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The ∞-Eigenvalue Problem

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Abstract

The Euler-Lagrange equation of the nonlinear Rayleigh quotient

$$\left(\int_\Omega |\nabla u|^p\,dx\right) \Big/ \left(\int_\Omega |u|^p\,dx\right)$$

is

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \Lambda_p^p |u|^{p-2}u,$$

where Λ_p^p is the minimum value of the quotient. The limit as $p \to \infty$ of these equations is found to be

$$\max\left\{\Lambda_\infty - \frac{|\nabla u(x)|}{u(x)},\ \ \Delta_\infty u(x)\right\} = 0,$$

where the constant $\Lambda_\infty = \lim_{p\to\infty} \Lambda_p$ is the reciprocal of the maximum of the distance to the boundary of the domain Ω.

§0. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$. The minimum of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx}$$

among all functions with zero boundary values is the first eigenvalue of the Laplacian in the domain Ω. This minimum value λ is achieved by the unique positive solution, up to multiplication by constants, of the Euler-Lagrange equation $\Delta u + \lambda u = 0$

with zero boundary values. Given a number p, $1 < p < \infty$, consider minimizing the nonquadratic Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}.$$

This problem leads to a nonlinear Euler-Lagrange equation, except in the case $p = 2$. As expected, the cases $p = 1$ and $p = \infty$ present additional difficulties. The objective of this paper is to study the limiting case $p = \infty$.

Formally, one has to minimize the ratio

$$\frac{\|\nabla u\|_{\infty,\Omega}}{\|u\|_{\infty,\Omega}} = \lim_{p\to\infty} \frac{\|\nabla u\|_{p,\Omega}}{\|u\|_{p,\Omega}}.$$

The minimum is the reciprocal of the radius of the largest possible ball inscribed in the domain Ω. Unfortunately, this min-max problem has too many solutions. In fact, outside the largest possible ball inscribed in the domain, one can modify a solution rather freely without changing the ratio. A more careful limiting procedure as $p \to \infty$ is called for to identify the genuine ∞-eigenfunctions.

The correct Euler-Lagrange equation turns out to be

$$\max\left\{ \Lambda_\infty - \frac{|\nabla u(x)|}{u(x)}, \quad \Delta_\infty u(x) \right\} = 0. \tag{0.1}$$

That is, at each point $x \in \Omega$, the larger of the two expressions is zero. Here

$$\Lambda_\infty = \frac{1}{\max\{\operatorname{dist}(x, \partial\Omega) : x \in \Omega\}},$$

and

$$\Delta_\infty u(x) = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

is the so called ∞-Laplacian. A most peculiar feature is that the "principal frequency" Λ_∞ has such a simple geometric characterization. The presence of the operator Δ_∞ is natural but, at first sight, the dichotomy of the equation is astonishing.

The equation has to be properly interpreted in the viscosity sense. For example, when Ω is a ball, the only solution is the distance function

$$\delta(x) = \operatorname{distance}(x, \partial\Omega)$$

up to constant multiples. In this case, the distance function satisfies $\Delta_\infty \delta(x) = 0$ and $\Lambda_\infty < |\nabla \log \delta(x)|$ except at the center x_0 of the ball. Indeed, at the center $\Lambda_\infty = |\nabla \log \delta(x_0)|$ and $\Delta_\infty \delta(x_0) < 0$ in the viscosity sense. Notice that the second derivatives needed to evaluate $\Delta_\infty \delta(x_0)$ do not exist in the ordinary sense. This illustrates the usefulness of viscosity solutions as weak solutions of nonlinear partial differential equations.

It is easy to see (Section 2 below) that the distance function always satisfies the minimization problem. That is,

$$\Lambda_\infty = \frac{\|\nabla\delta\|_{\infty,\Omega}}{\|\delta\|_{\infty,\Omega}}.$$

However, the distance is often not a genuine ∞-eigenfunction, since it is not a solution of the Euler-Lagrange equation (0.1). This happens already when Ω is a square or a parallelepiped. Moreover, this example shows that the solution is not a concave function when Ω is convex, although its logarithm is, indeed, concave. This follows from SAKAGUCHI's generalization of the Brascamp-Lieb theorem; see [S].

What about the existence of positive solutions to the equation

$$\max\left\{\Lambda - \frac{|\nabla u(x)|}{u(x)},\ \ \Delta_\infty u(x)\right\} = 0 \tag{0.2}$$

for values of Λ other than Λ_∞? It follows from an appropriate Harnack inequality that $\Lambda \leqq \Lambda_\infty$. If, in addition, the solution has zero boundary values, then $\Lambda = \Lambda_\infty$ indeed. This later result lies deeper, its proof being based on a uniqueness result for the equation

$$\max\left\{\Lambda - |\nabla v|,\ \ \Delta_\infty v + |\nabla v|^4\right\} = 0$$

satisfied by $v = \log u$, where u satisfies (0.2).

In Section 1 we present the relevant definitions and first results, and prove the basic fact that limits of p-eigenfunctions are indeed viscosity solutions of (0.2). Note that in order to use the terminology in [CIL] we consider equation (0.2) with a minus sign in front. See equation (1.22) below. In Section 2 we present a proof of a comparison principle for the logarithms of genuine ∞-eigenfunctions. This is our main result; its proof is based on the construction of a new sensitive test function. An application of this comparison principle is presented in Section 3, where we prove that Λ_∞ is the only "right" Λ. We finish by presenting some explicit computations in the case of a square, which are discussed in Section 4.

