

THE ELASTICITY COMPLEX: COMPACT EMBEDDINGS AND REGULAR DECOMPOSITIONS

DIRK PAULY AND WALTER ZULEHNER

ABSTRACT. We investigate the Hilbert complex of elasticity involving spaces of symmetric tensor fields. For the involved tensor fields and operators we show closed ranges, Friedrichs/Poincaré type estimates, Helmholtz type decompositions, regular decompositions, regular potentials, finite cohomology groups, and, most importantly, new compact embedding results. Our results hold for general bounded strong Lipschitz domains of arbitrary topology and rely on a general functional analysis framework (FA-ToolBox). Moreover, we present a simple technique to prove the compact embeddings based on regular decompositions/potentials and Rellich's selection theorem, which can be easily adapted to any Hilbert complex.

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1. INTRODUCTION

This paper is devoted to a discussion of the elasticity complex

$$(1) \quad \mathbf{L}^2(\Omega) \xrightarrow{\text{sym } \nabla} \mathcal{L}_{\mathbb{S}}^2(\Omega) \xrightarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} \mathcal{L}_{\mathbb{S}}^2(\Omega) \xrightarrow{\text{Div}_{\mathbb{S}}} \mathbf{L}^2(\Omega),$$

which plays an important role in the analysis of the elasticity equations as well as in the construction and analysis of appropriate numerical methods.

The elasticity complex is a Hilbert complex, i.e., it consists of a sequence of Hilbert spaces, here involving $\mathbf{L}^2(\Omega)$ and $\mathcal{L}_{\mathbb{S}}^2(\Omega)$, the spaces of square-integrable vector fields and symmetric tensor fields on a domain $\Omega \subset \mathbb{R}^3$, and a sequence of densely defined and closed linear operators, here the (unbounded) differential operators $\text{sym } \nabla$, $\text{Rot Rot}_{\mathbb{S}}^{\top}$, $\text{Div}_{\mathbb{S}}$, which denote the symmetric part of the gradient of vector fields, the Saint-Venant operator, given by

$$\text{Rot Rot}_{\mathbb{S}}^{\top} \mathbf{M} = \text{Rot} \left((\text{Rot } \mathbf{M})^{\top} \right)$$

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for symmetric tensor fields \mathbf{M} , where Rot is the (row-wise) rotation of a tensor field, and the (row-wise) divergence of symmetric tensor fields, respectively.

The key property of a complex is that the superposition of two successive operators vanishes, here

$$\text{Rot Rot}_{\mathbb{S}}^{\top} \text{sym } \nabla = 0, \quad \text{Div}_{\mathbb{S}} \text{Rot Rot}_{\mathbb{S}}^{\top} = 0.$$

In particular, we will consider the operators in (1) on their respective domains of definition of fields with homogeneous boundary conditions

$$\mathring{\mathbf{H}}(\text{sym } \nabla, \Omega) = \mathring{\mathbf{H}}^1(\Omega) \subset \mathbf{L}^2(\Omega), \quad \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega) \subset \mathcal{L}_{\mathbb{S}}^2(\Omega), \quad \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Div}, \Omega) \subset \mathcal{L}_{\mathbb{S}}^2(\Omega),$$

which leads to the associated (primal) domain complex

$$\mathring{\mathbf{H}}^1(\Omega) \xrightarrow{\text{sym } \nabla} \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega) \xrightarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Div}, \Omega) \xrightarrow{\text{Div}_{\mathbb{S}}} \mathbf{L}^2(\Omega),$$

where $\mathring{\text{sym}} \nabla$, $\mathring{\text{Rot}} \text{Rot}_{\mathbb{S}}^{\top}$, $\mathring{\text{Div}}_{\mathbb{S}}$ indicate the restrictions of $\text{sym } \nabla$, $\text{Rot Rot}_{\mathbb{S}}^{\top}$, $\text{Div}_{\mathbb{S}}$ to these domains of definition. This domain complex consists of bounded linear operators on their respective domains of definition, which are Hilbert spaces if equipped with the respective graph norms.

The corresponding dual domain complex is given by

$$\mathbf{L}^2(\Omega) \xleftarrow{-\text{Div}_{\mathbb{S}}} \mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega) \xleftarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega) \xleftarrow{-\text{sym } \nabla} \mathbf{H}^1(\Omega)$$

with the respective domains of definition of fields with no boundary conditions

$$\mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega) \subset \mathcal{L}_{\mathbb{S}}^2(\Omega), \quad \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega) \subset \mathcal{L}_{\mathbb{S}}^2(\Omega), \quad \mathbf{H}(\text{sym } \nabla, \Omega) = \mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega),$$

see Section 2 for detailed definitions of the involved spaces and operators.

As novel and main result of this paper we will show that the embeddings

$$(2) \quad \begin{aligned} \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega) &\hookrightarrow \mathcal{L}_{\mathbb{S}}^2(\Omega), \\ \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega) &\hookrightarrow \mathcal{L}_{\mathbb{S}}^2(\Omega) \end{aligned}$$

are compact for bounded strong Lipschitz domains Ω , see Theorem 4.11. The proofs are based on the existence of regular potentials and regular decompositions, in particular, if Ω is additionally topologically trivial. From these compact embeddings, closed ranges of the involved operators, finite-dimensional cohomology groups, geometric Helmholtz type decompositions, and Friedrichs/Poincaré type estimates follow then by abstract functional analytic arguments. For details see the FA-ToolBox, e.g., from [25, Section 2.1] or [20, 21, 22, 23], and, in particular, Theorem 4.17 and the results presented in [26].

Similar results are well-known, e.g., for the classical de Rham domain complex in 3D

$$(3) \quad \begin{aligned} \mathring{\mathbf{H}}^1(\Omega) &\xrightarrow{\mathring{\nabla}} \mathring{\mathbf{H}}(\text{rot}, \Omega) \xrightarrow{\mathring{\text{rot}}} \mathring{\mathbf{H}}(\text{div}, \Omega) \xrightarrow{\mathring{\text{div}}} \mathbf{L}^2(\Omega), \\ \mathbf{L}^2(\Omega) &\xleftarrow{-\text{div}} \mathbf{H}(\text{div}, \Omega) \xleftarrow{\text{rot}} \mathbf{H}(\text{rot}, \Omega) \xleftarrow{-\nabla} \mathbf{H}^1(\Omega) \end{aligned}$$

with the gradient ∇ of scalar functions, the rotation rot and the divergence div of vector fields, on their respective domains of definition of fields with and without boundary conditions, and also for the classical de Rham complex in ND (or even on Riemannian manifolds) using differential forms. Note that in the de Rham complex (of any dimension) no second order operators occur, contrary to the elasticity complex. Here, the famous Weck selection theorems [31], i.e., the compactness of the embeddings

$$\mathring{\mathbf{H}}(\text{rot}, \Omega) \cap \mathbf{H}(\text{div}, \Omega) \hookrightarrow \mathbf{L}^2(\Omega), \quad \mathring{\mathbf{H}}(\text{div}, \Omega) \cap \mathbf{H}(\text{rot}, \Omega) \hookrightarrow \mathbf{L}^2(\Omega),$$

for bounded (possibly non-smooth) domains play the central role. In Weck's work [31] one finds the first result for piece-wise smooth domains, and in Weber's paper [30] the first proof being valid for strong Lipschitz domains is given. See Picard's contribution [29] for the first result for weak Lipschitz domains. For the most recent state of the art results holding for weak Lipschitz domains and even for mixed boundary conditions see [11] and [12, 13].

We will follow the approach taken in [25] but with some significant improvements. In [25] we investigated the Gradgrad-domain complex and its dual the divDiv-domain complex, i.e.,

$$(4) \quad \begin{array}{ccccccc} \mathring{H}^2(\Omega) & \xrightarrow{\nabla \nabla} & \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot}, \Omega) & \xrightarrow{\text{Rot}_{\mathbb{S}}} & \mathring{\mathcal{H}}_{\mathbb{T}}(\text{Div}, \Omega) & \xrightarrow{\text{Div}_{\mathbb{T}}} & \mathbf{L}^2(\Omega), \\ \mathbf{L}^2(\Omega) & \xleftarrow{\text{div Div}_{\mathbb{S}}} & \mathcal{H}_{\mathbb{S}}(\text{div Div}, \Omega) & \xleftarrow{\text{sym Rot}_{\mathbb{T}}} & \mathcal{H}_{\mathbb{T}}(\text{sym Rot}, \Omega) & \xleftarrow{-\text{dev } \nabla} & \mathbf{H}^1(\Omega), \end{array}$$

where the subscript \mathbb{T} refers to spaces and operators of deviatoric tensor fields, $\nabla \nabla$ is the Hessian of scalar functions. These complexes are related to the biharmonic equation and the Einstein-Bianchi equations from general relativity. We have shown that the embeddings¹

$$\begin{aligned} \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot}, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\text{div Div}, \Omega) &\hookrightarrow \mathcal{L}_{\mathbb{S}}^2(\Omega), \\ \mathring{\mathcal{H}}_{\mathbb{T}}(\text{Div}, \Omega) \cap \mathcal{H}_{\mathbb{T}}(\text{sym Rot}, \Omega) &\hookrightarrow \mathcal{L}_{\mathbb{T}}^2(\Omega) \end{aligned}$$

are compact, see [25, Lemma 3.22].

We emphasize that all these results fit into the common framework of Hilbert domain complexes of the form

$$(5) \quad \begin{array}{ccccc} D(A_0) & \xrightarrow{A_0} & D(A_1) & \xrightarrow{A_1} & H_2, \\ H_0 & \xleftarrow{A_0^*} & D(A_0^*) & \xleftarrow{A_1^*} & D(A_1^*) \end{array}$$

and compact embeddings

$$D(A_1) \cap D(A_0^*) \hookrightarrow H_1$$

for densely defined and closed (unbounded) linear operators

$$A_0 : D(A_0) \subset H_0 \rightarrow H_1, \quad A_1 : D(A_1) \subset H_1 \rightarrow H_2$$

on three Hilbert spaces H_0, H_1, H_2 with (Hilbert space) adjoints

$$A_0^* : D(A_0^*) \subset H_1 \rightarrow H_0, \quad A_1^* : D(A_1^*) \subset H_2 \rightarrow H_1.$$

The key property of a complex reads generally $A_1 A_0 = 0$, i.e.,

$$R(A_0) \subset N(A_1)$$

or equivalently $R(A_1^*) \subset N(A_0^*)$. Here and throughout this paper, the domain of definition, the kernel, and the range of a linear operator A are denoted by $D(A)$, $N(A)$, and $R(A)$, respectively.

Some important applications, such as proper solution theories for first or second order systems, various variational formulations, and a posteriori error estimates for the respective operators of a Hilbert complex are discussed in detail in [23].

The elasticity complex is subject to past, recent, and ongoing research. Among others, we cite the important works in [3], [4], [6, 5, 7, 8], [10], [16, 17] [1, 2], [14], [18, 19] as well as the vast literature cited therein. When writing this manuscript, cf. our arXiv-preprint [27], we were not aware of papers on the elasticity complex dealing with compact embeddings. After completion we found the recent arXiv-preprint [9] on which the following remark is about.

Remark 1.1. *In this paper as well as in [25] properties of the elasticity complex, the $\nabla \nabla$ -complex and the $\text{div Div}_{\mathbb{S}}$ -complex are derived from known properties of the de Rham complex based on the same principle. The strategy in our work is to show in a first step compact embeddings. Then all the other important properties like closed ranges and finite-dimensional cohomology groups follow by abstract functional analytic arguments. A different approach is taken in [9], which deals in a more general way with the question how to derive complexes and their properties from known complexes. The basic technique in [9] is to utilize the Bernstein-Gelfand-Gelfand (BGG) resolution, which can be seen as a primarily algebraic procedure for deriving new complexes. The first step in [9] for analyzing these new complexes is to verify that the cohomology groups of the new complexes are*

¹A closer inspection of the respective proof of [25, Lemma 3.22] shows immediately that even the embedding

$$\mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot}, \Omega) \cap \mathcal{H}_{\mathbb{S}}^{0,-1}(\text{div Div}, \Omega) \hookrightarrow \mathcal{L}_{\mathbb{S}}^2(\Omega)$$

is compact.

finite-dimensional provided the cohomology groups to the known complexes are finite dimensional. Then all the other important properties like closed ranges and compact embeddings are derived.

In the end the final results on properties like compact embeddings, closed ranges, and finite-dimensional cohomology groups, of the two approaches - the more algebraic approach in [9] and our more analytic approach - coincide to a large extent if applied to the elasticity complex, the $\nabla \nabla$ -complex and the $\text{div Div}_{\mathbb{S}}$ -complex.

In more technical details, the approach in this paper relies on the existence of regular potentials for the de Rham complex for topologically trivial domains, while the approach in [9], if applied to the elasticity complex, is based on the existence of cohomology spaces of smooth functions whose dimensions are independent of the degree of regularity of the involved Sobolev spaces. All these properties are available for the de Rham complex of Sobolev spaces without boundary conditions and with full Dirichlet boundary conditions, the situation considered in [9] as well as here. The situation for other boundary conditions is not completely clear to us and might differentiate the two approaches. In any case, we feel strongly that both approaches are very important and shed light on new complexes from two different angles.

The paper is organised as follows. We start in Section 2 by introducing the notations used in this paper. In Subsection 2.2.3 the concept of potential operators, which are the most important tool for the whole paper, is discussed on an abstract level. In Subsection 2.2.4 we present a new, elegant, abstract, and short way to prove compact embedding results for an arbitrary Hilbert complex. In Section 3 known results on potential operators for the de Rham complex are recalled. This will point out the way to prove the corresponding results for the elasticity complex. Section 4 contains the main results of the paper, the existence of potential operators and compact embeddings for the elasticity complex, and, as a consequence, related space decompositions. Readers who are interested in more technical details and additional and more sophisticated results are referred to [26], which also contains an extended list of results which directly follow from the FA-ToolBox. Finally, useful formulas are collected in Appendix A.

2. NOTATIONS AND PRELIMINARIES

Throughout the paper lower-case standard letters are used for denoting scalars and scalar functions, lower-case boldface letters for vectors and vector fields, and upper-case boldface letters for matrices/tensors and tensor fields.

We will use the following notations from linear algebra for $\mathbf{A} \in \mathbb{R}^{3 \times 3}$, $\mathbf{v} \in \mathbb{R}^3$:

$$\text{sym } \mathbf{A} := \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top), \quad \text{skw } := \frac{1}{2}(\mathbf{A} - \mathbf{A}^\top), \quad \text{tr } \mathbf{A} := \sum_{i=1}^3 A_{ii}, \quad \text{spn } \mathbf{v} := \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

The identity matrix is denoted by \mathbf{I} and \mathbb{S} is the set of symmetric matrices in $\mathbb{R}^{3 \times 3}$.

Moreover, $\nabla \varphi$ is used for the gradient of a scalar function φ , and $\nabla \phi$, $\text{rot } \phi$, $\text{div } \phi$ denote the Jacobian, rotation, divergence of a vector field ϕ , respectively. For a tensor field Φ , $\text{Rot } \Phi$ and $\text{Div } \Phi$ denote the tensor field and the vector field that result from a row-wise application of rot and div , respectively. Additionally, for smooth fields we introduce the differential operators $\text{sym } \nabla$, Rot^\top , Rot Rot^\top by

$$(\text{sym } \nabla) \phi := \text{sym}(\nabla \phi), \quad \text{Rot}^\top \Phi := (\text{Rot } \Phi)^\top, \quad (\text{Rot Rot}^\top) \Phi := \text{Rot}(\text{Rot}^\top \Phi).$$

Depending on the context these differential operators are understood in the strong sense or in the distributional sense.

2.1. Domains. Throughout this paper we suppose the following:

General Assumption 2.1. $\Omega \subset \mathbb{R}^3$ is a bounded strong Lipschitz domain with boundary Γ .

This means that Γ is locally a graph of some Lipschitz function.

Definition 2.2 (Trivial topology). *A bounded strong Lipschitz domain Ω with boundary Γ is called additionally topologically trivial, if Ω has no handles (also known as Ω being simply connected) and Γ is connected, i.e., all Betti numbers are trivial.*

To use standard localisation techniques, we introduce an open covering (U_ℓ) of $\overline{\Omega}$, such that all $\Omega_\ell := \Omega \cap U_\ell$ are topologically trivial (bounded) strong Lipschitz domains. As $\overline{\Omega}$ is compact, there exists a finite subcovering denoted by (U_1, \dots, U_L) . We fix a partition of unity $(\varphi_\ell)_{\ell=1, \dots, L}$ subordinate to $(U_\ell)_{\ell=1, \dots, L}$ with $\varphi_\ell \in \mathring{C}^\infty(U_\ell)$, $\ell = 1, \dots, L$, and, additionally, $\phi_\ell \in \mathring{C}^\infty(U_\ell)$ with $\phi_\ell|_{\text{supp } \varphi_\ell} = 1$, $\ell = 1, \dots, L$.

2.2. FA-ToolBox. We make use of the FA-ToolBox presented in detail in, e.g., [25] and [20, 21, 22, 23], together with an extension suited for (bounded linear) regular potential operators and regular decompositions in Subsection 2.2.3. In Subsection 2.2.4 we present a new, elegant, abstract, and short way to prove compact embedding results for an arbitrary Hilbert complex. This abstract result will be of fundamental importance for the analysis of the elasticity complex in Section 4.

2.2.1. Functional Analytical Setting. The inner product and the norm in a Hilbert space \mathbf{H} are denoted by $|\cdot|_{\mathbf{H}}$ and $\langle \cdot, \cdot \rangle_{\mathbf{H}}$, respectively. We use $\mathbf{V}^{\perp_{\mathbf{H}}}$ for the orthogonal complement of a subspace \mathbf{V} of \mathbf{H} . The notations $\mathbf{V}_1 + \mathbf{V}_2$, $\mathbf{V}_1 \dot{+} \mathbf{V}_2$, $\mathbf{V}_1 \oplus_{\mathbf{H}} \mathbf{V}_2$ indicate the sum, the direct sum, the orthogonal sum of two subspaces $\mathbf{V}_1, \mathbf{V}_2$ of \mathbf{H} , respectively. Throughout this paper, the domain of definition, the kernel, and the range of a linear operator A is denoted by $D(A)$, $N(A)$, and $R(A)$, respectively.

