and the optimal allocation; this proof structure is adapted in order to prove the upper bounds on the price of equitability with divisible goods and the price of proportionality with divisible chores. Instead, the upper bound for the general case of n agents is constructive; its proof follows by defining a proportional allocation starting from the optimal one. The price of equitability is slightly worse for two agents (i.e., $9/8$) and $\Theta(n)$ in general. Our lower bound for the price of proportionality implies the same lower bound for the price of envy-freeness; while a very simple upper bound of $n-1/2$ completes the picture for divisible goods. For indivisible goods, we present an exact bound of $n-1+1/n$ on the price of proportionality while we show that the price of envy-freeness is $\Theta(n)$ in this case. Although our upper bounds follow by very simple arguments, the lower bounds use quite involved constructions. The price of equitability is proved to be finite only for the case of two agents.

	Divisible goods			Indivisible goods		
Price of	LB	UB	$n=2$	LB	UB	$n=2$
Proportionality	$\Omega(\sqrt{n})$	$O(\sqrt{n})$	$8-4\sqrt{3}$	$n-1+1/n$	$n-1+1/n$	$3/2$
Envy-freeness	$\Omega(\sqrt{n})$	$n-1/2$		$\frac{3n+7}{9}-O(1/n)$	$n-1/2$	
Equitability	$\frac{(n+1)^2}{4n}$	n	$9/8$	∞	∞	2

Table 1: Summary of our results (lower and upper bounds) for fair division of goods.

Table 2 summarizes our results for fair division of chores. For divisible chores, the price of proportionality is $9/8$ for two agents and $\Theta(n)$ in general while the price of equitability is exactly n. For indivisible chores, we present an exact bound of n on the price of proportionality while both the price of envy-freeness and the price of equitability are infinite. These last results imply that in the case of indivisible chores, envy-freeness and equitability are usually incompatible with efficiency.

	Divisible chores			Indivisible chores		
Price of	LB	UB	$n=2$	LB	UB	$n=2$
Proportionality	$\frac{(n+1)^2}{4n}$	n	$9/8$	n	n	2
Envy-freeness	$\frac{(n+1)^2}{4n}$	∞		∞	∞	
Equitability	n	n	2	∞	∞	∞

Table 2: Summary of our results (lower and upper bounds) for fair division of chores.

The rest of our paper is structured as follows. We begin with formal definitions in Section 2. We present our results for divisible goods in Section 3 and for indivisible goods in Section 4, while the case of chores is considered in Section 5. Finally, we conclude in Section 6.

2 Definitions

In this section we define the model considered in the paper. We begin with the setting of divisible and indivisible goods and proceed with the corresponding definitions for the setting of divisible and indivisible chores. Throughout the paper, n denotes the number of agents.

For the setting with divisible goods, let $\mathcal{I}$ represent a cake. Each agent i has a *utility function* u_i defined over any (possibly empty) piece (i.e., subset) of the cake that corresponds to the happiness of the agent if she gets that particular piece; this function is non-negative, non-atomic, and additive. This implies that $u_i(\emptyset)=0$ and that $u_i(\mathcal{P}\cup\mathcal{P}')=u_i(\mathcal{P})+u_i(\mathcal{P}')$ for disjoint pieces $\mathcal{P},\mathcal{P}'\subseteq\mathcal{I}$. We assume that the utility functions are normalized, i.e., $u_i(\mathcal{I})=1$ for every agent i. An allocation $\mathcal{A}=(\mathcal{A}_1,\dots,\mathcal{A}_n)$ is a collection of disjoint pieces such that $\bigcup_i\{\mathcal{A}_i\}=\mathcal{I}$, where $\mathcal{A}_i$ is the (not necessarily contiguous) piece allocated to agent i.

In instances with indivisible goods, there is a set of items $\mathcal{I}$. Each agent i has a non-negative and additive utility function u_i over subsets of $\mathcal{I}$ such that $u_i(\emptyset)=0$ and $u_i(\mathcal{I})=1$. Again, the utility function of each agent corresponds to her happiness if she gets the particular subset of items. An allocation $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a collection of disjoint subsets of items such that $\bigcup_i \{\mathcal{A}_i\} = \mathcal{I}$, where $\mathcal{A}_i$ is the set of items allocated to agent i. We note that each item cannot be cut in pieces and has to be allocated as a whole to some agent.

Definition 1 *An allocation $\mathcal{A}$ of goods to n agents is* proportional *if $u_i(\mathcal{A}_i) \geq 1/n$, for any agent i. It is* envy-free *if $u_i(\mathcal{A}_i) \geq u_i(\mathcal{A}_j)$, for any pair of agents i, j with $i \neq j$. It is* equitable *if $u_i(\mathcal{A}_i) = u_j(\mathcal{A}_j)$, for any pair of agents i, j with $i \neq j$.*

Definition 2 *Given an allocation $\mathcal{A}$ of goods, its total utility is $\sum_i u_i(\mathcal{A}_i)$. Allocation $\mathcal{A}$ is* optimal *if $\sum_i u_i(\mathcal{A}_i) \geq \sum_i u_i(\mathcal{B}_i)$, for any other allocation $\mathcal{B}$. Also, a proportional (respectively, envy-free, equitable) allocation $\mathcal{A}$ is called a* best *proportional (respectively, envy-free, equitable) allocation if $\sum_i u_i(\mathcal{A}_i) \geq \sum_i u_i(\mathcal{B}_i)$, for any other proportional (respectively, envy-free, equitable) allocation $\mathcal{B}$.*

In order to quantify the impact of fairness on efficiency, we define the price of proportionality, the price of envy-freeness, and the price of equitability.

Definition 3 *Given an instance for the allocation of goods, its* price of proportionality *(respectively,* price of envy-freeness, price of equitability*) is defined as the ratio of the total utility of the optimal allocation over the total utility of the best proportional (respectively, best envy-free, best equitable) allocation. The price of proportionality (respectively, price of envy-freeness, price of equitability) of a class $\mathcal{C}$ of instances is then the supremum price of proportionality (respectively, price of envy-freeness, price of equitability) over all instances of $\mathcal{C}$.*

The above notions are defined accordingly if we consider fair allocation of chores. Let $\mathcal{I}$ represent a universe of divisible chores. Each agent i has a *disutility function* u_i defined over subsets of $\mathcal{I}$ which denotes the regret of the agent when she gets that particular subset. Similarly to the case of divisible goods, the disutility functions are non-negative, non-atomic, additive, and normalized. An allocation $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a collection of disjoint subsets of $\mathcal{I}$ with $\bigcup_i \{\mathcal{A}_i\} = \mathcal{I}$, where $\mathcal{A}_i$ is the set of (pieces of) chores allocated to agent i.

In instances with indivisible chores, there is a set of items $\mathcal{I}$. Each agent i has a non-negative and additive disutility function u_i over subsets of $\mathcal{I}$ such that $u_i(\emptyset) = 0$ and $u_i(\mathcal{I}) = 1$. Again, the disutility function of each agent corresponds to her regret if she gets the particular subset of items. An allocation $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a collection of disjoint subsets of items such that $\bigcup_i \{\mathcal{A}_i\} = \mathcal{I}$, where $\mathcal{A}_i$ is the set of items allocated to agent i. We note that each item cannot be cut in pieces and has to be allocated as a whole to some agent.