§1. Definitions and First Results

For a bounded domain Ω in $\mathbb{R}^n$, the distance function $\delta(x) = \text{distance}(x, \partial\Omega)$ is Lipschitz continuous, satisfies $|\nabla\delta(x)| = 1$ for a.e. $x \in \Omega$, and vanishes on the boundary of Ω. Let ϕ be any other Lipschitz continuous function vanishing on $\partial\Omega$. Fix $x \in \Omega$ and choose $y \in \partial\Omega$ such that $\delta(x) = |x - y|$. We have

$$|\phi(x)| = |\phi(x) - \phi(y)| \leqq \|\nabla\phi\|_\infty \delta(x).$$

Therefore,

$$\frac{\|\nabla\phi\|_\infty}{|\phi(x)|} \geqq \frac{\|\nabla\delta\|_\infty}{|\delta(x)|} \tag{1.1}$$

and we see that the distance function satisfies

$$\Lambda_\infty = \frac{\|\nabla\delta\|_\infty}{\|\delta\|_\infty} \leqq \frac{\|\nabla\phi\|_\infty}{\|\phi\|_\infty}, \tag{1.2}$$

for all $\phi \in W^{1,\infty}(\Omega)$ vanishing on $\partial\Omega$. The constant $\Lambda_\infty = 1/\|\delta\|_\infty$ depends only on the domain Ω, and for reasons that will be clear later on we think of Λ_∞ as the smallest ∞-eigenvalue of the domain Ω.

Consider the problem corresponding to (1.2) for finite $p > 1$:

$$\Lambda_p = \inf\left\{ \frac{\left(\frac{1}{|\Omega|}\int_\Omega |\nabla\phi(x)|^p\,dx\right)^{1/p}}{\left(\frac{1}{|\Omega|}\int_\Omega |\phi(x)|^p\,dx\right)^{1/p}} :\ \phi \in W_0^{1,p}(\Omega) \right\}. \tag{1.3}$$

There is a minimizer $u_p \in W_0^{1,p}(\Omega)$, unique up to a multiplicative constant, that satisfies the Euler equation

$$-\operatorname{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = \Lambda_p^p |u_p|^{p-2} u_p. \tag{1.4}$$

It is well known that $u_p > 0$ in Ω so that we can replace the right-hand side of (1.4) by u_p^{p-1}. References to these facts can be found in [L]. We normalize u_p by requiring that $\|u_p\|_p = 1$, where $\|f\|_p = \left(\frac{1}{|\Omega|}\int_\Omega |f|^p\,dx\right)^{1/p}$. The name given to Λ_∞ is justified by the following lemma.

1.5. Lemma.

$$\lim_{p\to\infty} \Lambda_p = \Lambda_\infty.$$

Proof. Using $\delta(x)$ as a test function in (1.3) we get

$$\Lambda_p \leqq \frac{1}{\left(\frac{1}{|\Omega|}\int_\Omega |\delta(x)|^p\,dx\right)^{1/p}},$$

which implies that

$$\limsup_{p\to\infty} \Lambda_p \leqq \Lambda_\infty.$$

Note that

$$\left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p(x)|^p\,dx\right)^{1/p} \leqq \Lambda_p$$

is uniformly bounded in p. Fix an exponent $m > n$. For $p > m$ by Hölder's inequality we have

$$\left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p(x)|^m\,dx\right)^{1/m} \leqq \Lambda_p.$$

We conclude that $\{u_p\}_{p\geqq m}$ is uniformly bounded in $W_0^{1,m}(\Omega)$. We can select a subsequence u_{p_i} that converges to a function denoted by u_∞ weakly in $W^{1,m}(\Omega)$ and uniformly in $C^\alpha(\Omega)$ for $\alpha = 1-n/m$. The limit function u_∞ is an ∞-superharmonic function as defined in [LM2], where it is also proved that nonnegative ∞-superharmonic functions satisfy an inequality of Harnack type that implies that $u_\infty(x) > 0$ for all $x \in \Omega$. For q large enough, using the weak lower semicontinuity of the L^q-norm and the fact that u_{p_i} converges to u_∞ weakly also in $W^{1,q}(\Omega)$, we have the inequality

$$\frac{\|\nabla u_\infty\|_q}{\|u_\infty\|_q} \leqq \liminf_{p_i \to \infty} \frac{\left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_{p_i}(x)|^q\,dx\right)^{1/q}}{\left(\frac{1}{|\Omega|}\int_\Omega |u_{p_i}(x)|^q\,dx\right)^{1/q}}.$$

Multiplying and dividing this inequality by $\left(\frac{1}{|\Omega|}\int_\Omega |u_{p_i}(x)|^{p_i}\,dx\right)^{1/p_i}$ and using Hölder's inequality we obtain

$$\frac{\|\nabla u_\infty\|_q}{\|u_\infty\|_q} \leqq \liminf_{p_i \to \infty} \left(\Lambda_{p_i} \frac{\|u_{p_i}\|_\infty}{\|u_{p_i}\|_q}\right).$$

We can take limits as $p_i \to \infty$ in the right-hand side to get

$$\frac{\|\nabla u_\infty\|_q}{\|u_\infty\|_q} \leqq \left(\liminf_{p_i \to \infty} \Lambda_{p_i}\right) \frac{\|u_\infty\|_\infty}{\|u_\infty\|_q}$$

for a fixed q. Letting $q \to \infty$ and using the minimizing property (1.2) we have

$$\Lambda_\infty \leqq \liminf_{p_i \to \infty} \Lambda_{p_i}.$$

This is enough to conclude the lemma, since we can apply this process to any subsequence of $\{u_p\}$. □

1.6. Remark. As a matter of fact, the above proof shows that any such u_∞ is extremal for the problem (1.2), that is,

$$\Lambda_\infty = \frac{\|\nabla u_\infty\|_\infty}{\|u_\infty\|_\infty}.$$

As we noted in the introduction, it is quite easy to find examples in which this minimum is attained by more than one function.

