The Einstein static universe in Loop Quantum Cosmology

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Abstract.
Loop Quantum Cosmology strongly modifies the high-energy dynamics of Friedman-Robertson-Walker models and removes the big-bang singularity. We investigate how LQC corrections affect the stability properties of the Einstein static universe. In General Relativity, the Einstein static model with positive cosmological constant Λ is unstable to homogeneous perturbations. Using dynamical systems methods, we show that LQC modifications can lead to an Einstein static model which is neutrally stable for a large enough positive value of Λ.

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1. Introduction

The Einstein static universe in General Relativity (GR) is a closed Friedman-Robertson-Walker (FRW) model that is unstable to homogeneous perturbations [1]. (Note that it is neutrally stable to inhomogeneous scalar perturbations with high enough sound speed and to vector and tensor perturbations [2].)

The stability of Einstein static models in high-energy modifications of GR is an interesting mathematical question, and is also relevant for scenarios in which the Einstein static is an initial state for a past-eternal inflationary cosmology, the so-called Emergent Universe scenario [3]. The standard model of cosmology (see e.g. [4]) has a flat infinite spatial geometry, and experiences a period of primordial inflationary expansion, which is preceded by a big bang singularity in the classical theory. Observations however do not prove that the geometry is flat: the Universe could have nonzero spatial curvature, as long as the late-time effect of this curvature is very small. In particular, a positive curvature allows for an "Emergent Universe" that originates asymptotically in the past as an Einstein static universe, and then inflates and later reheats to a hot big bang era. This model generalizes the Eddington-Lemaître model [5], and is a counter-example to the notion that inflation can never be past-eternal and thus cannot avoid an initial singularity – because it is closed, it avoids the theorems showing that inflation cannot be past eternal [6].

Generalizations of the Einstein static solution in high-energy modifications to GR have been considered in the Randall-Sundrum braneworld scenario [7] and in $f(R)$ theories [8]. The homogeneous stability of Einstein static models in $R + \alpha R^2$ gravity has also been investigated [9].

Another theory leading to high-energy modifications of GR is Loop Quantum Cosmology (LQC) [10], which is a canonical quantization of homogeneous cosmological spacetimes based on Loop Quantum Gravity [11]. The gravitational phase-space variables are an su(2) valued connection and conjugate triad, and the elementary variables underlying the quantization are the holonomies of the connection and the fluxes of the triad. The quantum theory obtained from LQC turns out to be inequivalent to Wheeler-de Witt quantization (the LQC polymer representation is different from the usual Wheeler-de Witt Schrodinger representation). Wheeler-de Witt quantization does not resolve the cosmological singularity, but in LQC a generic resolution of such singularities has been obtained. Initial quantizations of LQC lead to a regularization of the big bang singularity [12] resulting from the fact that the quantum Einstein equation is non-singular as well as from modifications to the scalar field energy density and dynamics. The modifications to the scalar field dynamics were based on effects arising from quantum inverse scale factor operators. Subsequently, the elucidation of the consequences of using holonomies as the basic variables has shown that gravity is modified [13], i.e., the structure of the Friedman equation, rather than solely the energy density of the matter field. These gravitational modifications typically become important at lower energy scales than the modifications to the scalar field dynamics.

Mulryne et al. [14] used the scalar-field modification approach to investigate the stability of the Einstein static model to homogeneous perturbations. They found that the new LQC Einstein static model is a centre fixed point in phase space, i.e. a neutrally stable point, for a massless scalar field with $w \equiv p_\phi/\rho_\phi = 1$. This modification of stability behaviour has important consequences for the Emergent Universe scenario, since it ameliorates the fine-tuning that arises from the fact that the Einstein static is an unstable saddle in GR.

Here we consider the same question, but using the LQC gravitational modifications, and neglecting the higher energy modifications to matter. We consider a perfect fluid with $p = w\rho$ and $w > -1$. We show that, for all $w > -1/3$, there is a new centre, i.e. a neutrally stable point representing an Einstein static model, but only if the cosmological constant Λ is above a critical scale,

$$\Lambda > 6.6\pi M_P^2\,. \qquad (1)$$

2. Critical points of closed FRW models in LQC

The loop quantum effects that we investigate manifest themselves in the form of a modification to the classical Friedmann equation. For the closed FRW model, the explicit form is given by [13]