Let $\mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_2$ be Hilbert spaces. We shall consider densely defined, closed, and *unbounded* linear operators

$$A : D(A) \subset \mathbf{H}_1 \rightarrow \mathbf{H}_2,$$

where unboundedness refers to the norms in \mathbf{H}_1 and \mathbf{H}_2 , as well as *bounded* linear operators

$$A : D(A) \rightarrow \mathbf{H}_2,$$

where boundedness refers to the norms in $D(A)$ and \mathbf{H}_2 with $D(A)$ as Hilbert space, equipped with the graph inner product. The different notations indicate the difference in the boundedness. Note that, in contrast to the Banach space adjoint of the bounded operator A , the *Hilbert space adjoint* of the unbounded operator A is given by the densely defined, closed, and unbounded linear operator

$$A^* : D(A^*) \subset \mathbf{H}_2 \rightarrow \mathbf{H}_1$$

defined by $D(A^*) = \{y \in \mathbf{H}_2 : \exists f \in \mathbf{H}_1 \forall x \in D(A) \langle Ax, y \rangle_{\mathbf{H}_2} = \langle x, f \rangle_{\mathbf{H}_1}\}$ and $A^*y = f$ for $y \in D(A^*)$, and hence characterised by

$$\forall y \in D(A^*) \quad \forall x \in D(A) \quad \langle Ax, y \rangle_{\mathbf{H}_2} = \langle x, A^*y \rangle_{\mathbf{H}_1}.$$

Remark 2.3 (Geometric Helmholtz type decompositions and reduced operators). *We recall that the unbounded operator A induces by $\overline{R(A^*)} = N(A)^{\perp_{\mathbf{H}_1}}$ and $\overline{R(A)} = N(A^*)^{\perp_{\mathbf{H}_2}}$ the Helmholtz type decompositions*

$$\mathbf{H}_1 = \overline{R(A^*)} \oplus_{\mathbf{H}_1} N(A), \quad \mathbf{H}_2 = \overline{R(A)} \oplus_{\mathbf{H}_2} N(A^*),$$

leading directly to the so-called reduced operators $\mathcal{A} := A|_{N(A)^{\perp_{\mathbf{H}_1}}}$ and $\mathcal{A}^* := A^*|_{N(A^*)^{\perp_{\mathbf{H}_2}}}$, i.e.,

$$\begin{aligned} \mathcal{A} : D(\mathcal{A}) \subset \overline{R(A^*)} &\rightarrow \overline{R(A)}, & D(\mathcal{A}) &:= D(A) \cap \overline{R(A^*)} = D(A) \cap N(A)^{\perp_{\mathbf{H}_1}}, \\ \mathcal{A}^* : D(\mathcal{A}^*) \subset \overline{R(A)} &\rightarrow \overline{R(A^*)}, & D(\mathcal{A}^*) &:= D(A^*) \cap \overline{R(A)} = D(A^*) \cap N(A^*)^{\perp_{\mathbf{H}_2}}. \end{aligned}$$

Then we have also the Helmholtz type decompositions

$$D(\mathcal{A}) = D(\mathcal{A}) \oplus_{\mathbf{H}_1} N(\mathcal{A}), \quad D(\mathcal{A}^*) = D(\mathcal{A}^*) \oplus_{\mathbf{H}_2} N(\mathcal{A}^*)$$

and it holds $R(\mathcal{A}) = \overline{R(A)}$ and $R(\mathcal{A}^*) = \overline{R(A^*)}$. Note that, as \mathcal{A} and \mathcal{A}^* are injective and by the closed graph theorem, \mathcal{A}^{-1} and $(\mathcal{A}^*)^{-1}$ are bounded if and only if the ranges $R(\mathcal{A})$ and $R(\mathcal{A}^*)$ are closed if and only if the Friedrichs/Poincaré type estimates

$$\begin{aligned} \exists c > 0 \quad \forall x \in D(\mathcal{A}) & & |x|_{\mathbf{H}_1} &\leq c |Ax|_{\mathbf{H}_2}, \\ \exists c > 0 \quad \forall y \in D(\mathcal{A}^*) & & |y|_{\mathbf{H}_2} &\leq c |A^*y|_{\mathbf{H}_1} \end{aligned}$$

hold, respectively. Moreover, by the closed range theorem, $R(\mathcal{A})$ is closed if and only if $R(\mathcal{A}^*)$ is closed.

2.2.2. *Hilbert Complexes and Compact Embeddings.* Our main interest are *Hilbert complexes*

$$\begin{array}{ccc} \mathsf{H}_0 & \xrightarrow{A_0} & \mathsf{H}_1 & \xrightarrow{A_1} & \mathsf{H}_2, \\ \mathsf{H}_0 & \xleftarrow{A_0^*} & \mathsf{H}_1 & \xleftarrow{A_1^*} & \mathsf{H}_2 \end{array}$$

of densely defined, closed, and unbounded linear operators

$$A_0 : D(A_0) \subset \mathsf{H}_0 \rightarrow \mathsf{H}_1, \quad A_1 : D(A_1) \subset \mathsf{H}_1 \rightarrow \mathsf{H}_2$$

with (Hilbert space) adjoints

$$A_0^* : D(A_0^*) \subset \mathsf{H}_1 \rightarrow \mathsf{H}_0, \quad A_1^* : D(A_1^*) \subset \mathsf{H}_2 \rightarrow \mathsf{H}_1,$$

and corresponding *compact embeddings*

$$D(A_1) \cap D(A_0^*) \hookrightarrow \mathsf{H}_1.$$

Equivalently, we shall denote these Hilbert complexes as *Hilbert domain complexes* of bounded linear operators and write

$$\begin{array}{ccc} D(A_0) & \xrightarrow{A_0} & D(A_1) & \xrightarrow{A_1} & \mathsf{H}_2, \\ \mathsf{H}_0 & \xleftarrow{A_0^*} & D(A_0^*) & \xleftarrow{A_1^*} & D(A_1^*), \end{array}$$

cf. (5). The *key property* of a complex reads generally

$$(6) \quad A_1 A_0 \subset 0,$$

i.e., $R(A_0) \subset N(A_1)$ or equivalently $R(A_1^*) \subset N(A_0^*)$. Throughout the rest of Section 2 we assume the following to hold.

General Assumption 2.4. $\mathsf{H}_0 \xrightarrow{A_0} \mathsf{H}_1 \xrightarrow{A_1} \mathsf{H}_2$ is a Hilbert complex of densely defined and closed (unbounded) linear operators $A_0 : D(A_0) \subset \mathsf{H}_0 \rightarrow \mathsf{H}_1$ and $A_1 : D(A_1) \subset \mathsf{H}_1 \rightarrow \mathsf{H}_2$. Note that the complex property (6) holds, i.e., $A_1 A_0 \subset 0$ and $A_0^* A_1^* \subset 0$.

2.2.3. *Potentials and Decompositions.* Let

$$A : D(A) \subset \mathsf{H}_1 \rightarrow \mathsf{H}_2$$

be a densely defined and closed (unbounded) linear operator.

Definition 2.5 (Potential operator). Let $\mathsf{H}_1^+ \subset D(A)$ be another Hilbert space. A bounded linear operator

$$(7) \quad P_A : R(A) \rightarrow \mathsf{H}_1^+ \quad \text{with} \quad A P_A = \text{id}_{R(A)}$$

is called a *potential operator associated to A*. In short, a potential operator is a bounded right inverse of A.

By definition we have, for each $y \in R(A)$,

$$(8) \quad y = A p \quad \text{with} \quad p := P_A y.$$

We call p a *potential* of y . For each $x \in D(A)$, we have the representation

$$(9) \quad x = p + x_0 \quad \text{with} \quad p := P_A A x \in R(P_A) \subset \mathsf{H}_1^+, \quad A p = A x, \quad \text{and} \quad x_0 := x - p \in N(A),$$

which leads to the following space decomposition.

Lemma 2.6 (Potential decomposition). Let P_A be a potential operator. Then the potential decomposition $D(A) = R(P_A) \dot{+} N(A) = \mathsf{H}_1^+ + N(A)$ and $R(A) = A R(P_A) = A \mathsf{H}_1^+$ hold.

Note that the decomposition (9) of Lemma 2.6 is stable (topological/continuous/bounded), i.e.,

$$|p|_{\mathsf{H}_1^+} \leq c^+ |A x|_{\mathsf{H}_2}, \quad |x_0|_{\mathsf{H}_1} \leq |x|_{\mathsf{H}_1} + c_1 c^+ |A x|_{\mathsf{H}_2},$$

where c^+ is the norm of P_A and c_1 is the norm of the continuous embedding $\mathsf{H}_1^+ \subset \mathsf{H}_1$, which we assume to hold.

Example 2.7 (Geometric potential operator). *Let $R(A)$ be closed. The reduced operator of A , cf. Remark 2.3,*

$$\mathcal{A} : D(\mathcal{A}) \subset R(A^*) \rightarrow R(A)$$

is linear and bijective. Its inverse $P_A := \mathcal{A}^{-1} : R(A) \rightarrow D(\mathcal{A})$ is bounded (closed graph theorem) and a right inverse of A . We call this potential operator the geometric potential operator associated to A . It is easy to see that the associated space decomposition is an orthogonal decomposition

$$D(A) = D(\mathcal{A}) \oplus_{H_1} N(A),$$

which we call a geometric (or Helmholtz type) decomposition. Note that the reduced bounded variant $\mathcal{A} : D(\mathcal{A}) \rightarrow R(A)$ is a topological isomorphism.

The essential tool used in this paper is the existence of potential operators, where H_1^+ is a smoother space than $D(\mathcal{A})$ allowing for a compact embedding $H_1^+ \hookrightarrow H_1$, realised by, e.g., the well known Rellich's selection theorem. In this case we speak of *regular potentials* and introduce the concept of *regular potential operators*:

Definition 2.8. *A potential operator $P_A : R(A) \rightarrow H_1^+$ with a compact embedding $H_1^+ \hookrightarrow H_1$ is called a regular potential operator.*

Recall our General Assumption 2.4 and the corresponding Hilbert domain complex

$$D(A_0) \xrightarrow{A_0} D(A_1) \xrightarrow{A_1} H_2.$$

Let

$$(10) \quad P_{A_0} : R(A_0) \rightarrow H_0^+, \quad P_{A_1} : R(A_1) \rightarrow H_1^+$$

be associated potential operators. Moreover, let us assume that the Hilbert complex is *exact*, i.e.,

$$(11) \quad R(A_0) = N(A_1).$$

Using (9) for A_1 and (8) for A_0 together with the exact complex property (11), we obtain, for each $x \in D(A_1)$, the representation

$$(12) \quad x = p_1 + A_0 p_0 \quad \text{with} \quad p_1 := P_{A_1} A_1 x, \quad A_1 p_1 = A_1 x, \quad p_0 := P_{A_0}(x - p_1),$$

which leads to a refined version of the space decomposition from above.

Lemma 2.9 (Refined potential decomposition). *Let $R(A_0) = N(A_1)$, and let P_{A_0}, P_{A_1} be potential operators. Then*

$$D(A_1) = R(P_{A_1}) \dot{+} A_0 R(P_{A_0}) = R(P_{A_1}) \dot{+} A_0 H_0^+ = H_1^+ + A_0 H_0^+$$

as well as $R(A_0) = A_0 R(P_{A_0}) = A_0 H_0^+$ and $R(A_1) = A_1 R(P_{A_1}) = A_1 H_1^+$.

Decomposition (12) can be rephrased as follows. For each $x \in D(A_1)$ we have

$$x = Q_{A_1} x + A_0 \tilde{Q}_{A_0} x \quad \text{with} \quad Q_{A_1} = P_{A_1} A_1 \quad \text{and} \quad \tilde{Q}_{A_0} = P_{A_0}(\text{id}_{D(A_1)} - Q_{A_1}).$$

Note that Q_{A_1} and $\text{id}_{D(A_1)} - Q_{A_1}$ are projections and hence Q_{A_1} and \tilde{Q}_{A_0} are bounded. More precisely, $\text{id}_{D(A_1)} - Q_{A_1}$ is a projection onto $N(A_1) = R(A_0)$. So, the operator \tilde{Q}_{A_0} primarily depends on A_0 . The dependency of \tilde{Q}_{A_0} on A_1 is notationally suppressed for simplicity. The operators \tilde{Q}_{A_0} and Q_{A_1} are bounded, since

$$\begin{aligned} |Q_{A_1} x|_{H_1^+} &= |p_1|_{H_1^+} \leq c_1^+ |A_1 x|_{H_2}, \\ |\tilde{Q}_{A_0} x|_{H_0^+} &= |p_0|_{H_0^+} \leq c_0^+ |x - p_1|_{H_1} \leq c_0^+ |x|_{H_1} + c_1 c_0^+ c_1^+ |A_1 x|_{H_2}, \end{aligned}$$

where c_l^+ is the norm of P_{A_l} , $l = 0, 1$.

Motivated by this, the concept of a *regular space decomposition* is introduced as follows (without any reference to the exactness property (11)).

Definition 2.10 (Regular decomposition). *A decomposition $D(A_1) = H_1^+ + A_0 H_0^+$ with compact embeddings $H_0^+ \hookrightarrow H_0$ and $H_1^+ \hookrightarrow H_1$ is called a regular decomposition, if there are bounded operators*

$$(13) \quad Q_{A_1} : D(A_1) \rightarrow H_1^+, \quad \tilde{Q}_{A_0} : D(A_1) \rightarrow H_0^+,$$

such that

$$(14) \quad x = Q_{A_1} x + A_0 \tilde{Q}_{A_0} x$$

holds for all $x \in D(A_1)$. The operators in (13) are called decomposition operators.

2.2.4. Compact Embeddings by Regular Decompositions. Regular decompositions are very powerful tools. In particular, compact embeddings can easily be proved, which then in combination with the FA-ToolBox, cf. [20, 21, 22, 23, 25], immediately lead to a comprehensive list of important results for the underlying Hilbert complex.

Theorem 2.11 (Compact embedding by regular decompositions). *Let $D(A_1) = H_1^+ + A_0 H_0^+$ be a regular decomposition. Then $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ is compact.*

Proof. Let $(x_n) \subset D(A_1) \cap D(A_0^*)$ be a bounded sequence, i.e., there exists $c > 0$ such that for all n we have $|x_n|_{H_1} + |A_1 x_n|_{H_2} + |A_0^* x_n|_{H_0} \leq c$. By assumption we decompose $x_n = p_{1,n} + A_0 p_{0,n}$ with $p_{1,n} \in H_1^+$ and $p_{0,n} \in H_0^+$ satisfying $|p_{1,n}|_{H_1^+} + |p_{0,n}|_{H_0^+} \leq c|x_n|_{D(A_1)} \leq c$. Hence $(p_{\ell,n}) \subset H_\ell^+$ is bounded in H_ℓ^+ , $\ell = 0, 1$, and thus we can extract convergent subsequences, again denoted by $(p_{\ell,n})$, such that $(p_{\ell,n})$ are convergent in H_ℓ , $\ell = 0, 1$. Then with $x_{n,m} := x_n - x_m$ and $p_{\ell,n,m} := p_{\ell,n} - p_{\ell,m}$ we get

$$\begin{aligned} |x_{n,m}|_{H_1}^2 &= \langle x_{n,m}, p_{1,n,m} + A_0 p_{0,n,m} \rangle_{H_1} = \langle x_{n,m}, p_{1,n,m} \rangle_{H_1} + \langle A_0^* x_{n,m}, p_{0,n,m} \rangle_{H_0} \\ &\leq c(|p_{1,n,m}|_{H_1} + |p_{0,n,m}|_{H_0}), \end{aligned}$$

which shows that (x_n) is a Cauchy sequence in H_1 . \square

There is a dual version of Theorem 2.11.

Corollary 2.12 (Compact embedding by regular decompositions). *Let $D(A_0^*) = H_1^+ + A_1^* H_2^+$ be a regular decomposition. Then $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ is compact.*

Proof. The proof follows by duality or it can be done analogously to the proof of Theorem 2.11 with

$$\begin{aligned} |x_{n,m}|_{H_1}^2 &= \langle x_{n,m}, p_{1,n,m} + A_1^* p_{2,n,m} \rangle_{H_1} = \langle x_{n,m}, p_{1,n,m} \rangle_{H_1} + \langle A_1 x_{n,m}, p_{2,n,m} \rangle_{H_2} \\ &\leq c(|p_{1,n,m}|_{H_1} + |p_{2,n,m}|_{H_2}), \end{aligned}$$

as crucial estimate. \square

As we have seen, regular potential operators lead quite naturally to regular decompositions if the Hilbert complex is exact. Conversely and quite generally, regular decompositions can always be used to construct regular potential operators.

Corollary 2.13. *Let $D(A_1) = H_1^+ + A_0 H_0^+$ be a regular decomposition. Then the operator*

$$P_{A_1} = Q_{A_1} \mathcal{A}_1^{-1} : R(A_1) \rightarrow H_1^+$$

is well-defined and a regular potential operator associated to A_1 .

Proof. From Theorem 2.11 and the FA-ToolBox, e.g., [25, Theorem 2.9], it follows that $R(A_1)$ is closed. Therefore, the reduced operator \mathcal{A}_1 of A_1 is linear and bijective with a bounded inverse, which makes the operator P_{A_1} well-defined. Let $x \in D(A_1)$. Applying A_1 to the regular decomposition

$$x = Q_{A_1} x + A_0 \tilde{Q}_{A_0} x$$

yields $A_1 x = A_1 Q_{A_1} x$. Therefore, $A_1 P_{A_1} = A_1 Q_{A_1} \mathcal{A}_1^{-1} = A_1 \mathcal{A}_1^{-1} = \text{id}_{R(A_1)}$. \square

Corollary 2.13 admits a dual version as well, cf. Corollary 2.12.