Definition 4 *An allocation $\mathcal{A}$ of chores to n agents is* proportional *if $u_i(\mathcal{A}_i) \leq 1/n$, for any agent i. It is* envy-free *if $u_i(\mathcal{A}_i) \leq u_i(\mathcal{A}_j)$, for any pair of agents i, j with $i \neq j$. It is* equitable *if $u_i(\mathcal{A}_i) = u_j(\mathcal{A}_j)$, for any pair of agents i, j with $i \neq j$.*

Definition 5 *Given an allocation $\mathcal{A}$ of chores, its total disutility is $\sum_i u_i(\mathcal{A}_i)$. Allocation $\mathcal{A}$ is* optimal *if $\sum_i u_i(\mathcal{A}_i) \leq \sum_i u_i(\mathcal{B}_i)$, for any other allocation $\mathcal{B}$. Also, a proportional (respectively, envy-free, equitable) allocation $\mathcal{A}$ is called a* best *proportional (respectively, envy-free, equitable) allocation if $\sum_i u_i(\mathcal{A}_i) \leq \sum_i u_i(\mathcal{B}_i)$, for any other proportional (respectively, envy-free, equitable) allocation $\mathcal{B}$.*

The definitions for the price of fairness are similar.

Definition 6 *Given an instance for the allocation of chores, its* price of proportionality *(respectively,* price of envy-freeness, price of equitability*) is defined as the ratio of the total disutility of the best proportional (respectively, best envy-free, best equitable) allocation over the total disutility of the optimal allocation. The price of proportionality (respectively, price of envy-freeness, price of equitability) of a class $\mathcal{C}$ of instances is then the supremum price of proportionality (respectively, price of envy-freeness, price of equitability) over all instances of $\mathcal{C}$.*

The classes of instances considered in this paper are defined by the number of agents, the type of items (whether they are goods or chores), and their divisibility properties (whether they are divisible or indivisible). We remark that, in order for the price of proportionality, envy-freeness, and equitability to be well-defined, in the case of indivisible items, we assume that the class of instances contains only those ones for which proportional, envy-free, and equitable allocations, respectively, do exist.

3 Fair division with divisible goods

In this section, we focus on divisible goods. As a warm-up, we begin with simple upper bounds for the price of envy-freeness and equitability.

Lemma 7 *For n agents and divisible goods, the price of envy-freeness is at most $n - 1/2$ and the price of equitability is at most n.*

Proof. Consider an instance and a corresponding optimal allocation. If this allocation is envy-free or equitable, then the price of envy-freeness or equitability, respectively, is 1. In the following, we assume that this is not the case.

An envy-free allocation is also proportional; so the total utility of the agents in any envy-free allocation is at least 1. Since the optimal allocation is not envy-free, at least one agent is envious, and has utility over the pieces of the cake she receives less than $1/2$. So, the total utility in the optimal allocation is at most $n - 1/2$.

Now consider the allocation in which each negligibly small piece of the cake is shared equally among the n agents. This is an equitable allocation of total utility equal to 1 while the optimal allocation has total utility at most n. ■

We continue with a tight (up to constant factors) result for the price of proportionality.

Theorem 8 *For n agents and divisible goods, the price of proportionality is $\Theta(\sqrt{n})$.*

Proof. Consider an instance with n agents and let $\mathcal{O}$ denote the optimal allocation and OPT be the total utility of $\mathcal{O}$. We partition the set of agents into two sets, namely L and S, so that if an agent obtains utility at least $1/\sqrt{n}$ in $\mathcal{O}$, then she belongs to L, otherwise she belongs to S. Clearly, $OPT \leq |L| + |S|/\sqrt{n}$. We now describe how to obtain a proportional allocation $\mathcal{A}$; we distinguish between two cases depending on $|L|$.

We first consider the case $|L| \geq \sqrt{n}$; hence, $|S| \leq n - \sqrt{n}$. Then, for any negligibly small item that is allocated to an agent $i \in L$ in $\mathcal{O}$, we allocate to i a fraction of $\sqrt{n}/n$ of the item, while we allocate to each agent $i \in S$ a fraction of $\frac{n-\sqrt{n}}{n|S|} \geq 1/n$. Furthermore, for any negligibly small item that is allocated to an agent $i \in S$ in $\mathcal{O}$, we allocate to each agent $i \in S$ a fraction of $1/|S| > 1/n$. In this way, all agents obtain a utility of at least $1/n$, while all items are fully allocated; hence, $\mathcal{A}$ is proportional. For every agent $i \in L$, her utility in $\mathcal{A}$ is exactly $1/\sqrt{n}$ times her utility in $\mathcal{O}$, while every agent $i \in S$ obtained a utility strictly less than $1/\sqrt{n}$ in $\mathcal{O}$ and obtains utility at least $1/n$ in $\mathcal{A}$. So, we conclude that the total utility in $\mathcal{A}$ is at least $1/\sqrt{n}$ times the optimal total utility.

Otherwise, let $|L| < \sqrt{n}$. Since $OPT \leq |L| + |S|/\sqrt{n}$, we obtain that $OPT < 2\sqrt{n} - 1$, while the total utility of any proportional allocation is at least 1. Hence, in both cases we obtain that the price of proportionality is $O(\sqrt{n})$. We continue by presenting a lower bound of $\Omega(\sqrt{n})$.

Consider the following instance with n agents and $m < n$ items. Agent i, for $i = 1, ..., m$, has utility 1 for item i and 0 for any other item, while agent i, for $i = m+1, ..., n$, has utility $1/m$ for any item. An example is depicted in Figure 1. In the optimal allocation, item i, for $i = 1, ..., m$, is allocated to agent i, and the total utility is m. Consider any proportional allocation and let x be the sum of the fractions of

9 agents

item 1	1	0	0	
item 2	0	1	0	1/3
item 3	0	0	1	

Figure 1: An example of the lower bound construction in the proof of Theorem 8 for 9 agents and three divisible items. The utility of the first three agents is explicitly depicted, while the remaining 6 have utility $1/3$ for each item. The optimal total utility is 3 while in a best proportional allocation, each of the last six agents obtains $1/9$ of each item while the remaining items are given to the first three agents. In this case, the total utility is $5/3$.

the items that are allocated to the last $n-m$ agents. The total utility of these agents is x/m. Clearly, $x \geq m(n-m)/n$, otherwise some of them would obtain a utility less than $1/n$ and the allocation would not be proportional. The first m agents get the remaining fraction of $m-x$ of the items and their total utility is at most $m-x$. The total utility of all agents is $m-x+x/m \leq \frac{m^2+n-m}{n}$. We conclude that the price of proportionality is at least $\frac{mn}{m^2+n-m}$ which becomes more than $\sqrt{n}/2$ by setting $n=m^2$. ■

Since every envy-free allocation is also proportional, the lower bound on the price of proportionality also holds for envy-freeness. Interestingly, in the case of two agents, there always exist almost optimal proportional allocations. The main idea in the proof of the upper bound in the next theorem is to compare the structure of a best proportional allocation to the structure of an optimal allocation in order to obtain information about the agents' utilities on the pieces of the cake they receive in both allocations. This information reveals the desired relation between the total utility in both allocations. Recall that in this case proportionality and envy-freeness are equivalent.