$$H^2 = \left(\frac{\kappa}{3}\rho + \frac{\Lambda}{3} - \frac{1}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} - \frac{\Lambda}{\kappa\rho_c} + \frac{3}{\kappa\rho_c a^2}\right), \qquad (2)$$

where $\kappa = 8\pi G = 8\pi/M_P^2$, and the critical LQC energy density is

$$\rho_c \approx 0.82 M_P^4\,. \qquad (3)$$

It is evident that the first term in parentheses is the classical right hand side of the Friedmann equation, with the quantum modifications appearing in the second term. The classical GR limit is achieved in the limit as ρ_c goes to infinity whence the second term approaches unity. The classical energy conservation equation continues to hold‡,

$$\dot\rho = -3H\rho\,(1+w)\ . \qquad (4)$$

Note that $H^2 \geq 0$ imposes the limits

$$\frac{3}{a^2} \leq \kappa\rho + \Lambda \leq \kappa\rho_c + \frac{3}{a^2}\,. \qquad (5)$$

The modified Raychaudhuri equation follows from Eqs. (4) and (2):

$$\dot H = -\frac{\kappa}{2}\rho\,(1+w)\left(1 - \frac{2\rho}{\rho_c} - \frac{2\Lambda}{\kappa\rho_c}\right) + \\ + \left[1 - \frac{2\rho}{\rho_c} - \frac{2\Lambda}{\kappa\rho_c} - \frac{3\rho(1+w)}{\rho_c}\right]\frac{1}{a^2} + \frac{6}{\kappa\rho_c a^4}. \qquad (6)$$

‡ Note that this means we are not considering the inverse volume effects that would modify the scalar field energy density and hence modify the energy conservation equation. For the closed model there are indications that the inverse volume effects are negligible if it is required that the universe reach macroscopic size [13].

We will find the critical points to the system of Eqs. (4), (6) as well as

$$\dot{a} = aH\,, \tag{7}$$

which follows from the definition of H. The solution space is a 2-dimensional surface in the three-dimensional (ρ, a, H) space, defined by the Friedman constraint (2). The system admits two critical points, which are static solutions $\dot{a} = \dot{H} = \dot{\rho} = 0$. The first critical point is the standard GR Einstein static universe, while the second is a new LQC Einstein static universe:

$$\rho_{GR} = \frac{2\Lambda}{\kappa(1+3w)}\,, \quad a^2_{GR} = \frac{2}{\kappa\rho_{GR}(1+w)}\,, \tag{8}$$

$$\rho_{LQ} = \frac{2(\Lambda - \kappa\rho_c)}{\kappa(1+3w)}\,, \quad a^2_{LQ} = \frac{2}{\kappa\rho_{LQ}(1+w)}\,. \tag{9}$$

The conditions under which these static solutions exist are summarized in Table 1, and follow from $a^2, \rho > 0$.

A remarkable feature of the new LQ fixed point is that it is possible to have *an Einstein static universe even for vanishing cosmological constant.* Indeed, as one can see from Eq. (9) and Table 2, when $\Lambda = 0$ the LQ fixed point exists and is unstable.

Table 1. Conditions for the existence of Einstein static critical points.

GR	$\Lambda > 0$	$w > -1/3$
	$\Lambda < 0$	$-1 < w < -1/3$
LQ	$\Lambda < \kappa\rho_c$	$-1 < w < -1/3$
	$\Lambda > \kappa\rho_c$	$w > -1/3$

For the system of equations (4), (6) and (7), linearized stability analysis fails to give complete information about the properties of the two critical points, when they both exist, since in all cases, for one of the two points there is always a pair of complex eigenvalues with vanishing real part. In this case the linearization theorem (see e.g. [15]) tells us that a fixed point which is a centre for the linearized system is not necessarily a centre for the full nonlinear system. In addition, the linearization of equations (4), (6) and (7) always leads to a third vanishing eigenvalue, simply because the actual dynamics is two-dimensional because of the modified Friedman equation (2). Therefore, using this constraint, it is convenient to rewrite the system reducing the number of equations to two, the dimension of the configuration space. Solving Eq. (2) for a^2, we find that

$$a^2 = f_\pm(\rho, H)\,, \tag{10}$$

where

$$f_\pm = \frac{3}{2}\frac{\left[2(\kappa\rho+\Lambda) + \kappa\rho_c\left(1 \pm \sqrt{1 - 12H^2/\kappa\rho_c}\right)\right]}{(\kappa\rho+\Lambda)^2 + \kappa\rho_c\,(3H^2 - \kappa\rho - \Lambda)}\,. \tag{11}$$