Corollary 2.14. *Let $D(A_0^*) = H_1^+ + A_1^* H_2^+$ be a regular decomposition. Then the operator*

$$P_{A_0^*} = Q_{A_0^*}(\mathcal{A}_0^*)^{-1}: R(A_0^*) \rightarrow H_1^+$$

is well-defined and a regular potential operator associated to A_0^ .*

Remark 2.15. *The decompositions (12) and (14) translate to the identities*

$$\begin{aligned} P_{A_1} A_1 + A_0 P_{A_0}(\text{id}_{D(A_1)} - P_{A_1} A_1) &= \text{id}_{D(A_1)}, \\ Q_{A_1} + A_0 \tilde{Q}_{A_0} &= \text{id}_{D(A_1)}. \end{aligned}$$

2.3. Sobolev Spaces. Let $m \in \mathbb{N}_0$. We have to introduce several Sobolev spaces. Generally, spaces of scalar functions will be denoted by standard upper-case letters, spaces of vector fields by uppercase boldface letters, and spaces of tensor fields by calligraphic boldface letters. For scalar spaces we have the standard notations $L^2(\Omega)$, $L_{\perp}^2(\Omega) := L^2(\Omega) \cap \mathbb{R}^{\perp L^2(\Omega)}$, and $H^m(\Omega)$, as well as for vector fields

$$\mathbf{L}^2(\Omega), \quad \mathbf{L}_{\perp}^2(\Omega) = \mathbf{L}^2(\Omega) \cap (\mathbb{R}^3)^{\perp L^2(\Omega)}, \quad \mathbf{H}^m(\Omega), \quad \mathbf{H}(\text{rot}, \Omega), \quad \mathbf{H}(\text{div}, \Omega).$$

Corresponding Sobolev spaces of tensor fields are $\mathcal{L}^2(\Omega)$, $\mathcal{H}^m(\Omega)$, $\mathcal{H}(\text{Rot}, \Omega)$, $\mathcal{H}(\text{Div}, \Omega)$. Symmetric tensor fields are introduced by

$$\begin{aligned} \mathcal{L}_{\mathbb{S}}^2(\Omega) &:= \{M \in \mathcal{L}^2(\Omega) : M = M^{\top}\}, & \mathcal{H}_{\mathbb{S}}(\text{Rot}, \Omega) &:= \mathcal{H}(\text{Rot}, \Omega) \cap \mathcal{L}_{\mathbb{S}}^2(\Omega), \\ \mathcal{H}_{\mathbb{S}}^m(\Omega) &:= \mathcal{H}^m(\Omega) \cap \mathcal{L}_{\mathbb{S}}^2(\Omega), & \mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega) &:= \mathcal{H}(\text{Div}, \Omega) \cap \mathcal{L}_{\mathbb{S}}^2(\Omega). \end{aligned}$$

Furthermore, we need a few Sobolev spaces suited for the elasticity complex, namely

$$\begin{aligned} \mathbf{H}(\nabla, \Omega) &:= \{v \in \mathbf{L}^2(\Omega) : \nabla v \in \mathcal{L}^2(\Omega)\} = \mathbf{H}^1(\Omega), \\ \mathbf{H}(\text{sym } \nabla, \Omega) &:= \{v \in \mathbf{L}^2(\Omega) : \text{sym } \nabla v \in \mathcal{L}^2(\Omega)\}, \\ \mathcal{H}(\text{Rot Rot}^{\top}, \Omega) &:= \{M \in \mathcal{L}^2(\Omega) : \text{Rot Rot}^{\top} M \in \mathcal{L}^2(\Omega)\}, \\ \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega) &:= \mathcal{H}(\text{Rot Rot}^{\top}, \Omega) \cap \mathcal{L}_{\mathbb{S}}^2(\Omega). \end{aligned}$$

Note that for $M \in \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega)$ we have $\text{Rot Rot}^{\top} M \in \mathcal{L}_{\mathbb{S}}^2(\Omega)$, see Lemma A.1 (vi').

Homogeneous boundary conditions are always introduced in the strong form by closure of the respective test fields $\dot{\mathcal{C}}^{\infty}(\Omega)$, $\dot{\mathcal{C}}^{\infty}(\Omega)$, $\dot{\mathcal{C}}^{\infty}(\Omega)$, $\dot{\mathcal{C}}_{\mathbb{S}}^{\infty}(\Omega)$ in the respective Sobolev norm, i.e.,

$$\begin{aligned} \dot{\mathbf{H}}^m(\Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathbf{H}^m(\Omega)}, & \dot{\mathbf{H}}^m(\Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathbf{H}^m(\Omega)}, \\ \dot{\mathcal{H}}^m(\Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathcal{H}^m(\Omega)}, & \dot{\mathcal{H}}_{\mathbb{S}}^m(\Omega) &:= \overline{\dot{\mathcal{C}}_{\mathbb{S}}^{\infty}(\Omega)}^{\mathcal{H}^m(\Omega)}, \\ \dot{\mathbf{H}}(\nabla, \Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathbf{H}(\nabla, \Omega)} = \dot{\mathbf{H}}^1(\Omega), & \dot{\mathbf{H}}(\text{sym } \nabla, \Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathbf{H}(\text{sym } \nabla, \Omega)}, \\ \dot{\mathbf{H}}(\text{rot}, \Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathbf{H}(\text{rot}, \Omega)}, & \dot{\mathbf{H}}(\text{div}, \Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathbf{H}(\text{div}, \Omega)}, \\ \dot{\mathcal{H}}(\text{Rot}, \Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathcal{H}(\text{Rot}, \Omega)}, & \dot{\mathcal{H}}(\text{Div}, \Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathcal{H}(\text{Div}, \Omega)}, \\ \dot{\mathcal{H}}_{\mathbb{S}}(\text{Rot}, \Omega) &:= \overline{\dot{\mathcal{C}}_{\mathbb{S}}^{\infty}(\Omega)}^{\mathcal{H}(\text{Rot}, \Omega)}, & \dot{\mathcal{H}}_{\mathbb{S}}(\text{Div}, \Omega) &:= \overline{\dot{\mathcal{C}}_{\mathbb{S}}^{\infty}(\Omega)}^{\mathcal{H}(\text{Div}, \Omega)}, \\ \dot{\mathcal{H}}(\text{Rot Rot}^{\top}, \Omega) &:= \overline{\dot{\mathcal{C}}^{\infty}(\Omega)}^{\mathcal{H}(\text{Rot Rot}^{\top}, \Omega)}, & \dot{\mathcal{H}}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega) &:= \overline{\dot{\mathcal{C}}_{\mathbb{S}}^{\infty}(\Omega)}^{\mathcal{H}(\text{Rot Rot}^{\top}, \Omega)}. \end{aligned}$$

We use the convention $\mathbf{H}(\Omega) = \mathbf{H}^0(\Omega) = L^2(\Omega) = \dot{\mathbf{H}}^0(\Omega) = \dot{\mathbf{H}}(\Omega)$ and the same for the vector and tensor valued spaces.

As usual $\mathbf{H}^{-m}(\Omega)$ denotes the dual of $\dot{\mathbf{H}}^m(\Omega)$, and we introduce $\mathbf{H}^{-m}(\Omega)$, $\mathcal{H}^{-m}(\Omega)$, and $\mathcal{H}_{\mathbb{S}}^{-m}(\Omega)$ as the dual of $\dot{\mathbf{H}}^m(\Omega)$, $\dot{\mathcal{H}}^m(\Omega)$, and $\dot{\mathcal{H}}_{\mathbb{S}}^m(\Omega)$, respectively.

2.4. Operators and Complexes. Generally, the restriction of a differential operator A to its domain of definition of fields with homogeneous boundary conditions is denoted by \mathring{A} , e.g., $\mathring{\nabla}$. For a differential operator A acting on tensor fields, its restriction to spaces of symmetric tensor fields is denoted by $A_{\mathbb{S}}$, e.g., $\text{Rot Rot}_{\mathbb{S}}^{\top}$ or $\text{Div}_{\mathbb{S}}$.

The classical primal and dual de Rham complexes (of unbounded operators) are given by

$$\begin{aligned} \mathbf{L}^2(\Omega) &\xrightarrow{\mathring{\nabla}} \mathbf{L}^2(\Omega) \xrightarrow{\mathring{\text{rot}}} \mathbf{L}^2(\Omega) \xrightarrow{\mathring{\text{div}}} \mathbf{L}^2(\Omega), \\ \mathbf{L}^2(\Omega) &\xleftarrow{-\mathring{\text{div}}} \mathbf{L}^2(\Omega) \xleftarrow{\mathring{\text{rot}}} \mathbf{L}^2(\Omega) \xleftarrow{-\mathring{\nabla}} \mathbf{L}^2(\Omega) \end{aligned}$$

with the densely defined and closed (unbounded) linear operators and their adjoints

$$\begin{aligned} A_0 &:= \mathring{\nabla} : D(\mathring{\nabla}) = \mathring{\mathbf{H}}^1(\Omega) \subset \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega); & u &\mapsto \nabla u, \\ A_1 &:= \mathring{\text{rot}} : D(\mathring{\text{rot}}) = \mathring{\mathbf{H}}(\text{rot}, \Omega) \subset \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega); & v &\mapsto \text{rot } v, \\ A_2 &:= \mathring{\text{div}} : D(\mathring{\text{div}}) = \mathring{\mathbf{H}}(\text{div}, \Omega) \subset \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega); & v &\mapsto \text{div } v, \\ A_0^* &= -\text{div} : D(\text{div}) = \mathbf{H}(\text{div}, \Omega) \subset \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega); & v &\mapsto -\text{div } v, \\ A_1^* &= \text{rot} : D(\text{rot}) = \mathbf{H}(\text{rot}, \Omega) \subset \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega); & v &\mapsto \text{rot } v, \\ A_2^* &= -\nabla : D(\nabla) = \mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega); & u &\mapsto -\nabla u. \end{aligned}$$

The classical primal and dual elasticity complexes (of unbounded operators) are given by

$$\begin{aligned} \mathbf{L}^2(\Omega) &\xrightarrow{\mathring{\text{sym}} \nabla} \mathcal{L}_{\mathbb{S}}^2(\Omega) \xrightarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} \mathcal{L}_{\mathbb{S}}^2(\Omega) \xrightarrow{\mathring{\text{Div}}_{\mathbb{S}}} \mathbf{L}^2(\Omega), \\ \mathbf{L}^2(\Omega) &\xleftarrow{-\mathring{\text{Div}}_{\mathbb{S}}} \mathcal{L}_{\mathbb{S}}^2(\Omega) \xleftarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} \mathcal{L}_{\mathbb{S}}^2(\Omega) \xleftarrow{-\mathring{\text{sym}} \nabla} \mathbf{L}^2(\Omega) \end{aligned}$$

with the densely defined and closed (unbounded) operators and their adjoints

$$\begin{aligned} A_0 &:= \mathring{\text{sym}} \nabla : D(\mathring{\text{sym}} \nabla) = \mathring{\mathbf{H}}^1(\Omega) \subset \mathbf{L}^2(\Omega) \rightarrow \mathcal{L}_{\mathbb{S}}^2(\Omega); & v &\mapsto \text{sym } \nabla v, \\ A_1 &:= \text{Rot Rot}_{\mathbb{S}}^{\top} : D(\text{Rot Rot}_{\mathbb{S}}^{\top}) = \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot Rot}_{\mathbb{S}}^{\top}, \Omega) \subset \mathcal{L}_{\mathbb{S}}^2(\Omega) \rightarrow \mathcal{L}_{\mathbb{S}}^2(\Omega); & M &\mapsto \text{Rot Rot}_{\mathbb{S}}^{\top} M, \\ A_2 &:= \mathring{\text{Div}}_{\mathbb{S}} : D(\mathring{\text{Div}}_{\mathbb{S}}) = \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Div}, \Omega) \subset \mathcal{L}_{\mathbb{S}}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega); & N &\mapsto \text{Div } N, \\ A_0^* &= -\text{Div}_{\mathbb{S}} : D(\text{Div}_{\mathbb{S}}) = \mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega) \subset \mathcal{L}_{\mathbb{S}}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega); & N &\mapsto -\text{Div } N, \\ A_1^* &= \text{Rot Rot}_{\mathbb{S}}^{\top} : D(\text{Rot Rot}_{\mathbb{S}}^{\top}) = \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}_{\mathbb{S}}^{\top}, \Omega) \subset \mathcal{L}_{\mathbb{S}}^2(\Omega) \rightarrow \mathcal{L}_{\mathbb{S}}^2(\Omega); & M &\mapsto \text{Rot Rot}_{\mathbb{S}}^{\top} M, \\ A_2^* &= -\mathring{\text{sym}} \nabla : D(\mathring{\text{sym}} \nabla) = \mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega) \rightarrow \mathcal{L}_{\mathbb{S}}^2(\Omega); & v &\mapsto -\mathring{\text{sym}} \nabla v. \end{aligned}$$

Note that we have by the Korn inequalities

$$\begin{aligned} D(\mathring{\text{sym}} \nabla) &= \mathring{\mathbf{H}}(\mathring{\text{sym}} \nabla, \Omega) = \mathring{\mathbf{H}}^1(\Omega) = \mathring{\mathbf{H}}(\nabla, \Omega) = D(\mathring{\nabla}), \\ D(\mathring{\text{sym}} \nabla) &= \mathbf{H}(\mathring{\text{sym}} \nabla, \Omega) = \mathbf{H}^1(\Omega) = \mathbf{H}(\nabla, \Omega) = D(\nabla). \end{aligned}$$

3. THE DE RHAM COMPLEX

In this section we recall those basic properties of the de Rham complex, which are needed in the subsequent section for the analysis of the elasticity complex.

The classical primal and dual de Rham domain complexes read in full length

$$(15) \quad \begin{aligned} \{0\} &\xrightarrow{\iota_{\{0\}}} \mathring{\mathbf{H}}^1(\Omega) \xrightarrow{\mathring{\nabla}} \mathring{\mathbf{H}}(\text{rot}, \Omega) \xrightarrow{\mathring{\text{rot}}} \mathring{\mathbf{H}}(\text{div}, \Omega) \xrightarrow{\mathring{\text{div}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}, \\ \{0\} &\xleftarrow{\pi_{\{0\}}} \mathbf{L}^2(\Omega) \xleftarrow{-\mathring{\text{div}}} \mathbf{H}(\text{div}, \Omega) \xleftarrow{\mathring{\text{rot}}} \mathbf{H}(\text{rot}, \Omega) \xleftarrow{-\mathring{\nabla}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}. \end{aligned}$$

Here $\iota_{\{0\}}$, $\pi_{\{0\}}$ and $\iota_{\mathbb{R}}$, $\pi_{\mathbb{R}}$ denote the canonical embeddings and $\mathbf{L}^2(\Omega)$ -orthogonal projections of/on the finite-dimensional subspaces $\{0\}$ and \mathbb{R} , respectively. These complexes are Hilbert complexes. The corresponding cohomology groups

$$\mathbf{Harm}_{\mathbb{D}}(\Omega) := N(\mathring{\text{rot}}) \cap N(\mathring{\text{div}}), \quad \mathbf{Harm}_{\mathbb{N}}(\Omega) := N(\mathring{\text{div}}) \cap N(\mathring{\text{rot}})$$

are called Dirichlet and Neumann fields, respectively. For a derivation of the de Rham complex by operator theoretical arguments, see [27].

More generally, we consider the following variants of the primal and dual de Rham complexes (of unbounded operators) for $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$:

$$\begin{array}{ccccccccc} \{0\} & \xrightarrow{\iota_{\{0\}}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\mathring{\nabla}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\mathring{\text{rot}}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\mathring{\text{div}}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\pi_{\mathbb{R}}} & \mathbb{R}, \\ \{0\} & \xleftarrow{\pi_{\{0\}}} & \mathbf{H}^n(\Omega) & \xleftarrow{-\text{div}} & \mathbf{H}^n(\Omega) & \xleftarrow{\text{rot}} & \mathbf{H}^n(\Omega) & \xleftarrow{-\nabla} & \mathbf{H}^n(\Omega) & \xleftarrow{\iota_{\mathbb{R}}} & \mathbb{R} \end{array}$$

with the associated domain complexes

$$(16) \quad \begin{array}{ccccccccc} \{0\} & \xrightarrow{\iota_{\{0\}}} & \mathring{\mathbf{H}}^{m+1}(\Omega) & \xrightarrow{\mathring{\nabla}} & \mathring{\mathbf{H}}^m(\text{rot}, \Omega) & \xrightarrow{\mathring{\text{rot}}} & \mathring{\mathbf{H}}^m(\text{div}, \Omega) & \xrightarrow{\mathring{\text{div}}} & \mathbf{H}^m(\Omega) & \xrightarrow{\pi_{\mathbb{R}}} & \mathbb{R}, \\ \{0\} & \xleftarrow{\pi_{\{0\}}} & \mathbf{H}^n(\Omega) & \xleftarrow{-\text{div}} & \mathbf{H}^n(\text{div}, \Omega) & \xleftarrow{\text{rot}} & \mathbf{H}^n(\text{rot}, \Omega) & \xleftarrow{-\nabla} & \mathbf{H}^{n+1}(\Omega) & \xleftarrow{\iota_{\mathbb{R}}} & \mathbb{R} \end{array}$$

and the Sobolev spaces

$$\begin{aligned} \mathring{\mathbf{H}}^m(\text{rot}, \Omega) &:= \{\mathbf{v} \in D(\mathring{\text{rot}}) \cap \mathring{\mathbf{H}}^m(\Omega) : \text{rot } \mathbf{v} \in \mathring{\mathbf{H}}^m(\Omega)\}, \\ \mathbf{H}^n(\text{rot}, \Omega) &:= \{\mathbf{v} \in \mathbf{H}^n(\Omega) : \text{rot } \mathbf{v} \in \mathbf{H}^n(\Omega)\}, \\ \mathring{\mathbf{H}}^m(\text{div}, \Omega) &:= \{\mathbf{v} \in D(\mathring{\text{div}}) \cap \mathring{\mathbf{H}}^m(\Omega) : \text{div } \mathbf{v} \in \mathring{\mathbf{H}}^m(\Omega)\}, \\ \mathbf{H}^n(\text{div}, \Omega) &:= \{\mathbf{v} \in \mathbf{H}^n(\Omega) : \text{div } \mathbf{v} \in \mathbf{H}^n(\Omega)\}. \end{aligned}$$

Note that in the definition of the spaces with boundary conditions the intersection with the respective domain of definition is superfluous for $m \geq 1$.