Suppose for a moment that the u_p are smooth so that we can differentiate (1.4) to get

$$-\left[|\nabla u_p|^{p-2}\Delta u_p + (p-2)|\nabla u_p|^{p-4}\Delta_\infty u_p\right] = \Lambda_p^p |u_p|^{p-2}u_p. \tag{1.7}$$

This equation is fully nonlinear and it makes sense to talk about viscosity subsolutions and supersolutions of it. The following lemma tells us that u_p is always a viscosity solution of (1.7). This is a somewhat delicate lemma since it is not clear that the comparison principle holds for equation (1.9) below.

1.8. Lemma. *A continuous weak (sub-)supersolution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the equation*

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \Lambda_p^p |u|^{p-2}u \tag{1.9}$$

is always a viscosity (sub-)supersolution of (1.7).

Before proving Lemma 1.8, let us recall the definition of viscosity (sub-)supersolution in our case. Let $z \in \mathbb{R}^n$, $X \in \mathbb{R}^n$ and S be a real symmetric matrix. Consider the continuous function

$$F_p(z, X, S) = -\left[|X|^{p-2}\operatorname{trace}(S) + (p-2)|X|^{p-4}\langle S \cdot X, X\rangle\right] - \Lambda_p^p |z|^{p-2}z.$$

Since we are interested in solutions of the partial differential equation

$$F_p(u, \nabla u, D^2u) = 0 \tag{1.10}$$

when $p \to \infty$, we always assume that p is large enough.

1.11. Definition. An upper semicontinuous function u defined in Ω is a *viscosity subsolution* of (1.10) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that
(a) $u(x_0) = \phi(x_0)$ and (b) $u(x) < \phi(x)$ if $x \neq x_0$, then

$$F_p\big(\phi(x_0), \nabla\phi(x_0)\big), D^2\phi(x_0)) \leqq 0.$$

1.12. Definition. A lower semicontinuous function u defined in Ω is a *viscosity supersolution* of (1.10) if whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that
(a) $u(x_0) = \phi(x_0)$ and (b) $u(x) > \phi(x)$ if $x \neq x_0$, then

$$F_p\big(\phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)\big) \geqq 0.$$

Condition (b) in both definitions can be relaxed quite a bit. The strict inequality is not really required and the condition only needs to hold in a neighborhood of x_0. We refer to [CIL] for the theory of viscosity solutions in general and to [Ju] for viscosity solutions of operators related to the ∞-Laplacian.

Proof of Lemma 1.8. We present the details for the case of supersolutions. Fix $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u(x_0) = \phi(x_0)$ and $u(x) > \phi(x)$ for $x \neq x_0$. We want to show that

$$\begin{aligned}-\big[|\nabla\phi(x_0)|^{p-2}\Delta\phi(x_0) + (p-2)|\nabla\phi(x_0)|^{p-4}\Delta_\infty\phi(x_0)\big]&\\ -\Lambda_p^p|\phi(x_0)|^{p-2}\phi(x_0) &\geqq 0.\end{aligned}$$

Suppose that this is not the case. Then, by continuity there exists a small $r > 0$ such that, if $|x - x_0| < r$, we have

$$-\big[|\nabla\phi(x)|^{p-2}\Delta\phi(x) + (p-2)|\nabla\phi(x)|^{p-4}\Delta_\infty\phi(x)\big] < \Lambda_p^p|\phi(x)|^{p-2}\phi(x).$$

Set $m = \inf\{u(x) - \phi(x)\colon |x - x_0| = r\} > 0$ and write $\Phi = \phi + \frac{1}{2}m$. The function Φ satisfies $\Phi < u$ on $\partial B(x_0, r)$, $\Phi(x_0) > u(x_0)$ and

$$-\operatorname{div}\left(|\nabla\Phi(x)|^{p-2}\nabla\Phi(x)\right) < \Lambda_p^p|\phi(x)|^{p-2}\phi(x). \tag{1.13}$$

The function $(\Phi - u)^+$ extended as the zero function outside of the ball $B(x_0, r)$ is a good test function for equation (1.9). Since we are assuming that u is a weak supersolution, we get

$$(1.14)\quad \int_{\{\Phi>u\}} |\nabla u|^{p-2}\langle \nabla u, \nabla(\Phi - u)\rangle\, dx \geqq \Lambda_p^p \int_{\{\Phi>u\}} |u|^{p-2}u(\Phi - u)\, dx.$$

Multiply both sides of (1.13) by $(\Phi - u)^+$ and integrate the product by parts to obtain

$$(1.15)\quad \int_{\{\Phi>u\}} |\nabla \Phi|^{p-2}\langle \nabla \Phi, \nabla(\Phi - u)\rangle\, dx < \Lambda_p^p \int_{\{\Phi>u\}} |\phi|^{p-2}\phi(\Phi - u)\, dx.$$