Substituting this into Eq. (6), we find two branches for the time derivative of the Hubble parameter,

$$\text{LQ} : \dot{\rho} = -3H\rho\,(1+w) \quad \text{and} \quad \dot{H} = F_+(\rho, H)\,, \tag{12}$$

$$\mathrm{GR}: \dot{\rho} = -3H\rho\,(1+w) \quad \text{and} \quad \dot{H} = F_-(\rho, H)\,, \tag{13}$$

where

$$F_\pm = -\frac{\kappa}{2}(1+w)\rho\left(1-\frac{2\rho}{\rho_c}-\frac{2\Lambda}{\kappa\rho_c}\right)+\frac{6}{\kappa\rho_c f_\pm^2}+ \\ +\frac{1}{f_\pm}\left[1-\frac{2\rho}{\rho_c}-\frac{2\Lambda}{\kappa\rho_c}-3(1+w)\frac{\rho}{\rho_c}\right]. \tag{14}$$

In the classical limit,

$$\lim_{\rho_c\to\infty} f_+ = 0\,, \tag{15}$$

$$\lim_{\rho_c\to\infty} f_- = \frac{3}{\kappa\rho - 3H^2+\Lambda}\,, \tag{16}$$

where the second equation is the GR Friedman equation. This shows how the two branches in Eq. (10) recover the GR and the new quantum static solution. In addition,

$$\lim_{\rho_c\to\infty} F_+ = \infty\,, \tag{17}$$

$$\lim_{\rho_c\to\infty} F_- = -H^2 - \frac{\kappa\rho}{6}(1+w)+\frac{\Lambda}{3}\,, \tag{18}$$

where the first limit is consistent with Eq. (15), and the second limit gives the GR Raychaudhuri equation.

The system (12) admits the static solution with ρ_{LQ} as in Eq. (9). Substituting this into Eq. (11) reproduces a_{LQ}^2 as in Eq. (9). We evaluate the eigenvalues of the Jacobian matrix at this point, to find the two eigenvalues

$$\lambda_{LQ} = \pm\sqrt{(\kappa\rho_c-\Lambda)(1+w)}\,. \tag{19}$$

Thus the LQ fixed point is either unstable (of the saddle kind), when $\kappa\rho_c > \Lambda$ and $-1 < w < -1/3$, or a centre for the linearized system, i.e. a neutrally stable fixed point, when $\kappa\rho_c < \Lambda$ and $w > -1/3$. (The limits take into account the conditions in Table 1.) Again, for the latter point the linearized analysis is not sufficient and therefore we turn to a numerical analysis in the next section.

For the system Eq. (13), we find the GR static solution, and the eigenvalues of the linearized system are

$$\lambda_{GR} = \pm\sqrt{\Lambda(1+w)}\,. \tag{20}$$

These are real with opposite signs for $\Lambda > 0$ and $w > -1/3$, so that the fixed point is unstable (of the saddle type). For $\Lambda < 0$ and $-1 < w < -1/3$, the fixed point is a centre, as confirmed by numerical analysis. It does not appear to be widely known that the GR Einstein static universe can be neutrally stable when $\Lambda < 0$ (see also [9]).

The results of the linearized stability analysis are summarized in Table 2.

3. Numerical integration

In order to extend the linearized stability analysis, we perform numerical integrations of the systems (12) and (13). We also integrate the nonlinear system of Eqs. (4), (6)

Table 2. Eigenvalues for the critical points in Table 1.

			λ_1	λ_2
GR				
$\Lambda > 0$	and	$w > -1/3$	> 0	< 0
$\Lambda < 0$	and	$-1 < w < -1/3$	$\mathrm{Re}(\lambda_1) = 0$	$\mathrm{Re}(\lambda_2) = 0$
			$\mathrm{Im}(\lambda_1) > 0$	$\mathrm{Im}(\lambda_2) < 0$
LQ				
$\Lambda < \kappa\rho_c$	and	$-1 < w < -1/3$	> 0	< 0
$\Lambda > \kappa\rho_c$	and	$w > -1/3$	$\mathrm{Re}(\lambda_1) = 0$	$\mathrm{Re}(\lambda_2) = 0$
			$\mathrm{Im}(\lambda_1) > 0$	$\mathrm{Im}(\lambda_2) < 0$

and (7), with initial conditions fulfilling the Friedman constraint (2), in order to show the full configuration space diagrams and gain a better understanding of the dynamics. We take $\Lambda > 0$ and $w = 1$.