For the next lemma we introduce the following notation. Similar to $\mathbf{L}_{\perp}^2(\Omega)$ we set for $m \in \mathbb{N}_0$

$$\mathring{\mathbf{H}}_{\perp}^m(\Omega) := \mathring{\mathbf{H}}^m(\Omega) \cap \mathbb{R}^{\perp \mathbf{L}^2(\Omega)}.$$

We introduce kernel Sobolev spaces by

$$\begin{aligned} \mathring{\mathbf{H}}_0^m(\text{rot}, \Omega) &:= \{\mathbf{v} \in \mathring{\mathbf{H}}^m(\text{rot}, \Omega) : \text{rot } \mathbf{v} = 0\}, & \mathbf{H}_0^n(\text{rot}, \Omega) &:= \{\mathbf{v} \in \mathbf{H}^n(\text{rot}, \Omega) : \text{rot } \mathbf{v} = 0\}, \\ \mathring{\mathbf{H}}_0^m(\text{div}, \Omega) &:= \{\mathbf{v} \in \mathring{\mathbf{H}}^m(\text{div}, \Omega) : \text{div } \mathbf{v} = 0\}, & \mathbf{H}_0^n(\text{div}, \Omega) &:= \{\mathbf{v} \in \mathbf{H}^n(\text{div}, \Omega) : \text{div } \mathbf{v} = 0\}. \end{aligned}$$

Note that for $m = n = 0$ we have

$$\begin{aligned} \mathring{\mathbf{H}}_0^0(\text{rot}, \Omega) &= N(\mathring{\text{rot}}), & \mathbf{H}_0^0(\text{rot}, \Omega) &= N(\text{rot}), \\ \mathring{\mathbf{H}}_0^0(\text{div}, \Omega) &= N(\mathring{\text{div}}), & \mathbf{H}_0^0(\text{div}, \Omega) &= N(\text{div}). \end{aligned}$$

We often skip the upper right index 0, e.g., $\mathring{\mathbf{H}}_0^m(\text{rot}, \Omega) = \mathring{\mathbf{H}}^m(\text{rot}, \Omega) = N(\mathring{\text{rot}})$.

The existence of regular potential operators for the classical de Rham complex is well known. Here we cite results from [30, 15], and [25], see also the corresponding literature in [25].

Lemma 3.1 (Regular potential operators). *Let $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, and let Ω be additionally topologically trivial. Then there exist linear and bounded operators*

$$\begin{aligned} P_{\mathring{\nabla}}^m : \mathring{\mathbf{H}}_0^m(\text{rot}, \Omega) &\rightarrow \mathring{\mathbf{H}}^{m+1}(\Omega), & \nabla P_{\mathring{\nabla}}^m &= \text{id}_{\mathring{\mathbf{H}}_0^m(\text{rot}, \Omega)}, \\ P_{\nabla}^n : \mathbf{H}_0^n(\text{rot}, \Omega) &\rightarrow \mathbf{H}^{n+1}(\Omega), & \nabla P_{\nabla}^n &= \text{id}_{\mathbf{H}_0^n(\text{rot}, \Omega)}, \\ P_{\mathring{\text{rot}}}^m : \mathring{\mathbf{H}}_0^m(\text{div}, \Omega) &\rightarrow \mathring{\mathbf{H}}^{m+1}(\Omega), & \text{rot } P_{\mathring{\text{rot}}}^m &= \text{id}_{\mathring{\mathbf{H}}_0^m(\text{div}, \Omega)}, \\ P_{\text{rot}}^n : \mathbf{H}_0^n(\text{div}, \Omega) &\rightarrow \mathbf{H}^{n+1}(\Omega), & \text{rot } P_{\text{rot}}^n &= \text{id}_{\mathbf{H}_0^n(\text{div}, \Omega)}, \\ P_{\mathring{\text{div}}}^m : \mathring{\mathbf{H}}_{\perp}^m(\Omega) &\rightarrow \mathring{\mathbf{H}}^{m+1}(\Omega), & \text{div } P_{\mathring{\text{div}}}^m &= \text{id}_{\mathring{\mathbf{H}}_{\perp}^m(\Omega)}, \\ P_{\text{div}}^n : \mathbf{H}^n(\Omega) &\rightarrow \mathbf{H}^{n+1}(\Omega), & \text{div } P_{\text{div}}^n &= \text{id}_{\mathbf{H}^n(\Omega)}. \end{aligned}$$

These operators are regular potential operators associated to the domain complexes (16). In particular, all ranges are closed.

All results of Section 3 extend literally to the case of the vector de Rham complex (of unbounded operators)

$$(17) \quad \begin{array}{ccccccccc} \{0\} & \xrightarrow{\iota_{\{0\}}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\mathring{\nabla}} & \mathring{\mathcal{H}}^m(\Omega) & \xrightarrow{\mathring{\text{Rot}}} & \mathring{\mathcal{H}}^m(\Omega) & \xrightarrow{\mathring{\text{Div}}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\pi_{\mathbb{R}^3}} & \mathbb{R}^3, \\ \{0\} & \xleftarrow{\pi_{\{0\}}} & \mathbf{H}^n(\Omega) & \xleftarrow{-\text{Div}} & \mathcal{H}^n(\Omega) & \xleftarrow{\text{Rot}} & \mathcal{H}^n(\Omega) & \xleftarrow{-\nabla} & \mathbf{H}^n(\Omega) & \xleftarrow{\iota_{\mathbb{R}^3}} & \mathbb{R}^3 \end{array}$$

with the necessary notational adaptations as formulated in Sections 2.3, 2.4. Observe that here $\mathring{\nabla}$ (resp. ∇) denotes the Jacobian of vector fields.

4. THE ELASTICITY COMPLEX

The classical primal and dual elasticity domain complexes read in full length

$$(18) \quad \begin{array}{ccccccccc} \{0\} & \xrightarrow{\iota_{\{0\}}} & \mathring{\mathbf{H}}^1(\Omega) & \xrightarrow{\text{sym } \mathring{\nabla}} & \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot Rot}^\top, \Omega) & \xrightarrow{\text{Rot Rot}_{\mathbb{S}}^\top} & \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Div}, \Omega) & \xrightarrow{\mathring{\text{Div}}_{\mathbb{S}}} & \mathbf{L}^2(\Omega) & \xrightarrow{\pi_{\mathbf{RM}}} & \mathbf{RM}, \\ \{0\} & \xleftarrow{\pi_{\{0\}}} & \mathbf{L}^2(\Omega) & \xleftarrow{-\text{Div}_{\mathbb{S}}} & \mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega) & \xleftarrow{\text{Rot Rot}_{\mathbb{S}}^\top} & \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^\top, \Omega) & \xleftarrow{-\text{sym } \nabla} & \mathbf{H}^1(\Omega) & \xleftarrow{\iota_{\mathbf{RM}}} & \mathbf{RM}. \end{array}$$

Here $\iota_{\{0\}}$, $\pi_{\{0\}}$ and $\iota_{\mathbf{RM}}$, $\pi_{\mathbf{RM}}$ denote the canonical embeddings and $\mathbf{L}^2(\Omega)$ -orthogonal projections of/on the finite-dimensional subspaces $\{0\} \subset \mathbb{R}^3$ and the space of rigid motions

$$\mathbf{RM} = \{Q\mathbf{x} + \mathbf{q} : Q \in \mathbb{R}^{3 \times 3} \text{ skew}, \mathbf{q} \in \mathbb{R}^3\}, \quad \dim \mathbf{RM} = 6,$$

respectively. These complexes are Hilbert complexes. The corresponding cohomology groups

$$\mathbf{Harm}_{\mathbb{S},D}(\Omega) := N(\text{Rot Rot}_{\mathbb{S}}^\top) \cap N(\text{Div}_{\mathbb{S}}), \quad \mathbf{Harm}_{\mathbb{S},N}(\Omega) := N(\mathring{\text{Div}}_{\mathbb{S}}) \cap N(\text{Rot Rot}_{\mathbb{S}}^\top)$$

will be called generalized Dirichlet and Neumann tensors of the elasticity complex, respectively. For a comprehensive derivation of the elasticity complex by operator theoretical arguments, see [27].

More generally, we will discuss the following variants of the primal and dual elasticity complexes (of unbounded operators) for $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$:

$$\begin{array}{ccccccccc} \{0\} & \xrightarrow{\iota_{\{0\}}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\text{sym } \mathring{\nabla}} & \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega) & \xrightarrow{\text{Rot Rot}_{\mathbb{S}}^\top} & \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega) & \xrightarrow{\mathring{\text{Div}}_{\mathbb{S}}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\pi_{\mathbf{RM}}} & \mathbf{RM}, \\ \{0\} & \xleftarrow{\pi_{\{0\}}} & \mathbf{H}^n(\Omega) & \xleftarrow{-\text{Div}_{\mathbb{S}}} & \mathcal{H}_{\mathbb{S}}^n(\Omega) & \xleftarrow{\text{Rot Rot}_{\mathbb{S}}^\top} & \mathcal{H}_{\mathbb{S}}^n(\Omega) & \xleftarrow{-\text{sym } \nabla} & \mathbf{H}^n(\Omega) & \xleftarrow{\iota_{\mathbf{RM}}} & \mathbf{RM} \end{array}$$

with the associated domain complexes

$$(19) \quad \begin{array}{ccccccccc} \{0\} & \xrightarrow{\iota_{\{0\}}} & \mathring{\mathbf{H}}^{m+1}(\Omega) & \xrightarrow{\text{sym } \mathring{\nabla}} & \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^\top, \Omega) & \xrightarrow{\text{Rot Rot}_{\mathbb{S}}^\top} & \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega) & \xrightarrow{\mathring{\text{Div}}_{\mathbb{S}}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\pi_{\mathbf{RM}}} & \mathbf{RM}, \\ \{0\} & \xleftarrow{\pi_{\{0\}}} & \mathbf{H}^n(\Omega) & \xleftarrow{-\text{Div}_{\mathbb{S}}} & \mathcal{H}_{\mathbb{S}}^n(\text{Div}, \Omega) & \xleftarrow{\text{Rot Rot}_{\mathbb{S}}^\top} & \mathcal{H}_{\mathbb{S}}^n(\text{Rot Rot}^\top, \Omega) & \xleftarrow{-\text{sym } \nabla} & \mathbf{H}^{n+1}(\Omega) & \xleftarrow{\iota_{\mathbf{RM}}} & \mathbf{RM} \end{array}$$

and the Sobolev spaces

$$\begin{aligned} \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^\top, \Omega) &:= \{M \in D(\text{Rot Rot}_{\mathbb{S}}^\top) \cap \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega) : \text{Rot Rot}^\top M \in \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega)\}, \\ \mathcal{H}_{\mathbb{S}}^n(\text{Rot Rot}^\top, \Omega) &:= \{M \in \mathcal{H}_{\mathbb{S}}^n(\Omega) : \text{Rot Rot}^\top M \in \mathcal{H}_{\mathbb{S}}^n(\Omega)\}, \\ \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega) &:= \{N \in D(\mathring{\text{Div}}_{\mathbb{S}}) \cap \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega) : \text{Div } N \in \mathring{\mathbf{H}}^m(\Omega)\}, \\ \mathcal{H}_{\mathbb{S}}^n(\text{Div}, \Omega) &:= \{N \in \mathcal{H}_{\mathbb{S}}^n(\Omega) : \text{Div } N \in \mathbf{H}^n(\Omega)\}. \end{aligned}$$

Note that in the definition of the spaces $\mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^\top, \Omega)$ and $\mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega)$ the intersection with the domain of definition is superfluous for $m \geq 2$ and $m \geq 1$, respectively.

Additionally, we will also discuss the following variants of the elasticity complexes (of unbounded operators) for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$:

$$\begin{array}{ccccccccc} \{0\} & \xrightarrow{\iota_{\{0\}}} & \mathring{\mathbf{H}}^m(\Omega) & \xrightarrow{\text{sym } \mathring{\nabla}} & \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega) & \xrightarrow{\text{Rot Rot}_{\mathbb{S}}^\top} & \mathring{\mathcal{H}}_{\mathbb{S}}^{m-1}(\Omega) & \xrightarrow{\mathring{\text{Div}}_{\mathbb{S}}} & \mathring{\mathbf{H}}^{m-1}(\Omega) & \xrightarrow{\pi_{\mathbf{RM}}} & \mathbf{RM}, \\ \{0\} & \xleftarrow{\pi_{\{0\}}} & \mathbf{H}^{n-1}(\Omega) & \xleftarrow{-\text{Div}_{\mathbb{S}}} & \mathcal{H}_{\mathbb{S}}^{n-1}(\Omega) & \xleftarrow{\text{Rot Rot}_{\mathbb{S}}^\top} & \mathcal{H}_{\mathbb{S}}^n(\Omega) & \xleftarrow{-\text{sym } \nabla} & \mathbf{H}^n(\Omega) & \xleftarrow{\iota_{\mathbf{RM}}} & \mathbf{RM} \end{array}$$

with the associated domain complexes

$$(20) \quad \begin{array}{ccccccc} \{\mathbf{0}\} & \xrightarrow{\iota_{\{\mathbf{0}\}}} & \mathring{\mathbf{H}}^{m+1}(\Omega) & \xrightarrow{\text{sym} \nabla_{\mathbb{S}}} & \mathring{\mathcal{H}}_{\mathbb{S}}^{m,m-1}(\text{Rot Rot}^{\top}, \Omega) & \xrightarrow{\text{Rot} \mathring{\text{Rot}}_{\mathbb{S}}^{\top}} & \mathring{\mathcal{H}}_{\mathbb{S}}^{m-1}(\text{Div}, \Omega) & \xrightarrow{\text{Div}_{\mathbb{S}}} & \mathring{\mathbf{H}}^{m-1}(\Omega) & \xrightarrow{\pi_{\mathbf{RM}}} & \mathbf{RM}, \\ \{\mathbf{0}\} & \xleftarrow{\pi_{\{\mathbf{0}\}}} & \mathbf{H}^{n-1}(\Omega) & \xleftarrow{-\text{Div}_{\mathbb{S}}} & \mathcal{H}_{\mathbb{S}}^{n-1}(\text{Div}, \Omega) & \xleftarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} & \mathcal{H}_{\mathbb{S}}^{n,n-1}(\text{Rot Rot}^{\top}, \Omega) & \xleftarrow{\text{sym} \nabla_{\mathbb{S}}} & \mathbf{H}^{n+1}(\Omega) & \xleftarrow{\iota_{\mathbf{RM}}} & \mathbf{RM} \end{array}$$

and the slightly less regular Sobolev spaces

$$\begin{aligned} \mathring{\mathcal{H}}_{\mathbb{S}}^{m,m-1}(\text{Rot Rot}^{\top}, \Omega) &:= \{ \mathbf{M} \in D(\text{Rot} \mathring{\text{Rot}}_{\mathbb{S}}^{\top}) \cap \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega) : \text{Rot Rot}^{\top} \mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}^{m-1}(\Omega) \}, \\ \mathcal{H}_{\mathbb{S}}^{n,n-1}(\text{Rot Rot}^{\top}, \Omega) &:= \{ \mathbf{M} \in \mathcal{H}_{\mathbb{S}}^n(\Omega) : \text{Rot Rot}^{\top} \mathbf{M} \in \mathcal{H}_{\mathbb{S}}^{n-1}(\Omega) \}, \end{aligned}$$

where again the intersection with $D(\text{Rot} \mathring{\text{Rot}}_{\mathbb{S}}^{\top})$ is not needed for $m \geq 2$.

For our purposes the case $n = 0$ will be most important, i.e.,

$$\mathcal{H}_{\mathbb{S}}^{0,-1}(\text{Rot Rot}^{\top}, \Omega) = \{ \mathbf{M} \in \mathcal{L}_{\mathbb{S}}^2(\Omega) : \text{Rot Rot}^{\top} \mathbf{M} \in \mathcal{H}_{\mathbb{S}}^{-1}(\Omega) \}.$$

The first variants of the elasticity complexes are the more natural extension of the classical elasticity complex, while the second variants are better adapted to the technical difficulties due to the second order operator Rot Rot^{\top} , as we shall see.

Remark 4.1 (One operator). *Note that, e.g., the second order operator $\text{Rot Rot}_{\mathbb{S}}^{\top}$ is “one” operator and not a composition of the two first order operators Rot and Rot^{\top} . Similarly, the operators $\text{Rot} \mathring{\text{Rot}}_{\mathbb{S}}^{\top}$ or $\text{sym} \nabla$, $\text{sym} \nabla$ have to be understood in the same sense as one operator. This fact is underlined by some new results about Fredholm indices presented in [24]. The differential operators are of mixed order but cannot be seen as leading order type with relatively compact perturbation.*

We will now proceed to show the main result of this paper, the compact embeddings for the elasticity complex as formulated in Theorem 4.11. The proof is based on Theorem 2.11. Therefore, we have to construct appropriate regular decompositions. We do this in two main steps.

- In Section 4.1 we start with the construction of regular potential operators for the elasticity complex in the case of topologically trivial domains (Theorem 4.2). This immediately implies that the elasticity complex is an exact Hilbert complex in this case (Theorem 4.4). Then we will use these regular potential operators for constructing regular decompositions for the elasticity complex for topologically trivial domains (Corollary 4.6).
- In Section 4.2 we extend the results on regular decompositions for the elasticity complex to general strong Lipschitz domains by a partition of unity argument (Theorem 4.9). Based on these results it will be shown that the elasticity complexes are compact Hilbert complexes (Theorem 4.11). Additionally, we construct regular potentials even in this general case (Corollary 4.14) by applying Corollary 2.13.