Subtracting (1.14) from (1.15) we arrive at

$$\int_{\{\Phi>u\}} \langle |\nabla \Phi|^{p-2}\nabla\Phi - |\nabla u|^{p-2}\nabla u, \nabla(\Phi - u)\rangle\, dx$$
$$< \Lambda_p^p \int_{\{\Phi>u\}} \left(|\phi|^{p-2}\phi - |u|^{p-2}u\right)(\Phi - u)\, dx.$$

Since the left-hand side is bounded below by a positive constant, depending on p and n, times

$$\int_{\{\Phi>u\}} |\nabla\Phi - \nabla u|^p\, dx,$$

and the right-hand side is negative, we conclude that $\Phi \leqq u$ in $B(x_0, r)$, contradicting the fact that $\Phi(x_0) > u(x_0)$. □

Next, we compute the limit of the $F_p(z, X, S)$ as $p \to \infty$ in the viscosity sense. That is, we consider the sequence of viscosity solutions $\{u_p\}$ and we would like to find out what equation is satisfied by any cluster point of this sequence, which we denote by u_∞. Explicitly, we assume that for a subsequence $p_i \to \infty$ we have $\lim_{p_i\to\infty} u_{p_i} = u_\infty$ uniformly in Ω.

Fix a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and the inequality $u_\infty(x) > \phi(x)$ holds for $x \neq x_0$. Also fix $R > 0$ so that $B(x_0, 2R) \subset \Omega$. For $0 < r < R$ we certainly have

$$\inf\{u_\infty(x) - \phi(x) : x \in B(x_0, R) \setminus B(x_0, r)\} > 0.$$

Since $u_{p_i} \to u_\infty$ uniformly in the closure of $B(x_0, R)$, we conclude that for $i \geqq i_r$,

$$\inf\{u_{p_i}(x) - \phi(x) : x \in B(x_0, R)\} \setminus B(x_0, r) > u_{p_i}(x_0) - \phi(x_0).$$

Therefore, for such indices i, $u_{p_i} - \phi$ attains its minimum at a point $x_i \in B(x_0, r)$, and we see by letting $r \to 0$ that $x_i \to x_0$ as as $i \to \infty$. For notational simplicity we drop the subindices and write p_i for p_{i_r} and x_i for $x_{p_{i_r}}$. Since u_{p_i} is a viscosity supersolution of (1.7) we get

$$(1.16)\quad \begin{aligned} -\left[|\nabla\phi(x_i)|^{p_i-2}\Delta\phi(x_i) + (p_i - 2)|\nabla\phi(x_i)|^{p_i-4}\Delta_\infty\phi(x_i)\right] \\ \geqq \Lambda_{p_i}^{p_i}|u_{p_i}(x_i)|^{p_i-2}u_{p_i}(x_i). \end{aligned}$$

Recall that $u_\infty(x) > 0$, and so $u_{p_i}(x_i) > 0$ for large i, which itself implies that $|\nabla\phi(x_i)| \neq 0$ as follows from (1.16). Dividing by $|\nabla\phi(x_i)|^{p_i-4}$ and by $p_i - 2$ we arrive at

$$-\frac{|\nabla\phi(x_i)|^2\Delta\phi(x_i)}{p_i-2} - \Delta_\infty\phi(x_i) \geqq \left(\frac{\Lambda_{p_i}u_{p_i}(x_i)}{|\nabla\phi(x_i)|}\right)^{p_i-4}\frac{\Lambda_{p_i}^4 u_{p_i}(x_i)^3}{p_i-2}. \tag{1.17}$$

Suppose that $\frac{\Lambda_\infty\phi(x_0)}{|\nabla\phi(x_0)|} > 1$. Letting $p_i \to \infty$ we get a contradiction. Therefore we must have

$$\frac{\Lambda_\infty\phi(x_0)}{|\nabla\phi(x_0)|} \leqq 1. \tag{1.18}$$

Since the right-hand side of (1.17) is nonnegative, letting $p_i \to \infty$ we see that

$$-\Delta_\infty\phi(x_0) \geqq 0. \tag{1.19}$$

These two equations (1.18) and (1.19) can be combined into one by writing

$$\min\left\{|\nabla\phi(x_0)| - \Lambda_\infty\phi(x_0), -\Delta_\infty\phi(x_0)\right\} \geqq 0. \tag{1.20}$$

We have established that u_∞ is a viscosity supersolution of the equation

$$\min\left\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\right\} = 0.$$

It is therefore natural to define

$$F_\infty(z, X, S) = \min\left\{|X| - \Lambda_\infty z, -\langle S\cdot X, X\rangle\right\}.$$

We can now state the main theorem of this section:

1.21. Theorem. *A function u_∞ obtained as a limit of a subsequence of $\{u_p\}$ is a viscosity solution of the equation*

$$F_\infty(u, \nabla u, D^2u) = \min\left\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\right\} = 0. \tag{1.22}$$

Before finishing the proof of the theorem, note that

i) u_∞ is ∞-superharmonic, since

$$-\Delta_\infty u_\infty \geqq 0$$

in the viscosity sense, and

ii) $|\nabla u_\infty| \geqq \Lambda_\infty u_\infty$ in the viscosity sense. Moreover, at least heuristically, if one of these inequalities is strict, the other must be an equality.