First we consider the case $\Lambda > \kappa\rho_c$. The (H, ρ) plots in Figs. 1 and 2 are obtained by integrating the system (12) and (13) in some neighbourhood of the fixed points. The plots show that the GR Einstein static solution is a saddle while the LQ Einstein static solution is a centre.

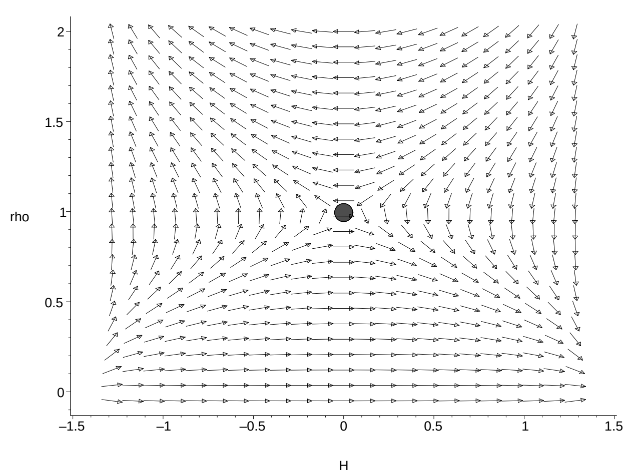

Figure 1. Dynamical behaviour of the system around the GR fixed point for the case $\Lambda > \kappa\rho_c$, with $\Lambda/\kappa = 2$, $w = 1$ (using units $M_P = 1$).

A better understanding can be obtained by plotting the whole 3D space (H, ρ, a) for a wide range of initial conditions, shown in Fig. 3. The trajectories lie on the 2-dimensional Friedman constraint surface.

The behaviour near the fixed point is in agreement with the linearized stability analysis, but new interesting features arise. For initial conditions far enough from the fixed point, there are trajectories that wrap around the Friedman tube, so that cyclic models are possible even if they are not related with the centre fixed point, since they

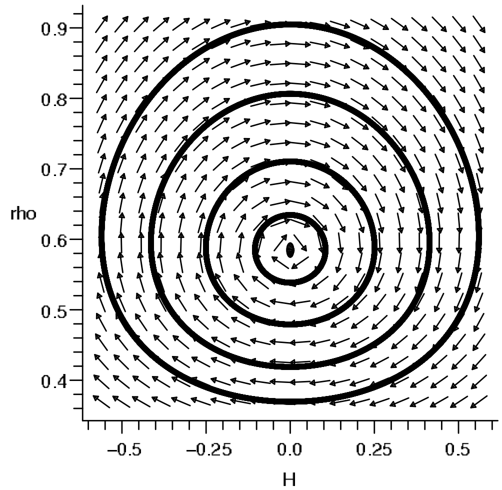

Figure 2. As in Fig. 1, for the LQ fixed point.

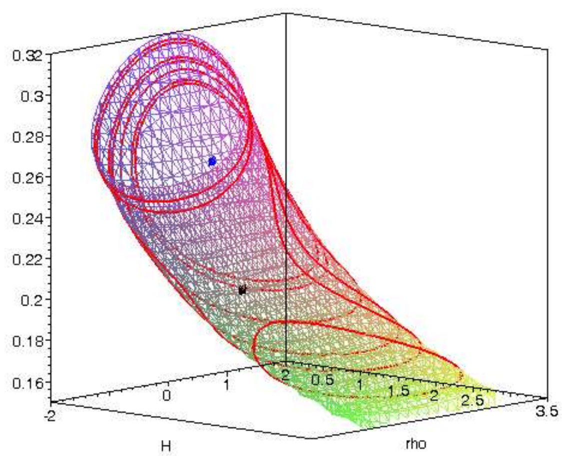

Figure 3. Trajectories on the Friedman constraint surface, for the same parameters as in Figs. 1 and 2. The GR fixed point is at the bottom of the Friedman surface, while the LQ point is at the top. Note that some trajectories wrap around the "tube" but cannot be shrunk continuously to the LQ fixed point.

cannot be shrunk to a point; see Fig. 4.