Theorem 9 *For two agents and divisible goods, the price of proportionality (or envy-freeness) is exactly* $8-4\sqrt{3} \approx 1.072$.

Proof. Consider an optimal allocation $\mathcal{O}$ and a proportional allocation $\mathcal{E}$ that maximizes the total utility of the agents. We partition the cake into four parts A, B, C, and D as follows:

- A is the part of the cake which is allocated to agent 1 in both $\mathcal{O}$ and $\mathcal{E}$,
- B is the part of the cake which is allocated to agent 2 in both $\mathcal{O}$ and $\mathcal{E}$,
- C is the part of the cake which is allocated to agent 1 in $\mathcal{O}$ and to agent 2 in $\mathcal{E}$, and
- D is the part of the cake which is allocated to agent 1 in $\mathcal{E}$ and to agent 2 in $\mathcal{O}$.

Since $\mathcal{O}$ maximizes the total utility, we have $u_1(A) \geq u_2(A)$, $u_1(B) \leq u_2(B)$, $u_1(C) \geq u_2(C)$, and $u_1(D) \leq u_2(D)$. First observe that if $u_1(C) = u_2(C)$ and $u_1(D) = u_2(D)$, then $\mathcal{E}$ has the same total utility with $\mathcal{O}$. So, in the following we assume that this is not the case.

We consider the case $u_1(C) > u_2(C)$; the other case is symmetric. In this case, we also have that $u_1(D) = u_2(D) = 0$. Assume otherwise that $u_2(D) > 0$. Then, there must be a subpart X of C for which agent 1 has utility x and agent 2 has utility at most $x \cdot u_2(C)/u_1(C)$ and a subpart Y of D for which agent 2 has utility x; note that since D is allocated to agent 2 in $\mathcal{O}$, agent 1 has utility at most

x for Y. Then, the allocation in which agent 1 gets parts A, X, and $D - Y$ and agent 2 gets parts B, $C - X$, and Y is proportional and has larger utility than $\mathcal{E}$.

Now, we claim that $u_2(A) = 1/2$. Clearly, since $\mathcal{E}$ is proportional, the utility of agent 2 in $\mathcal{E}$ is at least $1/2$, i.e., $u_2(B) + u_2(C) \geq 1/2$. Since the utilities of agent 2 sum up to 1 over the whole cake, we also have that $u_2(A) \leq 1/2$. If it were $u_2(A) < 1/2$, then we would have $u_2(B) + u_2(C) > 1/2$. Then, there would exist a subpart X of C for which agent 2 has utility x for some $x \leq 1/2 - u_2(A)$ and agent 1 has utility strictly larger than x. By allocating X to agent 1 instead of agent 2, we would obtain another proportional allocation with larger total utility.

Also, it holds that $u_2(A)/u_1(A) \leq u_2(C)/u_1(C)$. Otherwise, there would exist a subpart X of C for which agent 1 has utility x and agent 2 has utility $u_2(X)$ at most $x \cdot u_2(C)/u_1(C)$ and a subpart Y of A for which agent 1 has utility x and agent 2 has utility $u_2(Y)$ at least $x \cdot u_2(A)/u_1(A) > x \cdot u_2(C)/u_1(C) \geq u_2(X)$. By allocating the subpart X to agent 1 and subpart Y to agent 2, we would obtain another proportional allocation with larger total utility.

By the discussion above, we have $u_2(C) \geq \frac{u_1(C)}{2u_1(A)}$. We are now ready to bound the ratio of the total utility of $\mathcal{O}$ over the total utility of $\mathcal{E}$ which will give us the desired bound. We obtain that the price of proportionality is

$$\begin{aligned}
\frac{u_1(A) + u_2(B) + u_1(C)}{u_1(A) + u_2(B) + u_2(C)} &= \frac{u_1(A) + 1/2 - u_2(C) + u_1(C)}{u_1(A) + 1/2} \\
&\leq \frac{u_1(A) + 1/2 - \frac{u_1(C)}{2u_1(A)} + u_1(C)}{u_1(A) + 1/2} \\
&= \frac{u_1(A) + 1/2 + u_1(C)\left(1 - \frac{1}{2u_1(A)}\right)}{u_1(A) + 1/2} \\
&\leq \frac{u_1(A) + 1/2 + (1 - u_1(A))\left(1 - \frac{1}{2u_1(A)}\right)}{u_1(A) + 1/2}
\end{aligned}$$

where the last inequality follows since $u_1(A) \geq u_2(A) = 1/2$ and $u_1(C) \leq 1 - u_1(A)$. The last expression is maximized to $8 - 4\sqrt{3}$ for $u_1(A) = \frac{1+\sqrt{3}}{4}$ and the upper bound follows.

In order to prove the lower bound, it suffices to consider a cake consisting of two parts A and B. Agent 1 has utility $u_1(A) = 1$ and $u_1(B) = 0$ and agent 2 has utility $u_2(A) = \sqrt{3} - 1$ and $u_2(B) = 2 - \sqrt{3}$. The utilities of each agent within the parts are uniform. ■

We now study the price of equitability and show that when the number of agents is large, equitability may provably lead to less efficient allocations. The next lower bound matches the upper bound of Lemma 7 concerning equitability within a constant factor.

Theorem 10 *For n agents and divisible goods, the price of equitability is at least $\frac{(n+1)^2}{4n}$.*

Proof. We distinguish between the cases of odd and even n.

In the first case, there are $(n+1)/2$ items. Agent i for $i = 1, ..., (n+1)/2$, has utility 1 for item i and utility 0 for any other item. Agent i, for $i = (n+3)/2, ..., n$, has utility $2/(n+1)$ for any item. In the optimal allocation, each agent i, for $i = 1, ..., (n+1)/2$, gets item i and the total utility is $(n+1)/2$. We complete the proof of this case by showing that no equitable allocation in which each agent has a utility $\chi > 2/(n+1)$ exists. Assume otherwise; then, the total utility of the last $(n-1)/2$ agents is $(n-1)\chi/2 > (n-1)(n+1)$. Thus, there exists at least one item g such that a fraction of at least $(n-1)\chi/2$ of g is allocated to the last $(n-1)/2$ agents, and, therefore, agent g cannot obtain a utility larger than $1 - (n-1)\chi/2 < 2/(n+1)$; this contradicts the equitability assumption. So, the total utility of any equitable allocation is at most $2n/(n+1)$ and the proof of this case is complete.