Proof. It remains to be proved that u_∞ is a viscosity subsolution. The proof is similar to the supersolution case but not symmetric. Fix a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and the inequality $u_\infty(x) < \phi(x)$ holds for $x \neq x_0$. We want to check that

$$\min\{|\nabla\phi(x_0)| - \Lambda_\infty\phi(x_0), -\Delta_\infty\phi(x_0)\} \leqq 0.$$

Observe that if $\nabla\phi(x_0) = 0$, there is nothing to prove. As a matter of fact, we may assume that $|\nabla\phi(x_0)| - \Lambda_\infty\phi(x_0) > 0$. We now repeat the procedure that we followed in the supersolution case. The analogue of (1.16) is

$$-\left[|\nabla\phi(x_i)|^{p_i-2}\Delta\phi(x_i) + (p_i - 2)|\nabla\phi(x_i)|^{p_i-4}\Delta_\infty\phi(x_i)\right] \leqq \Lambda_p^p|u_{p_i}(x_i)|^{p-2}u_{p_i}(x_i),$$

and the analogue of (1.17) is

$$-\frac{|\nabla\phi(x_i)|^2\Delta\phi(x_i)}{p_i - 2} - \Delta_\infty\phi(x_i) \leqq \left(\frac{\Lambda_{p_i}u_{p_i}(x_i)}{|\nabla\phi(x_i)|}\right)^{p_i-4}\frac{\Lambda_{p_i}^4u_{p_i}(x_i)^3}{p_i - 2}.$$

Letting $p_i \to \infty$ we get $-\Delta_\infty\phi(x_0) \leqq 0$. □

§2. Comparison Principles

Consider again the equation (1.22):

$$F_\infty(u, \nabla u, D^2u) = \min\left\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\right\} = 0.$$

Note that $F_\infty(z, X, S)$ is decreasing in S and decreasing in z. In the language of [CIL], the function F_∞ is degenerate elliptic but it is not proper. Therefore, the usual tools to prove uniqueness to solutions to a Dirichlet problem associated with equation (1.22) do not apply. However, we know that every u_∞ is strictly positive. This suggests considering the equation that $v_\infty = \log(u_\infty)$ satisfies.

2.1. Lemma. *Let u be a nonnegative viscosity solution of* (1.22) *in* Ω. *Then* $v = \log(u)$ *is a viscosity solution of the equation*

$$\min\left\{|\nabla v| - \Lambda_\infty, -\Delta_\infty v - |\nabla v|^4\right\} = 0 \tag{2.2}$$

in Ω.

Proof. The lemma follows from a simple calculation. We provide the details in the supersolution case. Let $\phi \in C^2(\Omega)$ such that $v(x_0) = \phi(x_0)$ and $v(x) > \phi(x)$ for $x \neq x_0$. Write $\Phi(x) = e^{\phi(x)}$. Then Φ is a good test function for u at the point x_0. Therefore, we have

$$\min\left\{|\nabla\Phi(x_0)| - \Lambda_\infty\Phi(x_0), -\Delta_\infty\Phi(x_0)\right\} \geqq 0.$$

Writing this inequality in terms of ϕ we get

$$\min\left\{e^{\phi(x_0)}\big(|\nabla\phi(x_0)| - \Lambda_\infty\big), -e^{3\phi(x_0)}\big(\Delta_\infty\phi(x_0) + |\nabla\phi(x_0)|^4\big)\right\} \geqq 0,$$

from which the lemma follows easily. □

Since equation (2.2) is now proper, we can try to prove the comparison principle for solutions of (2.2). Because the equation is degenerate elliptic, the usual techniques of [CIL] need to be augmented. In the case of the ∞-harmonic equation $\Delta_\infty u = 0$ the comparison principle is given in JENSEN [J]. A nice proof of this comparison principle for the ∞-harmonic functions based on the "comparison principle for semicontinuous functions" is due to JUUTINEN [Ju]. Equation (2.2) is different on two counts. First, in the viscosity sense we have $|\nabla v| \geqq \Lambda_\infty$, which will make possible the uniqueness proof presented below and second, it contains the term $|\nabla v|^4$. The main result of this section is:

2.3. Theorem. *Let Ω be a bounded domain, let u be a viscosity subsolution of (2.2) in Ω and let v be viscosity supersolution of (2.2) in Ω. Suppose that both functions are continuous in $\overline{\Omega}$. Then, the following comparison principle holds:*

$$\sup_{x\in\overline{\Omega}} (u(x) - v(x)) = \sup_{x\in\partial\Omega} (u(x) - v(x)) \,. \tag{2.4}$$

Proof. Without loss of generality we may assume that u and v are positive by adding a large constant to both of them. We proceed by contradiction. Suppose that (2.4) does not hold. Then, we must have

$$\sup_{x\in\overline{\Omega}} (u(x) - v(x)) > \sup_{x\in\partial\Omega} (u(x) - v(x)) \,. \tag{2.5}$$

This inequality still holds if we replace v by a function w for which $\|v - w\|_{L^\infty(\Omega)}$ is small enough. We construct a function w that is a strict supersolution of (2.2), and then we apply the comparison for semicontinuous functions from [CIL].

2.6. Lemma. *Let $A > 1$ and $\alpha > 1$ be given. The function*

$$f(t) = \frac{1}{\alpha}\log\big(1 + A(e^{\alpha t} - 1)\big)$$

has the following properties:

(i) *$f(0) = 0$, $f'(t) > 1$ and $f''(t) < 0$ for all $t \geqq 0$,*

(ii) *f is invertible,*

(iii) *f satisfies the differential inequality*

$$1 - \frac{1}{f'(t)} + \frac{f''(t)}{(f'(t))^2} < 0,$$

(iv) *f is an approximation of the identity as $A \to 1^+$ in the sense that*

$$0 < f(t) - t < \frac{A-1}{\alpha}$$

for all $t \geqq 0$.