4.1. Topologically Trivial Domains: Regular Potentials and Decompositions. For the next theorem we introduce the following notations. Similarly to the orthogonality to \mathbb{R} and \mathbb{R}^3 indicated in the Hilbert spaces $\mathbf{L}_{\perp}^2(\Omega) = \mathbf{L}_{\perp_{\mathbb{R}}}^2(\Omega)$ and $\mathbf{L}_{\perp}^2(\Omega) = \mathbf{L}_{\perp_{\mathbb{R}^3}}^2(\Omega)$, respectively, we set

$$\mathbf{L}_{\perp_{\mathbf{RM}}}^2(\Omega) := \mathbf{L}^2(\Omega) \cap \mathbf{RM}^{\perp_{\mathbf{L}^2(\Omega)}}.$$

Let $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. We introduce kernel Sobolev spaces by

$$\begin{aligned} \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Rot Rot}^{\top}, \Omega) &:= \{ \mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^{\top}, \Omega) : \text{Rot Rot}^{\top} \mathbf{M} = 0 \}, \\ \mathcal{H}_{\mathbb{S},0}^n(\text{Rot Rot}^{\top}, \Omega) &:= \{ \mathbf{M} \in \mathcal{H}_{\mathbb{S}}^n(\text{Rot Rot}^{\top}, \Omega) : \text{Rot Rot}^{\top} \mathbf{M} = 0 \}, \\ \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Div}, \Omega) &:= \{ \mathbf{N} \in \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega) : \text{Div} \mathbf{N} = 0 \}, \\ \mathcal{H}_{\mathbb{S},0}^n(\text{Div}, \Omega) &:= \{ \mathbf{N} \in \mathcal{H}_{\mathbb{S}}^n(\text{Div}, \Omega) : \text{Div} \mathbf{N} = 0 \}. \end{aligned}$$

Note that for $m = n = 0$ we have

$$\begin{aligned} \mathring{\mathcal{H}}_{\mathbb{S},0}^0(\text{Rot Rot}^{\top}, \Omega) &= N(\text{Rot} \mathring{\text{Rot}}_{\mathbb{S}}^{\top}), & \mathcal{H}_{\mathbb{S},0}^0(\text{Rot Rot}^{\top}, \Omega) &= N(\text{Rot Rot}_{\mathbb{S}}^{\top}), \\ \mathring{\mathcal{H}}_{\mathbb{S},0}^0(\text{Div}, \Omega) &= N(\mathring{\text{Div}}_{\mathbb{S}}), & \mathcal{H}_{\mathbb{S},0}^0(\text{Div}, \Omega) &= N(\text{Div}_{\mathbb{S}}). \end{aligned}$$

Using the linear and bounded regular potential operators $P_{\nabla}^m, P_{\nabla}^n, P_{\text{rot}}^m, P_{\text{rot}}^n, P_{\text{div}}^m, P_{\text{div}}^n$ for the de Rham complex from Lemma 3.1 and their counterparts $P_{\nabla}^m, P_{\nabla}^n, P_{\text{Rot}}^m, P_{\text{Rot}}^n, P_{\text{Div}}^m, P_{\text{Div}}^n$ for the vector de Rham complex (17), we start now with our first regular potentials for the elasticity complex.

Theorem 4.2 (Regular potential operators). *Let Ω be additionally topologically trivial and let $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. Then there exist linear and bounded operators*

$$\begin{aligned} P_{\text{sym } \nabla}^m &: \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Rot Rot}^\top, \Omega) \rightarrow \mathring{\mathcal{H}}^{m+1}(\Omega), & \text{sym } \nabla P_{\text{sym } \nabla}^m &= \text{id}_{\mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Rot Rot}^\top, \Omega)}, \\ P_{\text{sym } \nabla}^n &: \mathcal{H}_{\mathbb{S},0}^n(\text{Rot Rot}^\top, \Omega) \rightarrow \mathcal{H}^{n+1}(\Omega), & \text{sym } \nabla P_{\text{sym } \nabla}^n &= \text{id}_{\mathcal{H}_{\mathbb{S},0}^n(\text{Rot Rot}^\top, \Omega)}, \\ P_{\text{Rot Rot}_S^\top}^m &: \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Div}, \Omega) \rightarrow \mathring{\mathcal{H}}_{\mathbb{S}}^{m+2}(\Omega), & \text{Rot Rot}^\top P_{\text{Rot Rot}_S^\top}^m &= \text{id}_{\mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Div}, \Omega)}, \\ P_{\text{Rot Rot}_S^\top}^n &: \mathcal{H}_{\mathbb{S},0}^n(\text{Div}, \Omega) \rightarrow \mathcal{H}_{\mathbb{S}}^{n+2}(\Omega), & \text{Rot Rot}^\top P_{\text{Rot Rot}_S^\top}^n &= \text{id}_{\mathcal{H}_{\mathbb{S},0}^n(\text{Div}, \Omega)}, \\ P_{\text{Div}_S}^m &: \mathring{\mathbf{H}}_{\perp\text{RM}}^m(\Omega) \rightarrow \mathring{\mathcal{H}}_{\mathbb{S}}^{m+1}(\Omega), & \text{Div } P_{\text{Div}_S}^m &= \text{id}_{\mathring{\mathbf{H}}_{\perp\text{RM}}^m(\Omega)}, \\ P_{\text{Div}_S}^n &: \mathbf{H}^n(\Omega) \rightarrow \mathcal{H}_{\mathbb{S}}^{n+1}(\Omega), & \text{Div } P_{\text{Div}_S}^n &= \text{id}_{\mathbf{H}^n(\Omega)} \end{aligned}$$

with

$$\begin{aligned} P_{\text{sym } \nabla}^m \mathbf{M} &= P_{\nabla}^m (\mathbf{M} + \text{spn } P_{\nabla}^{m-1} \text{Rot}^\top \mathbf{M}) \quad \text{for } m \geq 2, \\ P_{\text{sym } \nabla}^n \mathbf{M} &= P_{\nabla}^n (\mathbf{M} + \text{spn } P_{\nabla}^{n-1} \text{Rot}^\top \mathbf{M}), \\ P_{\text{Rot Rot}_S^\top}^m \mathbf{N} &= \text{sym } P_{\text{Rot}}^{m+1} ((P_{\text{Rot}}^m \mathbf{N})^\top - (\text{tr } P_{\text{Rot}}^m \mathbf{N}) \mathbf{I}), \\ P_{\text{Rot Rot}_S^\top}^n \mathbf{N} &= \text{sym } P_{\text{Rot}}^{n+1} ((P_{\text{Rot}}^n \mathbf{N})^\top - (\text{tr } P_{\text{Rot}}^n \mathbf{N}) \mathbf{I}), \\ P_{\text{Div}_S}^m \mathbf{v} &= \text{sym} (P_{\text{Div}}^m \mathbf{v} - 2 \text{Rot}^\top (P_{\text{Div}}^{m+1} \text{spn}^{-1} \text{skw } P_{\text{Div}}^m \mathbf{v})), \\ P_{\text{Div}_S}^n \mathbf{v} &= \text{sym} (P_{\text{Div}}^n \mathbf{v} - 2 \text{Rot}^\top (P_{\text{Div}}^{n+1} \text{spn}^{-1} \text{skw } P_{\text{Div}}^n \mathbf{v})). \end{aligned}$$

Proof. First, let us consider only $m = n \in \mathbb{N}_0$.

• Let $\mathbf{N} \in \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Div}, \Omega)$ resp. $\mathbf{N} \in \mathcal{H}_{\mathbb{S},0}^m(\text{Div}, \Omega)$. Then, with row-wise applications of Lemma 3.1, there exists $\mathbf{E} = P_{\text{Rot}}^m \mathbf{N} \in \mathring{\mathcal{H}}^{m+1}(\Omega)$ resp. $\mathbf{E} = P_{\text{Rot}}^m \mathbf{N} \in \mathcal{H}^{m+1}(\Omega)$ such that $\text{Rot } \mathbf{E} = \mathbf{N}$. As \mathbf{N} is symmetric we get with Lemma A.1 (v) for $\tilde{\mathbf{E}} := \mathbf{E}^\top - (\text{tr } \mathbf{E}) \mathbf{I} \in \mathring{\mathcal{H}}^{m+1}(\Omega)$ resp. $\mathcal{H}^{m+1}(\Omega)$

$$\text{Div } \tilde{\mathbf{E}} = \text{Div } \mathbf{E}^\top - \nabla \text{tr } \mathbf{E} = 2 \text{spn}^{-1} \text{skw } \text{Rot } \mathbf{E} = 2 \text{spn}^{-1} \text{skw } \mathbf{N} = 0.$$

Again row-wise applications of Lemma 3.1 show the existence of $\tilde{\mathbf{M}} = P_{\text{Rot}}^{m+1} \tilde{\mathbf{E}} \in \mathring{\mathcal{H}}^{m+2}(\Omega)$ resp. $\tilde{\mathbf{M}} = P_{\text{Rot}}^{m+1} \tilde{\mathbf{E}} \in \mathcal{H}^{m+2}(\Omega)$ such that $\text{Rot } \tilde{\mathbf{M}} = \tilde{\mathbf{E}}$. Hence

$$\mathbf{N} = \text{Rot } \mathbf{E} = \text{Rot} (\tilde{\mathbf{E}}^\top + (\text{tr } \mathbf{E}) \mathbf{I}) = \text{Rot Rot}^\top \tilde{\mathbf{M}} + \text{Rot} ((\text{tr } \mathbf{E}) \mathbf{I}).$$

Lemma A.1 (i) yields $\text{Rot} ((\text{tr } \mathbf{E}) \mathbf{I}) = -\text{spn } \nabla \text{tr } \mathbf{E}$, which is skew-symmetric. Therefore,

$$\mathbf{N} = \text{sym } \mathbf{N} = \text{sym } \text{Rot Rot}^\top \tilde{\mathbf{M}} = \text{Rot Rot}^\top \text{sym } \tilde{\mathbf{M}}$$

see Lemma A.1 (vi). The tensor field $\mathbf{M} := \text{sym } \tilde{\mathbf{M}} \in \mathring{\mathcal{H}}_{\mathbb{S}}^{m+2}(\Omega)$ resp. $\mathbf{M} := \text{sym } \tilde{\mathbf{M}} \in \mathcal{H}_{\mathbb{S}}^{m+2}(\Omega)$ solves $\text{Rot Rot}^\top \mathbf{M} = \mathbf{N}$ and is given by

$$\mathbf{M} = \text{sym } P^{m+1} \tilde{\mathbf{E}} = \text{sym } P^{m+1} ((P^m \mathbf{N})^\top - (\text{tr } P^m \mathbf{N}) \mathbf{I}) =: \hat{P}^m \mathbf{N},$$

where $P^k \in \{P_{\text{Rot}}^k, P_{\text{Rot}}^k\}$ and $\hat{P}^m \in \{P_{\text{Rot Rot}_S^\top}^m, P_{\text{Rot Rot}_S^\top}^m\}$.

• Let $\mathbf{v} \in \mathring{\mathbf{H}}_{\perp\text{RM}}^m(\Omega) \subset \mathbf{L}_{\perp\text{RM}}^2(\Omega)$ resp. $\mathbf{v} \in \mathbf{H}^m(\Omega)$. Row-wise applications of Lemma 3.1 show the existence of $\mathbf{E} = P_{\text{Div}_S}^m \mathbf{v} \in \mathring{\mathcal{H}}^{m+1}(\Omega)$ resp. $\mathbf{E} = P_{\text{Div}_S}^m \mathbf{v} \in \mathcal{H}^{m+1}(\Omega)$ such that $\text{Div } \mathbf{E} = \mathbf{v}$. Moreover, for $\mathbf{v} \in \mathbf{L}_{\perp\text{RM}}^2(\Omega)$, $\mathbf{E} \in \mathring{\mathcal{H}}^{m+1}(\Omega)$, and $\mathbf{b} \in \mathbb{R}^3$ we have that $\mathbf{A} := \text{spn } \mathbf{b}$ is skew-symmetric and hence

$$\begin{aligned} 0 &= \langle \mathbf{v}, \mathbf{Ax} \rangle_{\mathbf{L}^2(\Omega)} = \langle \text{Div } \mathbf{E}, \mathbf{Ax} \rangle_{\mathbf{L}^2(\Omega)} = -\langle \mathbf{E}, \nabla(\mathbf{Ax}) \rangle_{\mathcal{L}^2(\Omega)} \\ &= -\langle \mathbf{E}, \mathbf{A} \rangle_{\mathcal{L}^2(\Omega)} = -\langle \text{skw } \mathbf{E}, \text{spn } \mathbf{b} \rangle_{\mathcal{L}^2(\Omega)} = -2 \langle \text{spn}^{-1} \text{skw } \mathbf{E}, \mathbf{b} \rangle_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

using the relation $(\text{spn } \mathbf{d}) : (\text{spn } \mathbf{b}) = 2 \mathbf{d} \cdot \mathbf{b}$ for $\mathbf{d} = \text{spn}^{-1} \text{skw } \mathbf{E}$, i.e., $\text{spn}^{-1} \text{skw } \mathbf{E} \in \mathring{\mathbf{H}}_{\perp}^{m+1}(\Omega)$ resp. $\text{spn}^{-1} \text{skw } \mathbf{E} \in \mathbf{H}^{m+1}(\Omega)$. By Lemma 3.1 there is some $\tilde{\mathbf{E}} = \text{P}_{\text{Div}}^{m+1} \text{spn}^{-1} \text{skw } \mathbf{E} \in \mathring{\mathcal{H}}^{m+2}(\Omega)$ resp. $\tilde{\mathbf{E}} = \text{P}_{\text{Div}}^{m+1} \text{spn}^{-1} \text{skw } \mathbf{E} \in \mathcal{H}^{m+2}(\Omega)$ such that $\text{Div } \tilde{\mathbf{E}} = \text{spn}^{-1} \text{skw } \mathbf{E}$. By Lemma A.1 (iii), (v) we see

$$\text{Div skw } \mathbf{E} = -\text{rot spn}^{-1} \text{skw } \mathbf{E} = -\text{rot Div } \tilde{\mathbf{E}} = -2 \text{Div sym Rot } \tilde{\mathbf{E}}^{\top}.$$

Hence

$$\mathbf{v} = \text{Div } \mathbf{E} = \text{Div sym } \mathbf{E} + \text{Div skw } \mathbf{E} = \text{Div } \mathbf{N}$$

with $\mathbf{N} := \text{sym}(\mathbf{E} - 2 \text{Rot } \tilde{\mathbf{E}}^{\top}) \in \mathring{\mathcal{H}}_{\mathbb{S}}^{m+1}(\Omega)$ resp. $\mathcal{H}_{\mathbb{S}}^{m+1}(\Omega)$ given by

$$\begin{aligned} \mathbf{N} &= \text{sym}(\mathbf{E} - 2 \text{Rot}(\text{P}^{m+1} \text{spn}^{-1} \text{skw } \mathbf{E})^{\top}) \\ &= \text{sym}(\text{P}^m \mathbf{v} - 2 \text{Rot}^{\top}(\text{P}^{m+1} \text{spn}^{-1} \text{skw } \text{P}^m \mathbf{v})) =: \hat{\text{P}}^m \mathbf{v}, \end{aligned}$$

where $\text{P}^k \in \{\text{P}_{\text{Div}}^k, \text{P}_{\text{Div}_s}^k\}$ and $\hat{\text{P}} \in \{\text{P}_{\text{Div}_s}^m, \text{P}_{\text{Div}_s}^m\}$.

• Let $\mathbf{M} \in \mathcal{H}_{\mathbb{S},0}^m(\text{Rot Rot}^{\top}, \Omega)$. Then $\text{Rot}^{\top} \mathbf{M} \in \mathcal{H}^{m-1}(\Omega)$ with $\text{Rot}(\text{Rot}^{\top} \mathbf{M}) = 0$. Row-wise applications of Lemma 3.1 yield $\mathbf{u} = \text{P}_{\nabla}^{m-1} \text{Rot}^{\top} \mathbf{M} \in \mathbf{H}^m(\Omega)$ such that $\nabla \mathbf{u} = \text{Rot}^{\top} \mathbf{M}$. By Lemma A.1 (ii) we see $\text{div } \mathbf{u} = \text{tr } \nabla \mathbf{u} = \text{tr } \text{Rot}^{\top} \mathbf{M} = 0$ as \mathbf{M} is symmetric and thus by using Lemma A.1 (iv) we get $\nabla \mathbf{u} = -\text{Rot}^{\top} \text{spn } \mathbf{u}$. Therefore, $\text{Rot}^{\top}(\mathbf{M} + \text{spn } \mathbf{u}) = 0$ and again Lemma 3.1 gives some $\mathbf{v} = \text{P}_{\nabla}^m(\mathbf{M} + \text{spn } \mathbf{u}) \in \mathbf{H}^{m+1}(\Omega)$ with $\nabla \mathbf{v} = \mathbf{M} + \text{spn } \mathbf{u}$. As \mathbf{M} is symmetric we obtain $\mathbf{M} = \text{sym } \nabla \mathbf{v}$ and

$$\mathbf{v} = \text{P}_{\nabla}^m(\mathbf{M} + \text{spn } \mathbf{u}) = \text{P}_{\nabla}^m(\mathbf{M} + \text{spn } \text{P}_{\nabla}^{m-1} \text{Rot}^{\top} \mathbf{M}) =: \text{P}_{\text{sym } \nabla}^m \mathbf{M}.$$

Now, let $\mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Rot Rot}^{\top}, \Omega)$. We can extend \mathbf{M} by zero to $\tilde{\mathbf{M}} \in \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Rot Rot}^{\top}, B)$ (by definition), where B denotes an open ball containing $\bar{\Omega}$. This can be seen by approximating \mathbf{M} by a sequence $(\mathbf{M}_n) \subset \mathring{\mathcal{C}}_{\mathbb{S}}^{\infty}(\Omega)$ of test tensors with $\mathbf{M}_n \rightarrow \mathbf{M}$ in $\mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^{\top}, \Omega)$ resp. $\mathcal{H}^m(\Omega)$ and observing $\tilde{\mathbf{M}}_n \rightarrow \tilde{\mathbf{M}}$ in $\mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^{\top}, B)$ resp. $\mathcal{H}^m(B)$. In particular, we have $\tilde{\mathbf{M}} \in \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Rot Rot}^{\top}, B)$ and from the latter arguments (of the proof without boundary conditions) we obtain $\tilde{\mathbf{v}} := \text{P}_{\text{sym } \nabla, B}^m \tilde{\mathbf{M}} \in \mathbf{H}^{m+1}(B)$ with $\text{sym } \nabla \tilde{\mathbf{v}} = \tilde{\mathbf{M}}$ in B . As $\tilde{\mathbf{M}}$ vanishes in the complement of Ω , $\tilde{\mathbf{v}}$ is a rigid motion $\mathbf{r}_{\tilde{\mathbf{v}}} \in \mathbf{RM}$ in $B \setminus \bar{\Omega}$. Thus we have $\mathbf{v} := \tilde{\mathbf{v}} - \mathbf{r}_{\tilde{\mathbf{v}}} \in \mathbf{H}^{m+1}(B)$ and $\mathbf{v}|_{B \setminus \bar{\Omega}} = 0$, which implies $\mathbf{v} \in \mathring{\mathcal{H}}^{m+1}(\Omega)$ and $\text{sym } \nabla \mathbf{v} = \text{sym } \nabla \tilde{\mathbf{v}} = \mathbf{M}$ in Ω . Moreover,

$$\mathbf{v} = (\text{P}_{\text{sym } \nabla, B}^m \tilde{\mathbf{M}})|_{\Omega} - \mathbf{r}_{\tilde{\mathbf{v}}} =: \text{P}_{\text{sym } \nabla}^m \mathbf{M}.$$