In the case of even n, there are n items and agent i, for $i = 1, ..., n/2$, has utility 1 for item i and utility 0 for any other item. Agent i, for $i = n/2+1, ..., n$, has utility $1/n$ for any item. In the optimal allocation, agent i, for $i = 1, ..., n$, obtains item i and the total utility is $(n+1)/2$. Again, we show that no equitable allocation in which each agent has utility $\chi > 2/(n+1)$ exists. Assume otherwise; then the total utility of the first $n/2$ agents is $n\chi/2$ and, hence, a total fraction of $n\chi/2$ of the first $n/2$ items has been allocated to them. This leaves a total fraction of $n(2-\chi)/2$ items to be allocated to the last $n/2$ agents. Clearly, in any such allocation there is an agent g with $n/2+1 \leq g \leq n$ with utility at most $(2-\chi)/n < \chi$. This again contradicts the equitability assumption. So, the total utility of any equitable allocation is again at most $2n/(n+1)$ and the proof of this case is complete. ■

Our last result of this section concerns the simplest case with $n = 2$, for which we present a matching upper bound on the price of equitability. The proof is along similar lines with the proof of Theorem 9.

Theorem 11 *For two agents and divisible goods, the price of equitability is* $9/8$.

Proof. Consider an optimal allocation $\mathcal{O}$ and an equitable allocation $\mathcal{E}$ that maximizes the total utility of the agents. We partition the cake into four parts A, B, C, and D as follows:

- A is the part of the cake which is allocated to agent 1 in both $\mathcal{O}$ and $\mathcal{E}$,
- B is the part of the cake which is allocated to agent 2 in both $\mathcal{O}$ and $\mathcal{E}$,
- C is the part of the cake which is allocated to agent 1 in $\mathcal{O}$ and to agent 2 in $\mathcal{E}$, and
- D is the part of the cake which is allocated to agent 1 in $\mathcal{E}$ and to agent 2 in $\mathcal{O}$.

Since $\mathcal{O}$ maximizes the total utility, we have $u_1(A) \geq u_2(A)$, $u_1(B) \leq u_2(B)$, $u_1(C) \geq u_2(C)$, and $u_1(D) \leq u_2(D)$. First observe that if $u_1(C) = u_2(C)$ and $u_1(D) = u_2(D)$, then $\mathcal{E}$ has the same total utility with $\mathcal{O}$. So, in the following we assume that this is not the case.

We consider the case $u_1(C) > u_2(C)$; the other case is symmetric. In this case, we also have that $u_1(D) = u_2(D) = 0$. Assume otherwise that $u_2(D) > 0$. Then, there is a subpart X of C and a subpart Y of D such that $u_1(X) > u_2(X)$ and $0 < u_1(X) + u_2(X) = u_1(Y) + u_2(Y) \leq \min\{u_1(C) + u_2(C), u_1(D) + u_2(D)\}$. Also, $u_2(Y) \geq u_1(Y)$ since Y is allocated to agent 2 in $\mathcal{O}$. Equivalently, we have that $u_1(X) - u_1(Y) = u_2(Y) - u_2(X)$. We also claim that $u_1(X) - u_1(Y) > 0$. Assume otherwise; then, we would also have $u_2(Y) \geq u_1(Y) \geq u_1(X) > u_2(X)$ which implies that $u_2(Y) - u_2(X) > 0$ and contradicts the above equality. Hence, $u_1(X) - u_1(Y) = u_2(Y) - u_2(X) > 0$ and the allocation in which agent 1 gets part X instead of Y and agent 2 gets part Y instead of X is also equitable and has larger total utility than $\mathcal{E}$.

Since $\mathcal{E}$ is equitable, we have $u_1(A) = u_2(B) + u_2(C)$. Since the utilities of agent 2 sum up to 1, this implies that $u_2(A) = 1 - u_1(A)$. Since $u_1(A) \geq u_2(A)$, we also have that $u_1(A) \geq 1/2$.

Also, it holds that $u_2(A)/u_1(A) \leq u_2(C)/u_1(C)$. Otherwise, since $u_1(A), u_1(C) > 0$, there would exist a subpart X of C and a subpart Y of A such that $u_1(X), u_1(Y) > 0$, $\frac{u_2(Y)}{u_1(Y)} > \frac{u_2(X)}{u_1(X)}$ which implies that $u_2(Y) > 0$ and $\frac{u_1(X)}{u_1(Y)} > \frac{u_2(X)}{u_2(Y)}$, and $0 < u_1(X) + u_2(X) = u_1(Y) + u_2(Y) \leq \min\{u_1(A) + u_2(A), u_1(C) + u_2(C)\}$. The equality in the last expression is equivalent to $u_1(X) - u_1(Y) = u_2(Y) - u_2(X)$. We also claim that $u_1(X) - u_1(Y) > 0$. Assume otherwise; then we would also have $1 \geq \frac{u_1(X)}{u_1(Y)} > \frac{u_2(X)}{u_2(Y)}$ which implies that $u_2(Y) - u_2(X) > 0$ and contradicts the above equality. Hence, $u_1(X) - u_1(Y) = u_2(Y) - u_2(X) > 0$ and the allocation in which agent 1 gets part X instead of Y and agent 2 gets part Y instead of X is also equitable and has larger total utility than $\mathcal{E}$.

By the discussion above, we have $u_2(B) = u_1(A) - u_2(C)$ and $u_2(C) \geq u_1(C)\left(\frac{1}{u_1(A)} - 1\right)$. We are now ready to bound the ratio of the total utility of $\mathcal{O}$ over the total utility of $\mathcal{E}$ which will give us the desired bound. We obtain that the price of equitability is

$$\begin{aligned}
\frac{u_1(A)+u_2(B)+u_1(C)}{u_1(A)+u_2(B)+u_2(C)} &= \frac{2u_1(A)+u_1(C)-u_2(C)}{2u_1(A)} \\
&\leq \frac{2u_1(A)+u_1(C)-u_1(C)\left(\frac{1}{u_1(A)}-1\right)}{2u_1(A)} \\
&= \frac{2u_1(A)+u_1(C)\left(2-\frac{1}{u_1(A)}\right)}{2u_1(A)} \\
&\leq \frac{2u_1(A)+(1-u_1(A))\left(2-\frac{1}{u_1(A)}\right)}{2u_1(A)} \\
&= \frac{3u_1(A)-1}{2u_1(A)^2}.
\end{aligned}$$

The last inequality follows since $u_1(A) \geq 1/2$ and $u_1(C) \leq 1 - u_1(A)$. The last expression is maximized to $9/8$ for $u_1(A) = 2/3$ and the theorem follows. ■

4 Fair division with indivisible goods

We now turn our attention to indivisible goods. Again, we begin this section by simple upper-bounds for the price of proportionality and envy-freeness.

Lemma 12 *For n agents and indivisible goods, the price of proportionality is at most $n-1+1/n$ and the price of envy-freeness is at most $n-1/2$.*

Proof. Consider an instance and a corresponding optimal allocation. If this allocation is proportional or envy-free, then the price of proportionality or envy-freeness, respectively, is 1. In the following, we assume that this is not the case.

In any proportional allocation, each agent has utility at least $1/n$ on the items she receives and the total utility is at least 1. Since the optimal allocation is not proportional, some agent has utility less than $1/n$ and the total utility in the optimal allocation is at most $n-1+1/n$.