The proof of this lemma is elementary. Notice that f satisfies the differential equation

$$1-\frac{1}{f'(t)}+\frac{1}{\alpha}\frac{f''(t)}{(f'(t))^2}=0$$

so that (iii) follows from the fact that $1-1/\alpha>0$. Observe that if we write $f_A(t)=\frac{1}{\alpha}\log\left(1+A(e^{\alpha t}-1)\right)$, then for any positive A and B we have

$$f_A\circ f_B=f_{AB}.$$

In particular, $f_A^{-1}=f_{A^{-1}}$ since $AA^{-1}=1$ and f_1 is the identity.

By taking A close enough to 1, we can guarantee that $w=f(v)$ satisfies (2.5). The equation for which w is a supersolution is obtained as follows. Let $x_0\in\Omega$ and $\phi\in C^2(\Omega)$ be such that $w(x_0)=\phi(x_0)$ and $w(x)\geqq\phi(x)$ for $x\neq x_0$. Set

$$\Phi=f^{-1}(\phi),\quad\text{so that}\quad f(\Phi)=\phi.$$

Since f^{-1} is monotone, Φ is a good test function for v at the point x_0. Since v is a supersolution of (2.2), we have

$$\min\left\{|\nabla\Phi(x_0)|-\Lambda_\infty,\,-\Delta_\infty\Phi(x_0)-|\nabla\Phi(x_0)|^4\right\}\geqq 0. \tag{2.7}$$

Differentiating we obtain

$$\nabla\Phi=\frac{1}{f'(\Phi)}\nabla\phi,$$

$$D^2\Phi=\frac{1}{f'(\Phi)}D^2\phi-\frac{f''(\Phi)}{(f'(\Phi))^3}\left(\nabla\phi\otimes\nabla\phi\right).$$

From (2.7) we deduce that

$$|\nabla\Phi(x_0)|-\Lambda_\infty\geqq 0 \tag{2.8}$$

$$-\Delta_\infty\Phi(x_0)-|\nabla\Phi(x_0)|^4\geqq 0. \tag{2.9}$$

From (2.8) it follows that

$$|\nabla\phi(x_0)|\geqq f'(\Phi(x_0))\Lambda_\infty, \tag{2.10}$$

$$|\nabla\phi(x_0)|-\Lambda_\infty\geqq\left[f'(\Phi(x_0))-1\right]\Lambda_\infty. \tag{2.11}$$

We compute starting from (2.9). Omitting the point x_0 for notational simplicity, we obtain

$$-\left\langle\left(\frac{1}{f'(\Phi)}D^2\phi-\frac{f''(\Phi)}{(f'(\Phi))^3}\left(\nabla\phi\otimes\nabla\phi\right)\right)\frac{1}{f'(\Phi)}\nabla\phi,\,\frac{1}{f'(\Phi)}\nabla\phi\right\rangle$$
$$-\frac{1}{f'(\Phi)^4}|\nabla\phi|^4\geqq 0.$$

After elementary manipulations this inequality becomes

$$-\Delta_\infty\phi - \left[\frac{1}{f'} - \frac{f''}{(f')^2}\right]|\nabla\phi|^4 \geqq 0.$$

Thus, we have obtained the inequality

$$-\Delta_\infty\phi - |\nabla\phi|^4 \geqq -\left[1 - \frac{1}{f'} - \frac{f''}{(f')^2}\right]|\nabla\phi|^4.$$

Now using (iii) of Lemma 2.6, (2.10) and the fact that $\Phi(x_0) = v(x_0)$ we get

(2.12)

$$-\Delta_\infty\phi(x_0) - |\nabla\phi(x_0)|^4 \geqq -\left[1 - \frac{1}{f'(v(x_0))} - \frac{f''(v(x_0))}{(f'(v(x_0))^2}\right](f'(v(x_0)))^4\Lambda_\infty^4.$$

From (2.11) and (2.12) we deduce that

$$\min\{|\nabla\phi(x_0)| - \Lambda_\infty, -\Delta_\infty\phi(x_0) - |\nabla\phi(x_0)|^4\} \geqq \mu(x_0) > 0, \tag{2.13}$$

where

$$\mu(x) = \min\Big\{\big[f'(v(x)) - 1\big]\Lambda_\infty,\\ -\left[1 - \frac{1}{f'(v(x))} - \frac{f''(v(x))}{(f'(v(x))^2}\right]\big(f'(v(x))\big)^4\Lambda_\infty^4\Big\}.$$

Since $\mu(x) > 0$, inequality (2.13) expresses that w is a strict supersolution of (2.2).

We are now ready to complete the proof. Let (x_τ, y_τ) be a maximum point of

$$u(x) - w(y) - \frac{\tau}{2}|x-y|^2$$

in $\overline{\Omega} \times \overline{\Omega}$. From the results of [CIL] it follows that through a subsequence

$$x_{\tau_i} \to x_0 \in \overline{\Omega},$$

where x_0 is a maximum point of $u - w$ in $\overline{\Omega}$. By (2.5) x_0 is in fact an interior point of Ω. We note also that $y_{\tau_i} \to x_0$. From now on we just write τ for τ_i for notational simplicity. Applying the maximum principle for semicontinuous functions we get symmetric matrices $\mathbb{X}_\tau$, $\mathbb{Y}_\tau$ such that

$$(\tau(x_\tau - y_\tau), \mathbb{X}_\tau) \in \overline{D}^{2,+}u(x_\tau), \tag{2.14}$$

$$(\tau(x_\tau - y_\tau), \mathbb{Y}_\tau) \in \overline{D}^{2,-}w(y_\tau), \tag{2.15}$$

$$\langle\mathbb{X}_\tau\xi, \xi\rangle - \langle\mathbb{Y}_\tau\eta, \eta\rangle \leqq 3\tau|\xi - \eta|^2. \tag{2.16}$$

The maximum principle for semicontinuous functions as well as the definition of the semijets $\overline{D}^{2,+}$ and $\overline{D}^{2,-}$ can be found in [CIL].