For $m \geq 2$ we can literally following the corresponding proof without boundary conditions as well. In particular, we get $\text{Rot}^{\top} \mathbf{M} \in \mathring{\mathcal{H}}_0^{m-1}(\text{Rot}, \Omega)$, $\mathbf{u} \in \mathring{\mathbf{H}}^m(\Omega)$, and $\mathbf{M} - \text{spn } \mathbf{u} \in \mathring{\mathcal{H}}^m(\Omega)$. Thus

$$\mathring{\mathbf{H}}^{m+1}(\Omega) \ni \mathbf{v} = \text{P}_{\nabla}^m(\mathbf{M} + \text{spn } \mathbf{u}) = \text{P}_{\nabla}^m(\mathbf{M} + \text{spn } \text{P}_{\nabla}^{m-1} \text{Rot}^{\top} \mathbf{M}) =: \text{P}_{\text{sym } \nabla}^m \mathbf{M}.$$

• Finally, let $-n \in \mathbb{N}$. Then for $\mathbf{M} \in \mathcal{H}_{\mathbb{S},0}^n(\text{Rot Rot}^{\top}, \Omega)$, $\mathbf{N} \in \mathcal{H}_{\mathbb{S},0}^n(\text{Div}, \Omega)$, and $\mathbf{v} \in \mathbf{H}^n(\Omega)$ we literally follow the latter proofs (without boundary condition). This concludes the proof. \square

As a simple consequence of Theorem 4.2 we obtain:

Corollary 4.3 (Regular potentials). *Let Ω be additionally topologically trivial and let $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. Then*

$$\begin{aligned} \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Rot Rot}^{\top}, \Omega) &= \text{sym } \nabla \mathring{\mathbf{H}}^{m+1}(\Omega), \\ \mathcal{H}_{\mathbb{S},0}^n(\text{Rot Rot}^{\top}, \Omega) &= \text{sym } \nabla \mathbf{H}^{n+1}(\Omega), \\ \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Div}, \Omega) &= \text{Rot Rot}^{\top} \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^{\top}, \Omega) \\ &= \text{Rot Rot}^{\top} \mathring{\mathcal{H}}_{\mathbb{S}}^{m+1,m}(\text{Rot Rot}^{\top}, \Omega) = \text{Rot Rot}^{\top} \mathring{\mathcal{H}}_{\mathbb{S}}^{m+2}(\Omega), \\ \mathcal{H}_{\mathbb{S},0}^n(\text{Div}, \Omega) &= \text{Rot Rot}^{\top} \mathcal{H}_{\mathbb{S}}^n(\text{Rot Rot}^{\top}, \Omega) \\ &= \text{Rot Rot}^{\top} \mathcal{H}_{\mathbb{S}}^{n+1,n}(\text{Rot Rot}^{\top}, \Omega) = \text{Rot Rot}^{\top} \mathcal{H}_{\mathbb{S}}^{n+2}(\Omega), \end{aligned}$$

$$\begin{aligned}\mathring{\mathbf{H}}_{\perp\text{RM}}^m(\Omega) &= \text{Div } \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega) = \text{Div } \mathring{\mathcal{H}}_{\mathbb{S}}^{m+1}(\Omega), \\ \mathbf{H}^n(\Omega) &= \text{Div } \mathcal{H}_{\mathbb{S}}^n(\text{Div}, \Omega) = \text{Div } \mathcal{H}_{\mathbb{S}}^{n+1}(\Omega)\end{aligned}$$

hold. Moreover, the operators $P_{\text{sym } \nabla}^m, P_{\text{sym } \nabla}^n, P_{\text{Rot Rot}_{\mathbb{S}}^{\top}}^m, P_{\text{Rot Rot}_{\mathbb{S}}^{\top}}^n, P_{\text{Div}_{\mathbb{S}}}^m, P_{\text{Div}_{\mathbb{S}}}^n$ are regular potential operators associated to the domain complexes (19) and, for $m \geq 1$, the operators $P_{\text{sym } \nabla}^m, P_{\text{sym } \nabla}^n, P_{\text{Rot Rot}_{\mathbb{S}}^{\top}}^{m-1}, P_{\text{Rot Rot}_{\mathbb{S}}^{\top}}^{n-1}, P_{\text{Div}_{\mathbb{S}}}^{m-1}, P_{\text{Div}_{\mathbb{S}}}^{n-1}$ are regular potential operators associated to the domain complexes (20). In particular, all ranges are closed.

Proof. We have by Theorem 4.2, e.g.,

$$\begin{aligned}\mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Div}, \Omega) &= \text{Rot Rot}^{\top} P_{\text{Rot Rot}_{\mathbb{S}}^{\top}}^m \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Div}, \Omega) \subset \text{Rot Rot}^{\top} \mathring{\mathcal{H}}_{\mathbb{S}}^{m+2}(\Omega) \\ &\subset \text{Rot Rot}^{\top} \mathring{\mathcal{H}}_{\mathbb{S}}^{m+1,m}(\Omega) \subset \text{Rot Rot}^{\top} \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^{\top}, \Omega) \subset \mathring{\mathcal{H}}_{\mathbb{S},0}^m(\text{Div}, \Omega),\end{aligned}$$

which implies the third identity. The other identities follow completely analogously. The regularity follows from Rellich's selection theorem. \square

Corollary 4.3 immediately leads to the following result.

Theorem 4.4 (Closed Hilbert complexes). *Let $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ and let Ω be additionally topologically trivial. The domain complexes of elasticity, cf. (19), (20),*

$$\begin{aligned}\{0\} &\xrightarrow{\iota_{\{0\}}} \mathring{\mathbf{H}}^{m+1}(\Omega) \xrightarrow{\text{sym } \nabla} \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^{\top}, \Omega) \xrightarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega) \xrightarrow{\text{Div}_{\mathbb{S}}} \mathring{\mathbf{H}}^m(\Omega) \xrightarrow{\pi_{\text{RM}}} \mathbf{RM}, \\ \{0\} &\xleftarrow{\pi_{\{0\}}} \mathbf{H}^n(\Omega) \xleftarrow{-\text{Div}_{\mathbb{S}}} \mathcal{H}_{\mathbb{S}}^n(\text{Div}, \Omega) \xleftarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} \mathcal{H}_{\mathbb{S}}^n(\text{Rot Rot}^{\top}, \Omega) \xleftarrow{-\text{sym } \nabla} \mathbf{H}^{n+1}(\Omega) \xleftarrow{\iota_{\text{RM}}} \mathbf{RM},\end{aligned}$$

and, for $m \geq 1$,

$$\begin{aligned}\{0\} &\xrightarrow{\iota_{\{0\}}} \mathring{\mathbf{H}}^{m+1}(\Omega) \xrightarrow{\text{sym } \nabla} \mathring{\mathcal{H}}_{\mathbb{S}}^{m,m-1}(\text{Rot Rot}^{\top}, \Omega) \xrightarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} \mathring{\mathcal{H}}_{\mathbb{S}}^{m-1}(\text{Div}, \Omega) \xrightarrow{\text{Div}_{\mathbb{S}}} \mathring{\mathbf{H}}^{m-1}(\Omega) \xrightarrow{\pi_{\text{RM}}} \mathbf{RM}, \\ \{0\} &\xleftarrow{\pi_{\{0\}}} \mathbf{H}^{n-1}(\Omega) \xleftarrow{-\text{Div}_{\mathbb{S}}} \mathcal{H}_{\mathbb{S}}^{n-1}(\text{Div}, \Omega) \xleftarrow{\text{Rot Rot}_{\mathbb{S}}^{\top}} \mathcal{H}_{\mathbb{S}}^{n-1}(\text{Rot Rot}^{\top}, \Omega) \xleftarrow{-\text{sym } \nabla} \mathbf{H}^{n+1}(\Omega) \xleftarrow{\iota_{\text{RM}}} \mathbf{RM}\end{aligned}$$

are exact Hilbert complexes, all ranges are closed, and the operators from Theorem 4.2 are associated regular potential operators.

Remark 4.5 (Topology). *A closer inspection of Theorem 4.2 and Corollary 4.3 together with their proofs shows that the assumptions on the topologies can be weakened. In particular, in Theorem 4.2 the following holds true for the potential operators:*

- (i) $P_{\text{div}}^m, P_{\text{div}}^n, P_{\text{Div}}^m, P_{\text{Div}}^n$ and hence $P_{\text{Div}_{\mathbb{S}}}^m, P_{\text{Div}_{\mathbb{S}}}^n$ do not depend on the topology and all results involving these divergence operators hold for general bounded strong Lipschitz domains.
- (ii) $P_{\nabla}^m, P_{\text{rot}}^n, P_{\text{Rot}}^n$ and hence $P_{\text{sym } \nabla}^m, P_{\text{Rot Rot}_{\mathbb{S}}^{\top}}^n$ exist on the respective kernels if and only if
$$\mathbf{Harm}_{\mathbb{D}}(\Omega) = \{0\} \Leftrightarrow \Gamma \text{ is connected.}$$
- (iii) $P_{\nabla}^n, P_{\text{rot}}^m, P_{\text{Rot}}^m$ and hence $P_{\text{sym } \nabla}^n, P_{\text{Rot Rot}_{\mathbb{S}}^{\top}}^m$ exist on the respective kernels if and only if

$$\mathbf{Harm}_{\mathbb{N}}(\Omega) = \{0\} \Leftrightarrow \Omega \text{ has no handles.}$$

As the assertions for the potentials imply the triviality of the cohomology groups we have:

- (iv) $\mathcal{Harm}_{\mathbb{S},\mathbb{D}}(\Omega) = \{0\} \Leftrightarrow \mathbf{Harm}_{\mathbb{D}}(\Omega) = \{0\} \Leftrightarrow \Gamma \text{ is connected}$
- (v) $\mathcal{Harm}_{\mathbb{S},\mathbb{N}}(\Omega) = \{0\} \Leftrightarrow \mathbf{Harm}_{\mathbb{N}}(\Omega) = \{0\} \Leftrightarrow \Omega \text{ has no handles}$

From Lemma 2.6, Lemma 2.9, Theorem 4.2, and Corollary 4.3 we immediately obtain the following regular decompositions.

Corollary 4.6 (Regular decompositions). *Let Ω be additionally topologically trivial and let $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. Then the regular decompositions*

$$\begin{aligned}\mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^{\top}, \Omega) &= R(P_{\text{Rot Rot}_{\mathbb{S}}^{\top}}^m) \dot{+} \text{sym } \nabla R(P_{\text{sym } \nabla}^m) \\ &= R(P_{\text{Rot Rot}_{\mathbb{S}}^{\top}}^m) \dot{+} \text{sym } \nabla \mathring{\mathbf{H}}^{m+1}(\Omega) = \mathring{\mathcal{H}}_{\mathbb{S}}^{m+2}(\Omega) + \text{sym } \nabla \mathring{\mathbf{H}}^{m+1}(\Omega),\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_{\mathbb{S}}^n(\text{Rot Rot}^\top, \Omega) &= R(\mathbb{P}_{\text{Rot Rot}_{\mathbb{S}}^\top}^n) \dot{+} \text{sym} \nabla R(\mathbb{P}_{\text{sym} \nabla}^n) \\
&= R(\mathbb{P}_{\text{Rot Rot}_{\mathbb{S}}^\top}^n) \dot{+} \text{sym} \nabla \mathbf{H}^{n+1}(\Omega) = \mathcal{H}_{\mathbb{S}}^{n+2}(\Omega) + \text{sym} \nabla \mathbf{H}^{n+1}(\Omega), \\
\mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega) &= R(\mathbb{P}_{\text{Div}_{\mathbb{S}}}^m) \dot{+} \text{Rot Rot}^\top R(\mathbb{P}_{\text{Rot Rot}_{\mathbb{S}}^\top}^m) \\
&= R(\mathbb{P}_{\text{Div}_{\mathbb{S}}}^m) \dot{+} \text{Rot Rot}^\top \mathring{\mathcal{H}}_{\mathbb{S}}^{m+2}(\Omega) = \mathring{\mathcal{H}}_{\mathbb{S}}^{m+1}(\Omega) + \text{Rot Rot}^\top \mathring{\mathcal{H}}_{\mathbb{S}}^{m+2}(\Omega), \\
\mathcal{H}_{\mathbb{S}}^n(\text{Div}, \Omega) &= R(\mathbb{P}_{\text{Div}_{\mathbb{S}}}^n) \dot{+} \text{Rot Rot}^\top R(\mathbb{P}_{\text{Rot Rot}_{\mathbb{S}}^\top}^n) \\
&= R(\mathbb{P}_{\text{Div}_{\mathbb{S}}}^n) \dot{+} \text{Rot Rot}^\top \mathcal{H}_{\mathbb{S}}^{n+2}(\Omega) = \mathcal{H}_{\mathbb{S}}^{n+1}(\Omega) + \text{Rot Rot}^\top \mathcal{H}_{\mathbb{S}}^{n+2}(\Omega)
\end{aligned}$$

hold.

By similar arguments we also obtain the following non-standard regular decompositions.

Corollary 4.7 (Regular decompositions). *Let Ω be additionally topologically trivial and let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. Then the regular decompositions*

$$\begin{aligned}
\mathring{\mathcal{H}}_{\mathbb{S}}^{m,m-1}(\text{Rot Rot}^\top, \Omega) &= R(\mathbb{P}_{\text{Rot Rot}_{\mathbb{S}}^\top}^{m-1}) \dot{+} \text{sym} \nabla R(\mathbb{P}_{\text{sym} \nabla}^m) \\
&= R(\mathbb{P}_{\text{Rot Rot}_{\mathbb{S}}^\top}^{m-1}) \dot{+} \text{sym} \nabla \mathring{\mathbf{H}}^{m+1}(\Omega) = \mathring{\mathcal{H}}_{\mathbb{S}}^{m+1}(\Omega) + \text{sym} \nabla \mathring{\mathbf{H}}^{m+1}(\Omega), \\
\mathcal{H}_{\mathbb{S}}^{n,n-1}(\text{Rot Rot}^\top, \Omega) &= R(\mathbb{P}_{\text{Rot Rot}_{\mathbb{S}}^\top}^{n-1}) \dot{+} \text{sym} \nabla R(\mathbb{P}_{\text{sym} \nabla}^n) \\
&= R(\mathbb{P}_{\text{Rot Rot}_{\mathbb{S}}^\top}^{n-1}) \dot{+} \text{sym} \nabla \mathbf{H}^{n+1}(\Omega) = \mathcal{H}_{\mathbb{S}}^{n+1}(\Omega) + \text{sym} \nabla \mathbf{H}^{n+1}(\Omega)
\end{aligned}$$

hold.

4.2. General Strong Lipschitz Domains: Regular Decompositions and Potentials. To prove regular potentials and regular decompositions as well as more sophisticated compact embeddings for general bounded strong Lipschitz domains of arbitrary topology we start with a cutting lemma, showing a technical difficulty due to the “second order” nature of the operator Rot Rot^\top .

Lemma 4.8 (Cutting lemma). *Let $\varphi \in \mathring{C}^\infty(\mathbb{R}^3)$ and let $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$.*

(i) *If $\mathbf{N} \in \mathcal{H}_{\mathbb{S}}^n(\text{Div}, \Omega)$, then $\varphi \mathbf{N} \in \mathcal{H}_{\mathbb{S}}^n(\text{Div}, \Omega)$ and*

$$(21) \quad \text{Div}(\varphi \mathbf{N}) = \varphi \text{Div} \mathbf{N} + \mathbf{N} \nabla \varphi.$$

(i') *If $\mathbf{N} \in \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega)$, then $\varphi \mathbf{N} \in \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega)$ and (21) holds.*

(ii) *If $\mathbf{M} \in \mathcal{H}_{\mathbb{S}}^{n,n-1}(\text{Rot Rot}^\top, \Omega)$, then $\varphi \mathbf{M} \in \mathcal{H}_{\mathbb{S}}^{n,n-1}(\text{Rot Rot}^\top, \Omega)$ and*

$$(22) \quad \text{Rot Rot}^\top(\varphi \mathbf{M}) = \varphi \text{Rot Rot}^\top \mathbf{M} + 2 \text{sym}((\text{spn} \nabla \varphi) \text{Rot} \mathbf{M}) + \Psi(\nabla \nabla \varphi, \mathbf{M})$$

holds with an algebraic operator Ψ . In particular, this holds for $\mathbf{M} \in \mathcal{H}_{\mathbb{S}}^n(\text{Rot Rot}^\top, \Omega)$.