An envy-free allocation is also proportional; so the total utility of the agents in any envy-free allocation is at least 1. Since the optimal allocation is not envy-free, at least one agent is envious, and has utility over the items she receives less than $1/2$. So, the total utility in the optimal allocation is at most $n-1/2$. ■

In the following, we present lower bounds which are either exact or tight within a constant factor.

Theorem 13 *For n agents and indivisible goods, the price of proportionality is at least $n-1+1/n$.*

Proof. Consider the following instance with n agents and $2n-1$ items. Let $0<\epsilon<1/n$. For $i=1,...,n-1$, agent i has utility ϵ for item i, utility $1-1/n$ for item $i+1$, utility $1/n-\epsilon$ for item $n+i$ and utility 0 for all other items. The last agent has utility $1/n-\epsilon$ for items $1,2,...,n-1$, utility $1/n+(n-1)\epsilon$ for item n, and utility 0 for all other items.

We argue that the only proportional allocation assigns items i and $n+i$ to agent i for $i=1,...,n-1$, and item n to agent n. To see that, notice that each agent must get at least one of the first n items, regardless of what other items she obtains, in order to be proportional. Since there are n agents, each of them must get exactly one of the first n items. Now, consider agent n. It is obvious that she must get item n, since she has utility strictly less than $1/n$ for any other item. The only available items (with positive utility) left for agent $n-1$ are items $n-1$ and $2n-1$, and it is easy to see that both of them must be allocated to her. Using the same reasoning for agents $n-2, n-3, ..., 1$, we conclude that the only proportional allocation is the aforementioned one, which has total utility $1+(n-1)\epsilon$.

Now, the total utility of the optimal allocation is lower-bounded by the total utility of the allocation where agent i gets items $i+1$ and $n+i$, for $i=1,...,n-1$, and agent n gets the first item. The total utility obtained by this allocation is $(1-1/n+1/n-\epsilon)(n-1)+\frac{1}{n}-\epsilon = n-1+1/n-n\epsilon$. By selecting ϵ to be arbitrarily small, the theorem follows. ∎

We remark that in the construction in the proof of Theorem 13 we use instances with no envy-free allocation that cannot be used to prove bounds on the price of envy-freeness. The lower bound construction in the proof of Theorem 8 can be extended in order to yield a lower bound of $\Omega(\sqrt{n})$ for indivisible items as well. In the following we prove an even stronger and tight lower bound of $\Omega(n)$.

Theorem 14 *For n agents and indivisible goods, the price of envy-freeness is at least $\frac{3n+7}{9} - O(1/n)$.*

Proof. We construct the following instance with $n \geq 5$ agents. Let integers $\ell \geq 2$ and $k > \ell + 1$ be such that $n = k + \ell - 1$; we note that no such integers exist for $n < 5$. Furthermore, let the number of items m be such that $m = \ell(k+1)$.

We denote by U the $m \times n$ matrix of utilities, where the entry in the i-th column and j-th row denotes the utility of agent i for item j. Let U_1 be the $\ell k \times (k-1)$ upperleft submatrix, U_2 be the $\ell k \times \ell$ upperright submatrix, U_3 be the $\ell \times (k-1)$ lowerleft submatrix and U_4 be the $\ell \times \ell$ lowerright submatrix.

The utilities are defined as follows. Each entry in U_1 has value $\frac{1}{\ell(k+\ell-1)} + \epsilon$, while each entry in U_3 has value $\frac{\ell-1}{\ell(k+\ell-1)} - k\epsilon$, for some sufficiently small $\epsilon > 0$ (e.g., $\epsilon \leq 1/n^3$). Clearly, it holds that the sum of utilities for any of the first $k-1$ agents over the items is exactly 1. As far as submatrix U_2 is concerned, agent i, for $i = k, \ldots, k+\ell-1$, has utility $1/k - \epsilon$ for items $(i-k)k+1, \ldots, (i-k+1)k$ and 0 otherwise. Finally, each entry in U_4 has value $\frac{k\epsilon}{\ell}$. Clearly, it holds that the sum of utilities for any of the last ℓ agents over the items is exactly 1. This construction is depicted in Figure 2.

Consider the allocation where each agent $i \in \{k, \ldots, k+\ell-1\}$ gets each item among the first $k\ell$ ones with strictly positive utility in U_2, while the last ℓ items are allocated (arbitrarily) to the first $k-1$ agents. This is an optimal allocation with total utility

$$OPT = \ell - \ell k\epsilon + \frac{\ell(\ell-1)}{\ell(k+\ell-1)} - \ell k\epsilon = \ell + \frac{\ell-1}{k+\ell-1} - 2\ell k\epsilon.$$

We now consider the envy-free allocation and we argue about some important properties concerning its structure. First, due to the utilities in the $m \times (k-1)$ left submatrix of U, it is not hard to see that, for any agents $i, j \in \{1, \ldots, k-1\}$, agent i gets the same number of items as agent j, otherwise i would envy j or vice versa. This means that none of these $k-1$ agents gets any of the last ℓ items, since $k-1 > \ell$, i.e., these items are not enough so that all those $k-1$ agents receive the same number of these items.

Therefore, the last ℓ items are allocated to the last ℓ agents. Moreover, assume that one of the last ℓ agents (let i^* be this agent) does not obtain any of the first $k\ell$ items. Then, since i^*'s utility is at most $\ell\frac{k\epsilon}{\ell} = k\epsilon$, she would be envious of any agent that obtains one of the first $k\ell$ items that has positive value for i^*, since this value is $1/k - \epsilon$. So, we conclude that each of the last ℓ agents must obtain at least one of the first $k\ell$ items, so in total the last ℓ agents receive at least 2ℓ items. This leaves at most $\ell(k+1) - 2\ell = (k-1)\ell$ items for the first $k-1$ agents.

We now argue that each of the $k-1$ first agents must obtain at least ℓ items; thus, each of them must obtain exactly ℓ items, since the number of items should be equally divided among them. Assume otherwise and let i be an agent that receives at most $\ell-1$ items. Then, i's utility is at most $\frac{\ell-1}{\ell(k+\ell-1)} + (\ell-1)\epsilon$, which, for sufficiently small $\epsilon > 0$ is less than $1/n$; clearly, i is envious.