Since u is a subsolution of (2.2), we have

(2.17)

$$\min\left\{|\tau(x_\tau - y_\tau)| - \Lambda_\infty, -\tau^2\langle \mathbb{X}_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \tau^4|x_\tau - y_\tau|^4\right\} \leqq 0.$$

Since w is a strict supersolution of (2.2), we get from (2.13) that

(2.18)

$$\min\left\{|\tau(x_\tau - y_\tau)| - \Lambda_\infty, -\tau^2\langle \mathbb{Y}_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \tau^4|x_\tau - y_\tau|^4\right\} \geqq \mu(y_\tau) > 0.$$

We now subtract (2.17) from (2.18) to get

(2.19)

$$\begin{aligned}\mu(y_\tau) &\leqq \min\left\{|\tau(x_\tau - y_\tau)| - \Lambda_\infty, -\tau^2\langle \mathbb{Y}_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \tau^4|x_\tau - y_\tau|^4\right\}\\ &\quad - \min\left\{|\tau(x_\tau - y_\tau)| - \Lambda_\infty, -\tau^2\langle \mathbb{X}_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle - \tau^4|x_\tau - y_\tau|^4\right\}\\ &\leqq \tau^2 \max\left\{0, \langle(\mathbb{X}_\tau - \mathbb{Y}_\tau)(x_\tau - y_\tau), (x_\tau - y_\tau)\rangle\right\}\\ &= 0.\end{aligned}$$

Since $\mu(y_\tau) > 0$, we have arrived at a contradiction, and the theorem is thereby proved. □

2.20. Remark. It can be read off from the proof that the comparison principle also holds when *one* of the functions takes the value $-\infty$ on the whole boundary, as $\log u_\infty$ does for instance.

§3. The Principal Frequency of Δ_∞ in a Domain

As an application of the comparison principle (2.3) we are able to prove that Λ_∞ has a property typical of more conventional eigenvalue problems.

3.1. Theorem. *Let Ω be bounded domain in $\mathbb{R}^n$ satisfying $\partial\Omega = \partial\overline{\Omega}$. If u is a continuous positive solution in Ω of the equation*

$$\min\left\{|\nabla u| - \Lambda u, -\Delta_\infty u\right\} = 0, \tag{3.2}$$

with zero boundary values, then $\Lambda = \Lambda_\infty$.

Proof. Fix a point $x_0 \in \Omega$ so that

$$\delta(x_0) = \frac{1}{\Lambda_\infty}.$$

Without loss of generality we may assume that $x_0 = 0$. Suppose that $\Lambda > \Lambda_\infty$. Then the ball $B(0, 1/\Lambda)$ is strictly contained in Ω. Indeed it is away from $\partial\Omega$. Let $\rho(x)$ be the distance function to the boundary of the ball $B(0, 1/\Lambda)$. Both $C\rho(x)$ and $u(x)$ are solutions of (3.2) in $B(0, 1/\Lambda)$ for any positive constant C. By the comparison principle we have

$$\log C\rho(x) \leqq \log u(x)$$

in the ball $B(0, 1/\Lambda)$, leading to a contradiction as $C \to \infty$. Therefore we must have $\Lambda \leqq \Lambda_\infty$.

If $\Lambda < 0$, then $|\nabla u| - \Lambda u > 0$ because u is positive. Thus, equation (3.2) becomes $\Delta_\infty u = 0$ whose only solution with zero boundary values is the zero function. Therefore $\Lambda \geqq 0$.

We claim that $\Lambda \neq 0$. If not, equation (3.2) becomes

$$\min\left\{|\nabla u|, -\Delta_\infty u\right\} = 0. \tag{3.3}$$

Using the definition of a viscosity solution, we easily to check that, in fact, (3.3) is equivalent to $-\Delta_\infty u = 0$, again forcing u to vanish.

So far we have proved that $0 < \Lambda \leqq \Lambda_\infty$. Suppose that $\Lambda < \Lambda_\infty$ and denote $\Omega_\varepsilon = \{x \in \mathbb{R}^n \operatorname{dist}(x, \overline{\Omega}) < \varepsilon\}$. Since $\partial\Omega = \partial\overline{\Omega}$ and $\overline{\Omega}$ is compact, we have for small $\varepsilon > 0$ that $\Lambda_\infty(\Omega_\varepsilon) > \Lambda$. Now let Ω_Λ be the domain obtained by connecting Ω_ε to a ball of radius $1/\Lambda$ with a sufficiently narrow tube. For this new domain the reciprocal of the maximum of the distance from the boundary is now Λ and also $\overline{\Omega} \subset \Omega_\Lambda$. Consider a genuine ∞-eigenfunction of Ω_Λ, say u_Λ. Both Cu_Λ and u are solutions to the same equation in Ω. The comparison principle (2.3) can be used in this situation, since u_Λ is positive on $\partial\Omega$. It gives

$$\log u(x) \leqq \log Cu_\Lambda(x)$$

for $x \in \Omega$. We arrive at a contradiction by letting $C \to 0^+$. □

3.4. Remark. It is quite easy to give an example of a domain Ω and a number $0 < \Lambda < \Lambda_\infty$ for which the above argument cannot be applied. Nevertheless we think that the result itself is true even without the assumption $\partial\Omega = \partial\overline{\Omega}$.