(ii') *Let $m \geq 1$ and let $\mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}^{m,m-1}(\text{Rot Rot}^\top, \Omega)$. Then $\varphi \mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}^{m,m-1}(\text{Rot Rot}^\top, \Omega)$ together with (22). In particular, this holds for $\mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^\top, \Omega)$.*

Proof. (i) and (i') follow immediately from row-wise applications of the corresponding results for vector fields. Let us consider (ii). For vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$ and a matrix $\mathbf{Q} \in \mathbb{R}^{3 \times 3}$ it holds $\mathbf{p} \times \mathbf{q} = (\text{spn} \mathbf{p})\mathbf{q}$ and $\mathbf{p} \times \mathbf{Q} = -\mathbf{Q} \text{spn} \mathbf{p}$ with row-wise operation of the exterior product. Moreover, for $\mathbf{P}, \mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \in \mathring{C}^\infty(\mathbb{R}^3)$ we have

$$\text{Rot}(\mathbf{PQ}) = \mathbf{P} \text{Rot} \mathbf{Q} + (\nabla \mathbf{P}) \times \mathbf{Q}$$

with $(\nabla \mathbf{P}) \times \mathbf{Q} := ((\partial_2 \mathbf{P})\mathbf{q}_3 - (\partial_3 \mathbf{P})\mathbf{q}_2, (\partial_3 \mathbf{P})\mathbf{q}_1 - (\partial_1 \mathbf{P})\mathbf{q}_3, (\partial_1 \mathbf{P})\mathbf{q}_2 - (\partial_2 \mathbf{P})\mathbf{q}_1)$. Now, let $\mathbf{M} \in \mathring{C}^\infty(\mathbb{R}^3)$. By the standard row-wise formula for vector fields we compute

$$\text{Rot}(\varphi \mathbf{M}) = \varphi \text{Rot} \mathbf{M} - \mathbf{M} \text{spn} \nabla \varphi, \quad \text{Rot}^\top(\varphi \mathbf{M}) = \varphi \text{Rot}^\top \mathbf{M} + (\text{spn} \nabla \varphi) \mathbf{M}^\top$$

and

$$\begin{aligned}
&\text{Rot Rot}^\top(\varphi \mathbf{M}) \\
&= \varphi \text{Rot Rot}^\top \mathbf{M} - \text{Rot}^\top \mathbf{M} \text{spn} \nabla \varphi + (\text{spn} \nabla \varphi) \text{Rot} \mathbf{M}^\top + (\nabla(\text{spn} \nabla \varphi)) \times \mathbf{M}^\top
\end{aligned}$$

$$= \varphi \operatorname{Rot} \operatorname{Rot}^\top \mathbf{M} + ((\operatorname{spn} \nabla \varphi) \operatorname{Rot} \mathbf{M})^\top + (\operatorname{spn} \nabla \varphi) \operatorname{Rot} \mathbf{M}^\top + \Psi(\nabla \nabla \varphi, \mathbf{M}).$$

If \mathbf{M} is symmetric, we observe

$$\operatorname{Rot} \operatorname{Rot}^\top (\varphi \mathbf{M}) = \varphi \operatorname{Rot} \operatorname{Rot}^\top \mathbf{M} + 2 \operatorname{sym} ((\operatorname{spn} \nabla \varphi) \operatorname{Rot} \mathbf{M}) + \Psi(\nabla \nabla \varphi, \mathbf{M}).$$

Finally, the latter formulas extend to distributions as well. \square

Now we proceed by showing regular decompositions for the elasticity complexes.

Theorem 4.9 (Regular decompositions). *Let $m, n \in \mathbb{N}_0$. The regular decompositions in Corollary 4.6 and Corollary 4.7 hold in the sense that there exist bounded linear operators*

$$Q_{\operatorname{Rot} \operatorname{Rot}_S^\top}^{m, m-1} : \mathring{\mathcal{H}}_S^{m, m-1}(\operatorname{Rot} \operatorname{Rot}^\top, \Omega) \rightarrow \mathring{\mathcal{H}}_S^{m+1}(\Omega), \quad \tilde{Q}_{\operatorname{sym} \nabla}^{m, m-1} : \mathring{\mathcal{H}}_S^{m, m-1}(\operatorname{Rot} \operatorname{Rot}^\top, \Omega) \rightarrow \mathring{\mathbf{H}}^{m+1}(\Omega)$$

for $m \geq 1$, and

$$\begin{aligned} Q_{\operatorname{Rot} \operatorname{Rot}_S^\top}^{n, n-1} : \mathcal{H}_S^{n, n-1}(\operatorname{Rot} \operatorname{Rot}^\top, \Omega) &\rightarrow \mathcal{H}_S^{n+1}(\Omega), & \tilde{Q}_{\operatorname{sym} \nabla}^{n, n-1} : \mathcal{H}_S^{n, n-1}(\operatorname{Rot} \operatorname{Rot}^\top, \Omega) &\rightarrow \mathbf{H}^{n+1}(\Omega), \\ Q_{\operatorname{Div}_S}^m : \mathring{\mathcal{H}}_S^m(\operatorname{Div}, \Omega) &\rightarrow \mathring{\mathcal{H}}_S^{m+1}(\Omega), & \tilde{Q}_{\operatorname{Rot} \operatorname{Rot}_S^\top}^m : \mathring{\mathcal{H}}_S^m(\operatorname{Div}, \Omega) &\rightarrow \mathring{\mathcal{H}}_S^{m+2}(\Omega), \\ Q_{\operatorname{Div}_S}^n : \mathcal{H}_S^n(\operatorname{Div}, \Omega) &\rightarrow \mathcal{H}_S^{n+1}(\Omega), & \tilde{Q}_{\operatorname{Rot} \operatorname{Rot}_S^\top}^n : \mathcal{H}_S^n(\operatorname{Div}, \Omega) &\rightarrow \mathcal{H}_S^{n+2}(\Omega), \end{aligned}$$

such that

$$\begin{aligned} \mathring{\mathcal{H}}_S^{m, m-1}(\operatorname{Rot} \operatorname{Rot}^\top, \Omega) &= \mathring{\mathcal{H}}_S^{m+1}(\Omega) + \operatorname{sym} \nabla \mathring{\mathbf{H}}^{m+1}(\Omega) = R(Q_{\operatorname{Rot} \operatorname{Rot}_S^\top}^{m, m-1}) + \operatorname{sym} \nabla R(\tilde{Q}_{\operatorname{sym} \nabla}^{m, m-1}) \\ &= R(Q_{\operatorname{Rot} \operatorname{Rot}_S^\top}^{m, m-1}) + \mathring{\mathcal{H}}_{S,0}^m(\operatorname{Rot} \operatorname{Rot}^\top, \Omega) \end{aligned}$$

for $m \geq 1$, and

$$\begin{aligned} \mathcal{H}_S^{n, n-1}(\operatorname{Rot} \operatorname{Rot}^\top, \Omega) &= \mathcal{H}_S^{n+1}(\Omega) + \operatorname{sym} \nabla \mathbf{H}^{n+1}(\Omega) = R(Q_{\operatorname{Rot} \operatorname{Rot}_S^\top}^{n, n-1}) + \operatorname{sym} \nabla R(\tilde{Q}_{\operatorname{sym} \nabla}^{n, n-1}) \\ &= R(Q_{\operatorname{Rot} \operatorname{Rot}_S^\top}^{n, n-1}) + \mathcal{H}_{S,0}^n(\operatorname{Rot} \operatorname{Rot}^\top, \Omega), \\ \mathring{\mathcal{H}}_S^m(\operatorname{Div}, \Omega) &= \mathring{\mathcal{H}}_S^{m+1}(\Omega) + \operatorname{Rot} \operatorname{Rot}^\top \mathring{\mathcal{H}}_S^{m+2}(\Omega) = R(Q_{\operatorname{Div}_S}^m) + \operatorname{Rot} \operatorname{Rot}^\top R(\tilde{Q}_{\operatorname{Rot} \operatorname{Rot}_S^\top}^m) \\ &= R(Q_{\operatorname{Div}_S}^m) + \mathring{\mathcal{H}}_{S,0}^m(\operatorname{Div}, \Omega), \\ \mathcal{H}_S^n(\operatorname{Div}, \Omega) &= \mathcal{H}_S^{n+1}(\Omega) + \operatorname{Rot} \operatorname{Rot}^\top \mathcal{H}_S^{n+2}(\Omega) = R(Q_{\operatorname{Div}_S}^n) + \operatorname{Rot} \operatorname{Rot}^\top R(\tilde{Q}_{\operatorname{Rot} \operatorname{Rot}_S^\top}^n) \\ &= R(Q_{\operatorname{Div}_S}^n) + \mathcal{H}_{S,0}^n(\operatorname{Div}, \Omega). \end{aligned}$$

Proof. Let (φ_ℓ, U_ℓ) be the partition of unity together with $\phi_\ell \in \mathring{C}^\infty(U_\ell)$ from Section 2.1.

- Let $\mathbf{N} \in \mathring{\mathcal{H}}_S^m(\operatorname{Div}, \Omega)$. By Lemma 4.8, approximation, and Corollary 4.6 we have

$$\varphi_\ell \mathbf{N} \in \mathring{\mathcal{H}}_S^m(\operatorname{Div}, \Omega_\ell) = \mathring{\mathcal{H}}_S^{m+1}(\Omega_\ell) + \operatorname{Rot} \operatorname{Rot}^\top \mathring{\mathcal{H}}_S^{m+2}(\Omega_\ell).$$

Thus $\varphi_\ell \mathbf{N} = \mathbf{N}_\ell + \operatorname{Rot} \operatorname{Rot}^\top \mathbf{M}_\ell$ with $\mathbf{N}_\ell \in \mathring{\mathcal{H}}_S^{m+1}(\Omega_\ell)$ and $\mathbf{M}_\ell \in \mathring{\mathcal{H}}_S^{m+2}(\Omega_\ell)$. Extending \mathbf{N}_ℓ and \mathbf{M}_ℓ by zero gives $\tilde{\mathbf{N}}_\ell \in \mathring{\mathcal{H}}_S^{m+1}(\Omega)$ and $\tilde{\mathbf{M}}_\ell \in \mathring{\mathcal{H}}_S^{m+2}(\Omega)$ with

$$\mathbf{N} = \sum_{\ell=1}^L \varphi_\ell \mathbf{N} = \sum_{\ell=1}^L \tilde{\mathbf{N}}_\ell + \operatorname{Rot} \operatorname{Rot}^\top \sum_{\ell=1}^L \tilde{\mathbf{M}}_\ell \in \mathring{\mathcal{H}}_S^{m+1}(\Omega) + \operatorname{Rot} \operatorname{Rot}^\top \mathring{\mathcal{H}}_S^{m+2}(\Omega),$$

and all applied operations are continuous, showing the existence of $Q_{\operatorname{Div}_S}^m$ and $\tilde{Q}_{\operatorname{Rot} \operatorname{Rot}_S^\top}^m$.

- Let $\mathbf{N} \in \mathcal{H}_S^n(\operatorname{Div}, \Omega)$. As before we get

$$\varphi_\ell \mathbf{N} \in \mathcal{H}_S^n(\operatorname{Div}, \Omega_\ell) = \mathcal{H}_S^{n+1}(\Omega_\ell) + \operatorname{Rot} \operatorname{Rot}^\top \mathcal{H}_S^{n+2}(\Omega_\ell)$$

and $\varphi_\ell \mathbf{N} = \mathbf{N}_\ell + \operatorname{Rot} \operatorname{Rot}^\top \mathbf{M}_\ell$ with $\mathbf{N}_\ell \in \mathcal{H}_S^{n+1}(\Omega_\ell)$ and $\mathbf{M}_\ell \in \mathcal{H}_S^{n+2}(\Omega_\ell)$. In Ω_ℓ we observe with Lemma 4.8

$$\varphi_\ell \mathbf{N} = \varphi_\ell \phi_\ell \mathbf{N} = \phi_\ell \mathbf{N}_\ell + \phi_\ell \operatorname{Rot} \operatorname{Rot}^\top \mathbf{M}_\ell$$

Theorem 4.11 (Compact embeddings for the elasticity complex). *Let $m \in \mathbb{N}_0$. The embeddings*

$$\begin{aligned} \text{(i)} & \quad \mathring{\mathbf{H}}^{m+1}(\Omega) \hookrightarrow \mathring{\mathbf{H}}^m(\Omega), \\ \text{(ii)} & \quad \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^\top, \Omega) \cap \mathcal{H}_{\mathbb{S}}^m(\text{Div}, \Omega) \hookrightarrow \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega), \\ \text{(iii)} & \quad \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S}}^m(\text{Rot Rot}^\top, \Omega) \hookrightarrow \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega), \\ \text{(iv)} & \quad \mathbf{H}^{m+1}(\Omega) \hookrightarrow \mathbf{H}^m(\Omega) \end{aligned}$$

are compact and all ranges in the elasticity domain complexes are closed.

Proof. (i) and (iv) are just Rellich's selection theorems. With Theorem 4.9 and Rellich's selection theorem we can apply Theorem 2.11 (with $A_1 = \text{Div}_{\mathbb{S}}$) and Corollary 2.12 (with $A_0^* = \text{Div}_{\mathbb{S}}$) and get the compact embedding results (iii) and (ii) for the case $m = 0$. For (ii) and $m \geq 1$, let $(\mathbf{M}_k) \subset \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^\top, \Omega) \cap \mathcal{H}_{\mathbb{S}}^m(\text{Div}, \Omega)$ be bounded in $\mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Rot Rot}^\top, \Omega) \cap \mathcal{H}_{\mathbb{S}}^m(\text{Div}, \Omega)$. By assumption (w.l.o.g.) $(\mathbf{M}_k) \subset \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega)$ is a Cauchy sequence in $\mathring{\mathcal{H}}_{\mathbb{S}}^{m-1}(\Omega)$. Moreover, for all $|\alpha| = m$ we have $\partial^\alpha \mathbf{M}_k \in \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^\top, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega)$. As $\mathbf{M}_k, \text{Rot Rot}^\top \mathbf{M}_k \in \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega)$ and $\mathbf{M}_k \in D(\text{Rot Rot}_{\mathbb{S}}^\top)$ we observe for all $\Phi \in \mathring{\mathcal{C}}^\infty(\mathbb{R}^3)$

$$\begin{aligned} \langle \partial^\alpha \mathbf{M}_k, \text{Rot Rot}^\top \Phi \rangle_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} &= \pm \langle \mathbf{M}_k, \text{Rot Rot}^\top \partial^\alpha \Phi \rangle_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} \\ &= \pm \langle \text{Rot Rot}^\top \mathbf{M}_k, \partial^\alpha \Phi \rangle_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} = \langle \text{Rot Rot}^\top \partial^\alpha \mathbf{M}_k, \Phi \rangle_{\mathcal{L}_{\mathbb{S}}^2(\Omega)}. \end{aligned}$$

Thus $\partial^\alpha \mathbf{M}_k \in \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot Rot}^\top, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega)$ and hence w.l.o.g. $(\partial^\alpha \mathbf{M}_k)$ is a Cauchy sequence in $\mathcal{L}_{\mathbb{S}}^2(\Omega)$ by Theorem 4.11 for $m = 0$. We conclude that (\mathbf{M}_k) is a Cauchy sequence in $\mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega)$. Analogously we prove (iii) for $m \geq 1$. \square

Remark 4.12 (Compact embeddings for the elasticity complex). *Theorem 4.11 may also be proved by a variant of Theorem 2.11 (or Corollary 2.12). For this, let, e.g.,*

$$(\mathbf{M}_k) \subset \mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S}}^m(\text{Rot Rot}^\top, \Omega)$$

be a bounded sequence. In particular, (\mathbf{M}_k) is bounded in $\mathcal{H}_{\mathbb{S}}^{m,m-1}(\text{Rot Rot}^\top, \Omega)$. According to Theorem 4.9 we decompose $\mathbf{M}_k = \widetilde{\mathbf{M}}_k + \text{sym} \nabla \mathbf{v}_k$ with $\widetilde{\mathbf{M}}_k \in \mathcal{H}_{\mathbb{S}}^{m+1}(\Omega)$ and $\mathbf{v}_k \in \mathbf{H}^{m+1}(\Omega)$. By the boundedness of the potentials, $(\widetilde{\mathbf{M}}_k)$ and (\mathbf{v}_k) are bounded in $\mathcal{H}_{\mathbb{S}}^{m+1}(\Omega)$ and $\mathbf{H}^{m+1}(\Omega)$ and hence w.l.o.g. converge in $\mathcal{H}_{\mathbb{S}}^m(\Omega)$ and $\mathbf{H}^m(\Omega)$, respectively. Let α with $|\alpha| \leq m$. Then

$$\partial^\alpha \mathbf{M}_k \in \mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^\top, \Omega).$$

Moreover, as $\mathbf{M}_k \in \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega) \cap D(\text{Div}_{\mathbb{S}})$ and $\text{Div} \mathbf{M}_k \in \mathring{\mathbf{H}}^m(\Omega)$ we observe for all $\phi \in \mathring{\mathcal{C}}^\infty(\mathbb{R}^3)$

$$\begin{aligned} \langle \partial^\alpha \mathbf{M}_k, \text{sym} \nabla \phi \rangle_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} &= \pm \langle \mathbf{M}_k, \text{sym} \nabla \partial^\alpha \Phi \rangle_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} \\ &= \mp \langle \text{Div} \mathbf{M}_k, \partial^\alpha \phi \rangle_{\mathbf{L}^2(\Omega)} = - \langle \text{Div} \partial^\alpha \mathbf{M}_k, \Phi \rangle_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Thus $\partial^\alpha \mathbf{M}_k \in \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\text{Rot Rot}^\top, \Omega)$. With $\mathbf{M}_{k,l} := \mathbf{M}_k - \mathbf{M}_l$, $\widetilde{\mathbf{M}}_{k,l} := \widetilde{\mathbf{M}}_k - \widetilde{\mathbf{M}}_l$, and $\mathbf{v}_{k,l} := \mathbf{v}_k - \mathbf{v}_l$ we get

$$\begin{aligned} |\mathbf{M}_{k,l}|_{\mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega)}^2 &= \langle \mathbf{M}_{k,l}, \widetilde{\mathbf{M}}_{k,l} + \text{sym} \nabla \mathbf{v}_{k,l} \rangle_{\mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega)} = \langle \mathbf{M}_{k,l}, \widetilde{\mathbf{M}}_{k,l} \rangle_{\mathcal{H}_{\mathbb{S}}^m(\Omega)} - \langle \text{Div} \mathbf{M}_{k,l}, \mathbf{v}_{k,l} \rangle_{\mathbf{H}^m(\Omega)} \\ &\leq c(|\widetilde{\mathbf{M}}_{k,l}|_{\mathcal{H}_{\mathbb{S}}^m(\Omega)} + |\mathbf{v}_{k,l}|_{\mathbf{H}^m(\Omega)}) \rightarrow 0, \end{aligned}$$

finishing the proof. Alternatively, we can decompose $\mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega)$ using Theorem 4.9, which is even simpler.