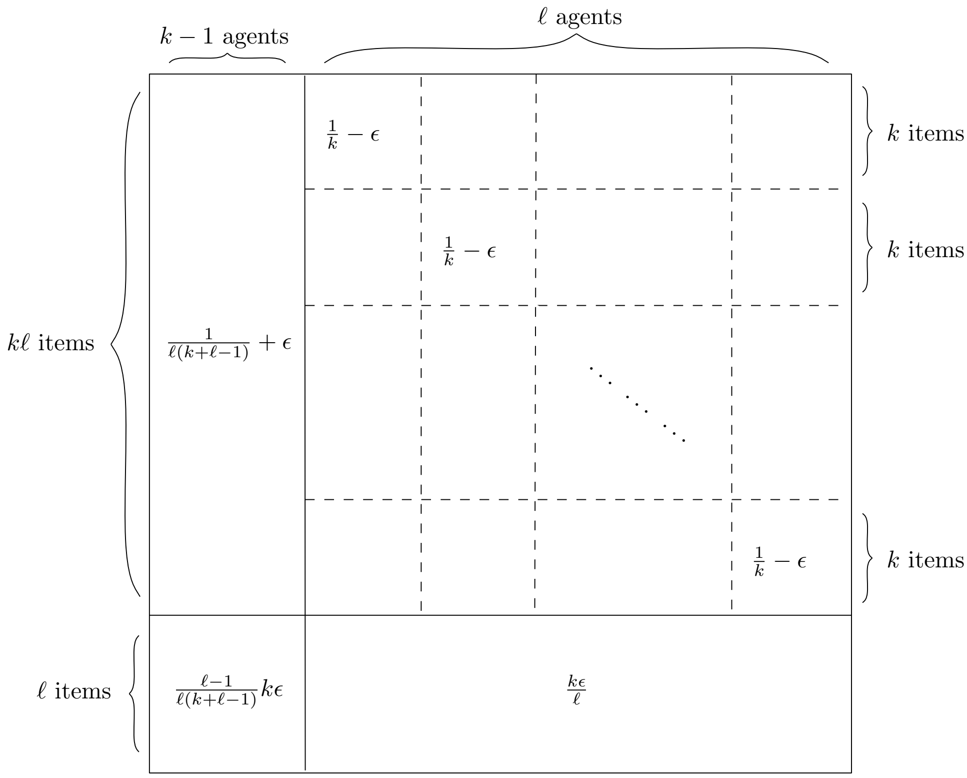

Figure 2: The lower bound construction in the proof of Theorem 14. For each submatrix, the utility of any agent for any item is provided, unless it is zero.

Therefore, we consider the following allocation. Each agent $i \in \{1, \ldots, k-1\}$ gets ℓ items from the first $k\ell$ ones, so that each of these ℓ items has strictly positive utility for exactly one of the last ℓ agents, and no item from the last ℓ ones. Furthermore, each agent $i' \in \{k, k+\ell-1\}$ gets exactly one item among the first $k\ell$ ones for which she has a strictly positive utility, while all the last ℓ items are allocated to the last ℓ agents. According to the properties above, this is the only envy-free allocation (up to relabeling agents and items) and the total utility is

$$\begin{aligned} EF &= \frac{\ell\,(k-1)}{\ell\,(k+\ell-1)} + \ell\,(k-1)\,\epsilon + \ell/k - \ell\epsilon + \frac{k\epsilon}{\ell}\ell \\ &= \ell/k + \frac{k-1}{k+\ell-1} + (\ell\,(k-2)+k)\,\epsilon. \end{aligned}$$

Therefore, we obtain that the price of envy-freeness is

$$\rho = \frac{OPT}{EF} = \frac{\ell + \frac{\ell-1}{k+\ell-1} - 2\ell k\epsilon}{\ell/k + \frac{k-1}{k+\ell-1} + (\ell\,(k-2)+k)\,\epsilon} = \frac{\ell + \frac{\ell-1}{k+\ell-1}}{\ell/k + \frac{k-1}{k+\ell-1}} - \epsilon'$$

for some $\epsilon' > 0$ that can become arbitrarily small by selecting ϵ to be arbitrarily small. When n is odd, we set $k = (n+3)/2$ and $\ell = (n-1)/2$ and the price of envy-freeness becomes

$$\rho = \frac{n^3 + 3n^2 - 3n - 9}{3n^2 + 2n + 3} - \epsilon'.$$

When n is even, we set $k = n/2 + 2$ and $\ell = n/2 - 1$ and the price of envy-freeness becomes

$$\rho = \frac{n^3 + 3n^2 - 4n}{3n^2 + 2n + 8} - \epsilon'.$$

In both cases, we obtain that

$$\rho = \frac{3n+7}{9} - O(1/n).$$

Finally, we note that for small values of n (i.e., $n = 3$ or $n = 4$) for which no k, ℓ satisfying $\ell \geq 2$ and $k > \ell + 1$ exist, we can ignore the constraint that $k > \ell + 1$ and construct the instance with the values $(k, \ell) = (2, 2)$ for $n = 3$ and $(k, \ell) = (3, 2)$ for $n = 4$. The lower bounds obtained are $7/4$ and $27/14$, respectively. ∎

Unfortunately, equitability may lead to arbitrarily inefficient allocations of indivisible goods when the number of agents is at least 3.

Theorem 15 *For n agents and indivisible goods, the price of equitability is 2 for $n = 2$ and infinite for $n > 2$.*

Proof. For $n > 2$, let ϵ be an arbitrarily small positive number and consider the following instance with n agents and n items. For $i = 1, ..., n-1$, agent i has utility ϵ for item i, utility $1 - \epsilon$ for item $i + 1$, and utility 0 for all other items. Agent n has utility $1 - 2\epsilon$ for item 1, utility ϵ for items $n - 1$ and n, and utility 0 for all other items. The total utility of the optimal allocation is $n - (n + 1)\epsilon$, which is obtained by allocating item $i + 1$ to agent i, for $i = 1, ..., n - 1$, and allocating item 1 to agent n. Clearly, this is not an equitable allocation and, furthermore, the only equitable allocation assigns item i to agent i, for $i = 1, ..., n$. The total utility of this allocation is $n\epsilon$ and the price of equitability is $\Omega(1/\epsilon)$; the statement for $n > 2$ follows.

For $n = 2$, the upper bound on the price of equitability holds since the optimal total utility is at most 2 and in an equitable allocation, each agent obtains a utility of at least $1/2$ (otherwise, the two agents would exchange bundles). The lower bound consists of four items a, b, c, and d. Agent 1 has utilities $1/2$, ϵ', $1/2 - \epsilon'$ and 0, respectively, while agent 2 has utilities $2\epsilon'$, $1/2$, 0, and $1/2 - 2\epsilon'$, respectively. In the optimal allocation, agent 1 obtains items a and c, while agent 2 obtains items b and d for a total utility of $2 - 3\epsilon'$. Clearly, in any equitable allocation, each agent obtains utility exactly $1/2$, and, hence, the statement for two agents follows by selecting ϵ' to be arbitrarily small. ∎

5 Fair division with chores

In this section, we study the allocation of divisible and indivisible chores. The next theorem states a tight bound on the price of proportionality for divisible chores.

Theorem 16 *For n agents and divisible chores, the price of proportionality is at most n and at least $\frac{(n+1)^2}{4n}$.*

Proof. The upper bound is obtained in a similar manner as in Theorem 7. Consider the optimal allocation. Clearly, if the optimal allocation is proportional, then the price of proportionality is 1. So, we can assume that at least one of the agents has disutility strictly larger than $1/n$ and this bounds the total disutility of the optimal allocation from below. By definition, in any proportional allocation, the disutility of each of the n agents is at most $1/n$. Hence, the total disutility of any proportional allocation is at most 1, which means that the price of proportionality is at most n.