§4. Examples

We now use the limit equation (1.22) to conclude that the distance function

$$\delta(x, y) = \frac{1 - (|x| + |y|)}{\sqrt{2}}$$

is not a genuine ∞-eigenfunction of the square

$$\Omega = \left\{(x, y) \colon |x| + |y| < 1\right\}$$

centered at the origin. In other words, $\delta(x, y)$ is not the limit of eigenfunctions $u_p(x, y)$ as $p \to \infty$. Note that $\Lambda_\infty = \sqrt{2}$. The ridge set of Ω (the set of points at which δ is not in C^1) consists of the intersection of Ω with the coordinate axes.

4.1. Proposition. *Along the ridge of Ω the distance function $\delta(x, y)$ is not a viscosity subsolution of*

$$\min\{|\nabla u| - \sqrt{2}\, u, -\Delta_\infty u\} = 0. \tag{4.2}$$

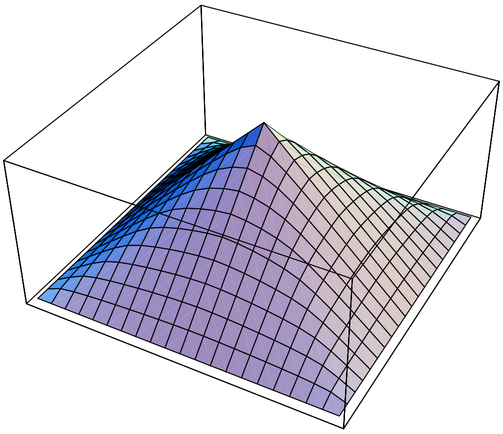

Fig. 4.1.

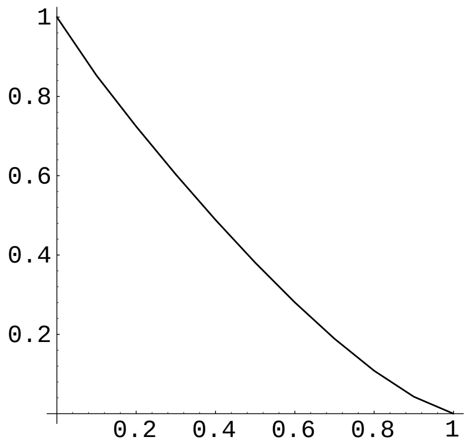

Fig. 4.2.

Proof. Select a point in the ridge, for example, the point $\left(0, \frac{1}{2}\right)$. We will exhibit a C^2 function $\phi(x, y)$ satisfying

$$\delta\left(0, \tfrac{1}{2}\right) = \phi\left(0, \tfrac{1}{2}\right) = \frac{1}{2\sqrt{2}}, \tag{4.3}$$

$$\delta(x, y) < \phi(x, y) \text{ in a neighborhood of } \left(0, \tfrac{1}{2}\right), \tag{4.4}$$

$$\min\left\{\left|\nabla\phi\left(0, \tfrac{1}{2}\right)\right| - \sqrt{2}\,\phi\left(0, \tfrac{1}{2}\right), -\Delta_\infty\phi\left(0, \tfrac{1}{2}\right)\right\} > 0. \tag{4.5}$$

This shows that $\delta(x, y)$ cannot be a subsolution of $\min\left\{|\nabla u| - \sqrt{2}\,u, -\Delta_\infty u\right\} = 0$. To find this ϕ start out with

$$\phi_0(x, y) = \frac{1}{2\sqrt{2}} + ax - \frac{1}{\sqrt{2}}\big(y - \tfrac{1}{2}\big) + bx^2 + cx(y - \tfrac{1}{2}) + d\big(y - \tfrac{1}{2}\big)^2$$

and require that

$$\frac{1 - (|x| + |y|)}{\sqrt{2}} < \phi_0(x, y)$$

in a neighborhood of $(0, \frac{1}{2})$. Elementary considerations show that the choice $a = 1/2\sqrt{2}$, $b = -1$, $c = 0$ and $d = 0$ gives us a function ϕ_0 satisfying (4.3), (4.4) with "$\leqq$" instead of "$<$", and (4.5). To get the strict inequality just consider $\phi(x, y) = \phi_0(x, y) + x^4 + (y - \frac{1}{2})^4$. □

In the case of the square one can prove uniqueness of smooth (C^1) genuine ∞-eigenfunctions. Although we believe that solutions are indeed of class C^1 off the center of the square, we have not yet been able to prove it.

Normalizing u_∞ so that $u_\infty(0, 0) = \delta(0, 0) = 1/\sqrt{2}$ we have

$$\frac{1}{\sqrt{2}} - \sqrt{x^2 + y^2} \leqq u_\infty(x, y) \leqq \delta(x, y)$$

by comparison. The lower bound is the distance to the largest inscribed circle. On the lines $x = \pm y$ we have equality $u_\infty = \delta$. This shows that u_∞ cannot be a concave function. However, $\log u_\infty$ is concave; cf. [S]. The graph of the solution on a square is shown in Figure 4.1 and the graph of the diagonal cross-section, showing the lack of concavity, is shown in Figure 4.2.

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