This behaviour can also be interpreted via a plot of the energy density against a. Defining

$$\rho_- = \frac{3}{\kappa a^2}\,,\ \rho_+ = \rho_c + \frac{3}{\kappa a^2}\,,\ \rho_m = \rho + \frac{\Lambda}{\kappa}\,, \tag{21}$$

Eq. (2) can be written in the form

$$H^2 = \frac{\kappa}{3\rho_c}\left(\rho_m - \rho_-\right)\left(\rho_+ - \rho_m\right). \tag{22}$$

When $\rho_m = \rho_-(a)$ or $\rho_+(a)$, then H=0, so the system undergoes a bounce or starts a recollapsing phase, respectively. This is illustrated in Fig. 5.

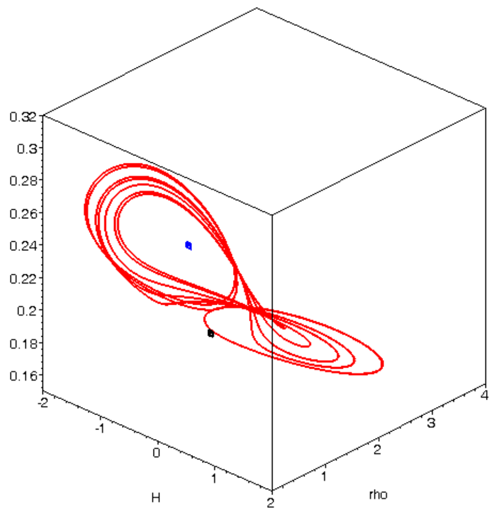

Figure 4. As in Fig. 3, showing trajectories that start far from the fixed points.

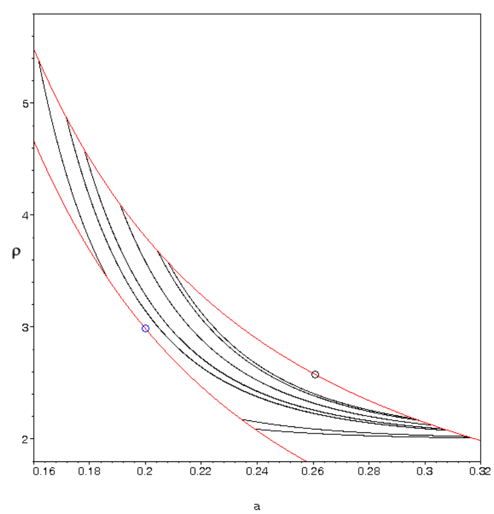

Figure 5. The upper red (grey) curve is $\rho_+(a)$ and $\rho_-(a)$ is the lower curve. Trajectories for several initial conditions are depicted in black. The GR fixed point is the box on the curve ρ_-, and the LQ point is the circle on the curve ρ_+.

The second case $\Lambda < \kappa\rho_c$ is illustrated in Fig. 6. For $w = 1$, only the GR fixed point is present and it is unstable. The trajectories are again wrapped around the Friedman surface. During a contracting phase they tend to a minimum of the scale factor and then, after a bounce, there is a phase of expansion.

We can integrate Eq. (4) for ρ as a function of a, and substitute the expression for $a(\rho)$ into the modified Friedman equation (2). Using the dimensionless variables,

$$x^2 = \frac{3\kappa\rho_c}{\Lambda^2} H^2 \,, \quad y = \frac{\kappa\rho}{\Lambda} \,, \quad B = \frac{\kappa\rho_c}{\Lambda} \,,$$

this leads to

$$x^2 = \left[y + 1 - C y^{2/3(w+1)}\right]\left[B - y - 1 + C y^{2/3(w+1)}\right], \tag{23}$$

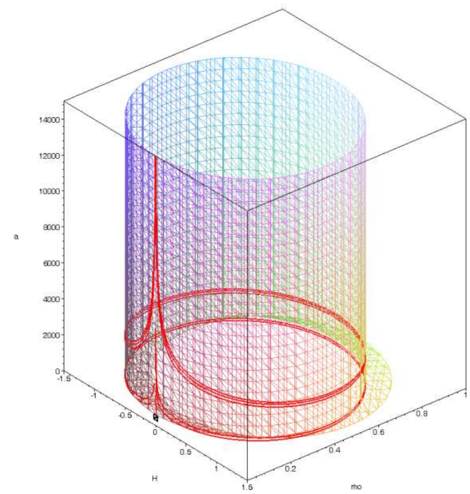

Figure 6. The case $\Lambda/\kappa = 4 \times 10^{-7}$, with $w = 1$.