We now present the lower bound. Consider the following instance with n agents and 2 items, where agents $1, \ldots, n-1$ have disutility 1 for item 1 and disutility 0 for item 2. Agent n has disutility $2/(n+1)$ for item 1, and disutility $(n-1)/(n+1)$ for item 2. In the optimal allocation, item 1 is allocated to agent n, whereas item 2 is allocated to one of the first $n - 1$ agents. The total disutility of this allocation is $2/(n + 1)$.

In a best proportional allocation, agent n will get a fraction of item 1, such that her disutility is exactly $1/n$, and the rest of the agents will in some way share the rest of item 1 as well as item 2 in such a way that they are proportional. In more detail, agent n obtains a fraction of $(n+1)/2n$ of the first item and the rest of the agents share the rest $(n-1)/2n$ fraction of the first item, as well as the whole item 2. The total disutility of this allocation is $(n+1)/2n$. We conclude that the price of proportionality is at least $\frac{(n+1)^2}{4n}$. ■

Since every envy-free allocation is also proportional, the lower bound on the price of proportionality also holds for envy-freeness. We also have a matching upper bound for proportionality (or envy-freeness) in the case $n = 2$. The proof of the next theorem is along similar lines with the proof of Theorem 9.

Theorem 17 *For two agents and divisible chores, the price of proportionality (or envy-freeness) is at most* $9/8$.

Proof. Consider an optimal allocation $\mathcal{O}$ and a proportional allocation $\mathcal{E}$ that minimizes the total disutility of the agents. We partition the cake into four parts A, B, C, and D as follows:

- A is the part of the cake which is allocated to agent 1 in both $\mathcal{O}$ and $\mathcal{E}$,
- B is the part of the cake which is allocated to agent 2 in both $\mathcal{O}$ and $\mathcal{E}$,
- C is the part of the cake which is allocated to agent 1 in $\mathcal{O}$ and to agent 2 in $\mathcal{E}$, and
- D is the part of the cake which is allocated to agent 1 in $\mathcal{E}$ and to agent 2 in $\mathcal{O}$.

Since $\mathcal{O}$ minimizes the total disutility, we have $u_1(A) \leq u_2(A)$, $u_1(B) \geq u_2(B)$, $u_1(C) \leq u_2(C)$, and $u_1(D) \geq u_2(D)$. First observe that if $u_1(C) = u_2(C)$ and $u_1(D) = u_2(D)$, then $\mathcal{E}$ has the same total disutility with $\mathcal{O}$. So, in the following we assume that this is not the case.

We consider the case $u_1(C) < u_2(C)$; the other case is symmetric. In this case, we also have that $u_1(D) = u_2(D) = 0$. Assume otherwise that $u_1(D) > 0$. Then, there must be a subpart X of C for which agent 2 has disutility x and agent 1 has disutility at most $x \cdot u_1(C)/u_2(C)$ and a subpart Y of D for which agent 1 has disutility x and agent 2 has disutility at most x. Then, the allocation in which agent 1 gets parts A, X, and $D - Y$ and agent 2 gets parts B, $C - X$, and Y is proportional and has smaller disutility than $\mathcal{E}$.

Now, we claim that $u_1(A) = 1/2$. Clearly, since $\mathcal{E}$ is proportional, the disutility of agent 1 in $\mathcal{E}$ is at most $1/2$, i.e., $u_1(A) \leq 1/2$. If it were $u_1(A) < 1/2$, then, there would exist a subpart X of C for which agent 1 has disutility x for some $x \leq 1/2 - u_1(A)$ and agent 2 has disutility strictly larger than x. By allocating X to agent 1 instead of agent 2, we would obtain another proportional allocation with smaller total disutility.

Also, observe that $u_1(A)/u_2(A) \leq u_1(C)/u_2(C)$. Otherwise, there would exist a subpart X of C for which agent 2 has disutility x and agent 1 has disutility $u_1(X)$ at most $x \cdot u_1(C)/u_2(C)$ and a subpart Y of A for which agent 2 has disutility x and agent 1 has disutility $u_1(Y)$ at least $x \cdot u_1(A)/u_2(A) > x \cdot u_1(C)/u_2(C) \geq u_1(X)$. By allocating X to agent 1 and Y to agent 2, we would obtain another proportional allocation with smaller total disutility.

By the discussion above, we have $u_1(C) \geq \frac{u_2(C)}{2u_2(A)}$ and, clearly, $u_2(B) = 1 - u_2(A) - u_2(C)$. We are now ready to bound the ratio of the total disutility of $\mathcal{E}$ over the total disutility of $\mathcal{O}$ which will give us the desired bound. We obtain that the price of proportionality is

$$\begin{aligned}\frac{u_1(A)+u_2(B)+u_2(C)}{u_1(A)+u_2(B)+u_1(C)} &= \frac{u_1(A)+1-u_2(A)-u_2(C)+u_2(C)}{u_1(A)+1-u_2(A)-u_2(C)+u_1(C)}\\ &= \frac{3/2-u_2(A)}{3/2-u_2(A)-u_2(C)+u_1(C)}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{3/2 - u_2(A)}{3/2 - u_2(A) - u_2(C) + \frac{u_2(C)}{2u_2(A)}} \\
&= \frac{3/2 - u_2(A)}{3/2 - u_2(A) - u_2(C)\left(1 - \frac{1}{2u_2(A)}\right)} \\
&\leq \frac{3/2 - u_2(A)}{3/2 - u_2(A) - (1 - u_2(A)\left(1 - \frac{1}{2u_2(A)}\right)} \\
&= 3u_2(A) - 2u_2(A)^2
\end{aligned}
$$

where the last inequality follows since $u_2(A) \geq u_1(A) = 1/2$ and $u_2(C) \leq 1 - u_2(A)$. The last expression is maximized to $9/8$ for $u_2(A) = 3/4$ and the theorem follows. ∎

We now turn our attention to the price of equitability and prove a tight bound.

Theorem 18 *For n agents and divisible chores, the price of equitability is n.*

Proof. We begin by proving the upper bound. By starting from an optimal allocation, we will show how to compute an equitable allocation with at most n times larger disutility. Consider a piece a in the optimal allocation and let $\{u_1(a), u_2(a), \ldots, u_n(a)\}$ be the vector denoting the disutility that each agent obtains if she gets a. Without loss of generality, let $u_1(a) = \min_i u_i(a)$ and $u_n(a) = \max_i u_i(a)$, i.e., piece a was allocated to agent 1. If $u_1(a) = 0$, then in the equitable allocation item a is allocated to agent 1, and all agents obtain a disutility of 0 from this item. Otherwise, if $u_1(a) > 0$, then each agent i gets a fraction $\chi \frac{u_n(a)}{u_i(a)}$ of piece a, where χ is such that $\sum_i \chi \frac{u_n(a)}{u_i(a)} = 1$. Clearly, every agent obtains a disutility of $\chi u_n(a)$ from a. Furthermore, it holds that $\chi u_n(a) \leq u_1(a)$, since agent 1 obtains at most the whole piece a; hence, a contributes to the equitable allocation a disutility of at most n times the disutility it contributes to the optimal allocation. By applying similar reasoning for all pieces of the optimal allocation, we can obtain the desired equitable allocation.

We now proceed to present the lower bound. Consider the following instance with n agents and 2 items. Agent i, for $i = 1, \ldots, n-1$, has disutility 1 for item 1 and disutility 0 for item 2, while agent n has disutility ϵ for item 1, and disutility $1 - \epsilon$ for item 2, for an arbitrarily small ϵ. In the optimal allocation, item 1 is allocated to agent n, whereas item 2 is allocated to one of the first $n-1$ agents. The total disutility of this allocation is ϵ.

In the best equitable allocation, we suppose without loss of generality that the second item is shared among the first $n-1$ agents. Also, each of the first $n-1$ agents must get the same fraction of item 1; let that fraction be χ. Then, agent n will get fraction $1-(n-1)\chi$ of item 1. Hence, $\chi = (1-(n-1)\chi)\,\epsilon$, which implies that $\chi = \frac{\epsilon}{1+(n-1)\epsilon}$, and the total disutility of the best equitable allocation is $\frac{n\epsilon}{1+(n-1)\epsilon}$. Since the optimal disutility is ϵ, we conclude that the price of equitability is at least $\frac{n}{1+(n-1)\epsilon}$. The lower bound follows by selecting ϵ to be arbitrarily small. ∎

Finally, we consider the case of indivisible chores. Although the price of proportionality is bounded, the price of envy-freeness and equitability is infinite.

Theorem 19 *For n agents and indivisible chores, the price of proportionality is n.*

Proof. The upper bound for the price of proportionality with divisible items can be easily extended to the indivisible case. We continue by presenting the lower bound.

Consider the following instance with n agents and $2n-1$ items. For $i = 1, ..., n-1$, agent i has disutility ϵ for items $1, \ldots, n-1$, disutility $1/n + (n-1)\epsilon$ for item n, and disutility $1/n - 2\epsilon$ for any other item. The last agent has disutility $1/n - \epsilon$ for items $1, ..., n-1$, disutility $1/n$ for item n, and disutility ϵ for items $n+1, ..., 2n-1$.

In the optimal allocation, agent i, for $i = 1, ..., n-1$, obtains item i, and agent n obtains items $n, n+1, ..., 2n-1$. The total disutility of the optimal allocation is $1/n + (2n-2)\epsilon$. It is not hard to see that in any proportional allocation, agent i, for $i = 1, \dots, n-1$, obtains exactly one of the first $n-1$ and exactly one of the last $n-1$ items, while agent n obtains item n. The total disutility of this allocation is $1-(n-1)\epsilon$ and the theorem follows by letting ϵ be arbitrarily small. ■

Theorem 20 *For n agents and indivisible chores, the price of envy-freeness (for $n \geq 3$) and equitability (for $n \geq 2$) is infinite.*

Proof. We begin with the case of envy-freeness. Consider the following instance with n agents and $2n$ items. Let $\epsilon < 1/(2n)$. For $i = 1, \dots, n-2$, agent i has disutility $1/n$ for the first n items and disutility 0 for every other item. Agent $n-1$ has disutility 0 for the first $n-1$ items, disutility ϵ for item n, disutility $1/n$ for items $n+1, \dots, 2n-1$ and disutility $1/n - \epsilon$ for item $2n$. Finally, agent n has disutility 0 for the first $n-1$ items, disutility $1/(2n)$ for items n and $2n$, and disutility $1/n$ for items $n+1, \dots, 2n-1$.

Clearly, the optimal allocation has total disutility ϵ and is obtained by allocating items $n+1, \dots, 2n$ to agents $1, \dots, n-2$, item n to agent $n-1$, and items $1, \dots, n-1$ either to agent $n-1$, or to agent n. In each case, agent $n-1$ envies agent n. Furthermore, the allocation in which agent i, for $i = 1, \dots, n$ gets items i and $i+n$ is envy-free. The remark that concludes this proof is that there cannot exist an envy-free allocation having negligible disutility (i.e., less than $1/(2n)$). This holds since, in any allocation of negligible disutility, item n would be allocated to agent $n-1$ and agent n would not get any of the last $n+1$ items. Hence, agent $n-1$ would envy agent n, since her disutility on the items allocated to agent n would be zero.

Now, we prove the lower bound regarding the price of equitability. Consider the following instance with n agents and $n+2$ items. Agent 1 has disutility $1/2$ for item 1, disutility $1/2 - \epsilon'$ for item 3, disutility ϵ' for item 4 and disutility 0 for all other items. Agent 2 has disutility $\epsilon'/4$ for item 1, disutility $1/2 - \epsilon'/4$ for item 2, disutility $3\epsilon'/4$ for item 3, disutility $1/2 - 3\epsilon'/4$ for item 4, and disutility 0 for all other items. For $i = 3, \dots, n$, agent i has disutility $1/2$ for item $i+2$ and disutility $\frac{1}{2(n+1)}$ for all other items.

Clearly, the optimal allocation has total disutility $2\epsilon'$ and is obtained by allocating items 2 and 4 to agent 1, items 1 and 3 to agent 2 and items $5, \dots, n+2$ to any of the first two agents. Since the last $n-2$ agents have strictly positive disutility for any item, in an equitable allocation the first two agents must get some of the first four items so that they have strictly positive disutility. It is not hard to see that in the only equitable allocation, agent 1 gets items 1 and 2, agent 2 gets items 3 and 4 (or vice versa), and agent i gets item $i+2$, for $i = 3, \dots, n$. Thus, each agent has a disutility of $1/2$ and the theorem follows by letting ϵ' be arbitrarily small. ■

6 Conclusions

We have studied the impact of fairness on the efficiency of allocations by considering divisible and indivisible items, both for the case of goods and chores. We have considered different measures of fairness, like proportionality, envy-freeness and equitability, and our results provide a rather complete picture of the decrease of the efficiency in all cases.

Our work has essentially left open the correct bound for the price of envy-freeness in the cases of divisible goods and chores. Although equitability seems to be worse than the other two fairness properties as far as efficiency is concerned, it is not clear whether proportionality is better than envy-freeness in the case of divisible goods and chores. It is tempting to conjecture that this is not the case but we have been unable to prove such a claim.

An interesting variation of the model for divisible items that we consider in this paper is to include the restriction that each agent gets a contiguous part of the cake. Bounds on the price of fairness for this model were recently obtained in [3]. Another interesting research direction is to quantify the impact of efficiency on fairness. For example, what is the minimum degree of envy (e.g., measured as the number of agents that are envious) among allocations that approximate the optimal one within a factor of α?

We also remark that we have made the assumption that agents have normalized utilities. In a sense, we have adopted a definition of efficiency that does not discriminate between agents. It would also be interesting to extend the definition of efficiency by removing this assumption and determine bounds on the price of fairness for the classes of instances considered in the current paper. We expect that these bounds will depend also on the relative (dis)utilities of the agents on the whole set of items.

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