where C is a constant of integration. This expression allows us to get a better understanding of the particular case depicted in Figs. 3 and 4, where both the fixed points are present, by finding the separatrix. The separatrix is a curve (actually, a union of orbits) which marks the boundary of regions where the dynamical behaviour of the system is different. In this case the separatrix is the junction of the stable and unstable manifold with the GR Einstein static hyperbolic fixed point. First we solve Eq. (23) for C, then we substitute numerical values for the parameters $w = 1$ and $B = .4121769562$ (using $M_P = 1$), and we also substitute $x = 0$ and $y = 1/2$ for the GR fixed point. This produces a numerical value for C. Through this procedure, the equation of the separatrix is implicitly given by Eq. (23) with fixed values of the parameters C, w and B.

More insight can be acquired via the plot of the separatrix projected onto the (x, y)-plane, as shown in Fig. 7. The curves depicted are the separatrix, which wraps around the Friedman surface, and some other trajectories which are first integrals obtained for different values of the integration constants. The closed loops around the Friedman surface are outside the region marked by the separatrix; the other curve is not closed around the tube, and it can be shrunk continuously to the LQ fixed point.

4. Conclusions

We have shown that LQC gravitational modifications to the Friedman equation lead to a new high-energy critical point for the Einstein static universe. This LQC Einstein static model is an unstable saddle (like the standard GR Einstein static solution) for sufficiently negative pressure and a sub-critical cosmological constant, i.e.,

$$-1 < w < -\frac{1}{3}, \quad \Lambda < \kappa\rho_c \,. \tag{24}$$

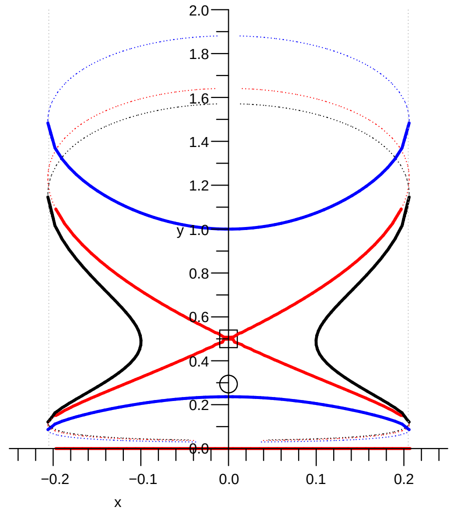

Figure 7. Projection of the phase space onto the (x, y)-plane for the case of Figs. 3 and 4. It represents the trajectories as seen from the bottom of the Friedman surface. Both the GR fixed point (a box) and the LQ fixed point (a circle) exist. The two vertical lines are the edges of the Friedman surface. The red curve represents the two branches of the separatrix.

If w is large enough and the cosmological constant is above the critical value, then the LQC Einstein static model is a centre:

$$w > -\frac{1}{3}, \quad \Lambda > \kappa\rho_c \,. \tag{25}$$

This neutrally stable behaviour is in strong contrast to the GR case, where the Einstein static model is unstable for all (positive) values of Λ.

Modified stability of the Einstein static model is also found for $R + \alpha R^2$ gravity [9] and for the LQC case with matter modifications, rather than gravitational modifications [14]. This illustrates the general point that high-energy modifications to GR, which typically strongly modify the big bang singularity, also modify the dynamical nature of the non-singular Einstein static universe.

Our result means that the fine-tuning problem for the Emergent Universe scenario is qualitatively changed by LQC gravitational modifications. In GR, the Einstein static is an unstable saddle, so that severe fine-tuning is required if the Einstein static is to be the initial state for a past-eternal inflationary cosmology. With LQC gravitational modifications, the initial state becomes a centre when Eq. (25) holds, thus ameliorating the fine-tuning. The same point was made in Mulryne et al. [14] in the case of LQC matter modifications. In that case however, the analysis is restricted to a massless scalar field ($w = 1$), and the centre exists for all Λ. Our analysis applies for all perfect fluids with $w > -1$, since we focus on the LQC gravitational modifications. These gravitational modifications operate typically at a lower energy scale than the matter modifications. Clearly, from the standpoint of the Emergent Universe scenario, one needs a further mechanism to break the infinite cycles about the centre fixed point, leading to a subsequent inflationary phase as discussed in [16] for a general class of braneworld models sourced by a scalar field with a constant potential. This mechanism will be the subject of future work.

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