Remark 4.12 shows that we even have the following compact embeddings:

Theorem 4.13 (More compact embeddings for the elasticity complex). *Let $m \in \mathbb{N}_0$. The embeddings*

$$\mathring{\mathcal{H}}_{\mathbb{S}}^m(\text{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S}}^{m,m-1}(\text{Rot Rot}^\top, \Omega) \hookrightarrow \mathring{\mathcal{H}}_{\mathbb{S}}^m(\Omega)$$

are compact. In particular, for $m = 0$

$$\mathring{\mathcal{H}}_{\mathbb{S}}(\text{Div}, \Omega) \cap \mathcal{H}_{\mathbb{S}}^{0,-1}(\text{Rot Rot}^\top, \Omega) \hookrightarrow \mathcal{L}_{\mathbb{S}}^2(\Omega)$$

is compact. Similarly, for $m \geq 1$ also the embeddings

$$\mathring{\mathcal{H}}_S^{m,m-1}(\text{Rot Rot}^\top, \Omega) \cap \mathcal{H}_S^m(\text{Div}, \Omega) \hookrightarrow \mathring{\mathcal{H}}_S^m(\Omega)$$

are compact.

From Corollary 2.13 we immediately obtain the existence of regular potentials also in the case of general strong Lipschitz domains.

Corollary 4.14 (Regular potentials). *Let $m, n \in \mathbb{N}_0$. The regular potential representations in Corollary 4.3 hold in the sense*

$$\begin{aligned} \text{Rot Rot}^\top \mathring{\mathcal{H}}_S^{m,m-1}(\text{Rot Rot}^\top, \Omega) &= \text{Rot Rot}^\top \mathring{\mathcal{H}}_S^{m+1}(\Omega), \quad m \geq 1, \\ \text{Rot Rot}^\top \mathring{\mathcal{H}}_S^m(\text{Rot Rot}^\top, \Omega) &= \text{Rot Rot}^\top \mathring{\mathcal{H}}_S^{m+1,m}(\text{Rot Rot}^\top, \Omega) = \text{Rot Rot}^\top \mathring{\mathcal{H}}_S^{m+2}(\Omega), \quad m \geq 1, \\ \text{Rot Rot}^\top \mathcal{H}_S^{n,n-1}(\text{Rot Rot}^\top, \Omega) &= \text{Rot Rot}^\top \mathcal{H}_S^{n+1}(\Omega), \\ \text{Rot Rot}^\top \mathcal{H}_S^n(\text{Rot Rot}^\top, \Omega) &= \text{Rot Rot}^\top \mathcal{H}_S^{n+1,n}(\text{Rot Rot}^\top, \Omega) = \text{Rot Rot}^\top \mathcal{H}_S^{n+2}(\Omega), \\ \mathring{\mathbf{H}}_{\perp\text{RM}}^m(\Omega) &= \text{Div } \mathring{\mathcal{H}}_S^m(\text{Div}, \Omega) = \text{Div } \mathring{\mathcal{H}}_S^{m+1}(\Omega), \\ \mathbf{H}^n(\Omega) &= \text{Div } \mathcal{H}_S^n(\text{Div}, \Omega) = \text{Div } \mathcal{H}_S^{n+1}(\Omega) \end{aligned}$$

with linear and bounded regular potential operators, cf. Theorem 4.2,

$$\begin{aligned} P_{\text{Rot Rot}_S^\top}^{m,m-1} : \text{Rot Rot}^\top \mathring{\mathcal{H}}_S^{m,m-1}(\text{Rot Rot}^\top, \Omega) &\rightarrow \mathring{\mathcal{H}}_S^{m+1}(\Omega), \quad m \geq 1, \\ P_{\text{Rot Rot}_S^\top}^m : \text{Rot Rot}^\top \mathring{\mathcal{H}}_S^m(\text{Rot Rot}^\top, \Omega) &\rightarrow \mathring{\mathcal{H}}_S^{m+2}(\Omega), \quad m \geq 1, \\ P_{\text{Rot Rot}_S^\top}^{n,n-1} : \text{Rot Rot}^\top \mathcal{H}_S^{n,n-1}(\text{Rot Rot}^\top, \Omega) &\rightarrow \mathcal{H}_S^{n+1}(\Omega), \\ P_{\text{Rot Rot}_S^\top}^n : \text{Rot Rot}^\top \mathcal{H}_S^n(\text{Rot Rot}^\top, \Omega) &\rightarrow \mathcal{H}_S^{n+2}(\Omega), \\ P_{\text{Div}_S}^m : \mathring{\mathbf{H}}_{\perp\text{RM}}^m(\Omega) &\rightarrow \mathring{\mathcal{H}}_S^{m+1}(\Omega), \\ P_{\text{Div}_S}^n : \mathbf{H}^n(\Omega) &\rightarrow \mathcal{H}_S^{n+1}(\Omega). \end{aligned}$$

Lemma 2.6 yields:

Corollary 4.15 (Direct regular decompositions). *Let $m, n \in \mathbb{N}_0$. Some of the regular decompositions in Theorem 4.9 and Remark 4.10 are direct. More precisely,*

$$\begin{aligned} \mathring{\mathcal{H}}_S^{m,m-1}(\text{Rot Rot}^\top, \Omega) &= R(P_{\text{Rot Rot}_S^\top}^{m,m-1}) \dot{+} \mathring{\mathcal{H}}_{S,0}^m(\text{Rot Rot}^\top, \Omega), \quad m \geq 1, \\ \mathring{\mathcal{H}}_S^m(\text{Rot Rot}^\top, \Omega) &= R(P_{\text{Rot Rot}_S^\top}^m) \dot{+} \mathring{\mathcal{H}}_{S,0}^{m+1}(\text{Rot Rot}^\top, \Omega), \quad m \geq 1, \\ \mathcal{H}_S^{n,n-1}(\text{Rot Rot}^\top, \Omega) &= R(P_{\text{Rot Rot}_S^\top}^{n,n-1}) \dot{+} \mathcal{H}_{S,0}^n(\text{Rot Rot}^\top, \Omega), \\ \mathcal{H}_S^n(\text{Rot Rot}^\top, \Omega) &= R(P_{\text{Rot Rot}_S^\top}^n) \dot{+} \mathcal{H}_{S,0}^{n+1}(\text{Rot Rot}^\top, \Omega), \\ \mathring{\mathcal{H}}_S^m(\text{Div}, \Omega) &= R(P_{\text{Div}_S}^m) \dot{+} \mathring{\mathcal{H}}_{S,0}^m(\text{Div}, \Omega), \\ \mathcal{H}_S^n(\text{Div}, \Omega) &= R(P_{\text{Div}_S}^n) \dot{+} \mathcal{H}_{S,0}^{n+1}(\text{Div}, \Omega). \end{aligned}$$

Theorem 4.11 and Theorem 4.13 show:

Theorem 4.16 (Compact elasticity Hilbert complexes). *Let $m, n \in \mathbb{N}_0$. The elasticity domain complexes*

$$\begin{aligned} \{0\} &\xrightarrow{\iota_{\{0\}}} \mathring{\mathbf{H}}^{m+1}(\Omega) \xrightarrow{\text{sym } \nabla} \mathring{\mathcal{H}}_S^m(\text{Rot Rot}^\top, \Omega) \xrightarrow{\text{Rot Rot}_S^\top} \mathring{\mathcal{H}}_S^m(\text{Div}, \Omega) \xrightarrow{\text{Div}_S} \mathring{\mathbf{H}}^m(\Omega) \xrightarrow{\pi_{\text{RM}}} \text{RM}, \\ \{0\} &\xleftarrow{\pi_{\{0\}}} \mathbf{H}^n(\Omega) \xleftarrow{-\text{Div}_S} \mathcal{H}_S^n(\text{Div}, \Omega) \xleftarrow{\text{Rot Rot}_S^\top} \mathcal{H}_S^n(\text{Rot Rot}^\top, \Omega) \xleftarrow{-\text{sym } \nabla} \mathbf{H}^{n+1}(\Omega) \xleftarrow{\iota_{\text{RM}}} \text{RM} \end{aligned}$$

and, for $m \geq 1$

$$\begin{aligned} \{0\} &\xrightarrow{\iota_{\{0\}}} \mathring{\mathbf{H}}^{m+1}(\Omega) \xrightarrow{\text{sym } \nabla} \mathring{\mathcal{H}}_{\mathbb{S}}^{m,m-1}(\text{Rot Rot}^\top, \Omega) \xrightarrow{\text{Rot } \mathring{\text{Rot}}_{\mathbb{S}}^\top} \mathring{\mathcal{H}}_{\mathbb{S}}^{m-1}(\text{Div}, \Omega) \xrightarrow{\text{Div}_{\mathbb{S}}} \mathring{\mathbf{H}}^{m-1}(\Omega) \xrightarrow{\pi_{\mathbf{RM}}} \mathbf{RM}, \\ \{0\} &\xleftarrow{\pi_{\{0\}}} \mathbf{H}^{n-1}(\Omega) \xleftarrow{-\text{Div}_{\mathbb{S}}} \mathcal{H}_{\mathbb{S}}^{n-1}(\text{Div}, \Omega) \xleftarrow{\text{Rot } \mathring{\text{Rot}}_{\mathbb{S}}^\top} \mathcal{H}_{\mathbb{S}}^{n,n-1}(\text{Rot Rot}^\top, \Omega) \xleftarrow{\text{sym } \nabla} \mathring{\mathbf{H}}^{n+1}(\Omega) \xleftarrow{\iota_{\mathbf{RM}}} \mathbf{RM} \end{aligned}$$

are compact Hilbert complexes, all ranges are closed, and the linear and bounded operators from Theorem 4.9 and Corollary 4.14 are associated regular decomposition and potential operators, respectively. If Ω is additionally topologically trivial, then the elasticity complexes are exact.

Based on Theorem 4.11, the FA-ToolBox (Section 2.2 and [20, 21, 22, 23, 25]) immediately provides a long list of implications such as geometric Helmholtz type decompositions, regular decompositions, Friedrichs/Poincaré type estimates, closed ranges, see [27] for a comprehensive list of such results.

Finally, we want to highlight some of these important and useful implications.

Theorem 4.17 (Mini FA-ToolBox for the elasticity complex). *It holds:*

- (i) All ranges of the elasticity complexes are closed.
- (ii) The cohomology groups of generalised Dirichlet and Neumann tensors

$$\mathcal{H}\text{arm}_{\mathbb{S},\text{D}}(\Omega) = N(\text{Rot } \mathring{\text{Rot}}_{\mathbb{S}}^\top) \cap N(\text{Div}_{\mathbb{S}}), \quad \mathcal{H}\text{arm}_{\mathbb{S},\text{N}}(\Omega) = N(\mathring{\text{Div}}_{\mathbb{S}}) \cap N(\text{Rot } \mathring{\text{Rot}}_{\mathbb{S}}^\top)$$

are finite-dimensional.

- (iii) The geometric Helmholtz type decompositions

$$\begin{aligned} \mathcal{L}_{\mathbb{S}}^2(\Omega) &= R(\text{sym } \mathring{\nabla}) \oplus \mathcal{L}_{\mathbb{S}}^2(\Omega) \mathcal{H}\text{arm}_{\mathbb{S},\text{D}}(\Omega) \oplus \mathcal{L}_{\mathbb{S}}^2(\Omega) R(\text{Rot } \mathring{\text{Rot}}_{\mathbb{S}}^\top), \\ \mathcal{L}_{\mathbb{S}}^2(\Omega) &= R(\text{Rot } \mathring{\text{Rot}}_{\mathbb{S}}^\top) \oplus \mathcal{L}_{\mathbb{S}}^2(\Omega) \mathcal{H}\text{arm}_{\mathbb{S},\text{N}}(\Omega) \oplus \mathcal{L}_{\mathbb{S}}^2(\Omega) R(\text{sym } \nabla) \end{aligned}$$

hold.

- (iv) There exist (optimal) $c_0, c_1, c_2 > 0$ such that the Friedrichs/Poincaré type estimates

$$\begin{aligned} \forall \mathbf{v} \in \mathring{\mathbf{H}}^1(\Omega) & \quad |\mathbf{v}|_{\mathbf{L}^2(\Omega)} \leq c_0 |\text{sym } \nabla \mathbf{v}|_{\mathcal{L}_{\mathbb{S}}^2(\Omega)}, \\ \forall \mathbf{N} \in \mathcal{H}_{\mathbb{S}}(\text{Div}, \Omega) \cap R(\text{sym } \mathring{\nabla}) & \quad |\mathbf{N}|_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} \leq c_0 |\text{Div } \mathbf{N}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Rot } \mathring{\text{Rot}}^\top, \Omega) \cap R(\text{Rot } \mathring{\text{Rot}}_{\mathbb{S}}^\top) & \quad |\mathbf{M}|_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} \leq c_1 |\text{Rot } \mathring{\text{Rot}}^\top \mathbf{M}|_{\mathcal{L}_{\mathbb{S}}^2(\Omega)}, \\ \forall \mathbf{M} \in \mathcal{H}_{\mathbb{S}}(\text{Rot } \mathring{\text{Rot}}^\top, \Omega) \cap R(\text{Rot } \mathring{\text{Rot}}_{\mathbb{S}}^\top) & \quad |\mathbf{M}|_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} \leq c_1 |\text{Rot } \mathring{\text{Rot}}^\top \mathbf{M}|_{\mathcal{L}_{\mathbb{S}}^2(\Omega)}, \\ \forall \mathbf{N} \in \mathring{\mathcal{H}}_{\mathbb{S}}(\text{Div}, \Omega) \cap R(\text{sym } \nabla) & \quad |\mathbf{N}|_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} \leq c_2 |\text{Div } \mathbf{N}|_{\mathbf{L}^2(\Omega)}, \\ \forall \mathbf{v} \in \mathbf{H}_{\perp}^1(\Omega) & \quad |\mathbf{v}|_{\mathbf{L}^2(\Omega)} \leq c_2 |\text{sym } \nabla \mathbf{v}|_{\mathcal{L}_{\mathbb{S}}^2(\Omega)} \end{aligned}$$

hold.

Remark 4.18 (Dirichlet and Neumann tensors). *As shown in [24], the dimensions of the Dirichlet and Neumann tensor fields of the elasticity complexes are given by multiples of the corresponding Dirichlet and Neumann vector fields of the de Rham complexes, respectively. More precisely,*

$$\dim \mathcal{H}\text{arm}_{\mathbb{S},\text{D}}(\Omega) = \dim \mathbf{RM} \cdot \dim \mathbf{H}\text{arm}_{\text{D}}(\Omega), \quad \dim \mathcal{H}\text{arm}_{\mathbb{S},\text{N}}(\Omega) = \dim \mathbf{RM} \cdot \dim \mathbf{H}\text{arm}_{\text{N}}(\Omega),$$

where $\dim \mathbf{RM} = 6$, $\dim \mathbf{H}\text{arm}_{\text{D}}(\Omega) + 1$ is the number of connected components of $\mathbb{R}^3 \setminus \overline{\Omega}$ resp. Γ , and $\dim \mathbf{H}\text{arm}_{\text{N}}(\Omega)$ is the number of “handles” of Ω , see [28, 24] for exact definitions. See also [14] for $\dim \mathcal{H}\text{arm}_{\mathbb{S},\text{N}}(\Omega)$.

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APPENDIX A. SOME FORMULAS

Lemma A.1. *For smooth functions, vector fields, and tensor fields we have*

- (i) $\text{Rot}(u \mathbf{I}) = -\text{spn } \nabla u$,
- (ii) $\text{tr Rot } \mathbf{M} = 2 \text{div spn}^{-1} \text{skw } \mathbf{M}$,

- (ii') $\text{tr Rot } \mathbf{M} = 0$, if \mathbf{M} is symmetric,
- (iii) $\text{Div spn } \mathbf{v} = -\text{rot } \mathbf{v}$,
- (iii') $\text{Div skw } \mathbf{M} = -\text{rot } \mathbf{v}$ for $\mathbf{v} = \text{spn}^{-1} \text{skw } \mathbf{M}$,
- (iv) $\text{Rot spn } \mathbf{v} = (\text{div } \mathbf{v}) \mathbf{I} - (\nabla \mathbf{v})^\top$,
- (iv') $\text{Rot skw } \mathbf{M} = (\text{div } \mathbf{v}) \mathbf{I} - (\nabla \mathbf{v})^\top$ for $\mathbf{v} = \text{spn}^{-1} \text{skw } \mathbf{M}$,
- (v) $\text{skw Rot } \mathbf{M} = \text{spn } \mathbf{v}$ with $2\mathbf{v} = \text{Div } \mathbf{M}^\top - \nabla \text{tr } \mathbf{M}$,
- (v') $2 \text{Div sym Rot } \mathbf{M} = -2 \text{Div skw Rot } \mathbf{M} = \text{rot Div } \mathbf{M}^\top$,
- (vi) $\text{skw Rot Rot}^\top \mathbf{M} = \text{Rot Rot}^\top \text{skw } \mathbf{M}$,
- (vi') $\text{sym Rot Rot}^\top \mathbf{M} = \text{Rot Rot}^\top \text{sym } \mathbf{M}$.

These formulas hold for distributions as well.

Proof. For (i)–(v') see [25, Lemma A.1]. We show (vi), (vi'). For \mathbf{M} and $\mathbf{v} = \text{spn}^{-1} \text{skw } \mathbf{M}$ it follows from (iv) and (i) that

$$\text{Rot}(\text{Rot skw } \mathbf{M})^\top = \text{Rot}((\text{div } \mathbf{v}) \mathbf{I}) = -\text{spn } \nabla \text{div } \mathbf{v}.$$

Conversely, (v) and (ii) imply

$$2 \text{skw Rot}(\text{Rot } \mathbf{M})^\top = \text{spn}(\text{Div Rot } \mathbf{M} - \nabla \text{tr Rot } \mathbf{M}) = -2 \text{spn } \nabla \text{div } \mathbf{v},$$

showing (vi). (vi') easily follows by

$$\begin{aligned} \text{sym Rot Rot}^\top \mathbf{M} &= \text{Rot Rot}^\top \mathbf{M} - \text{skw Rot Rot}^\top \mathbf{M} \\ &= \text{Rot Rot}^\top \mathbf{M} - \text{Rot Rot}^\top \text{skw } \mathbf{M} = \text{Rot Rot}^\top \text{sym } \mathbf{M}, \end{aligned}$$

completing the proof. □

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, CAMPUS ESSEN, GERMANY

Email address, Dirk Pauly: dirk.pauly@uni-due.de

JOHANN RADON INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS, AUSTRIAN ACADEMY OF SCIENCES,
ALTENBERGER STRASSE 69, 4040 LINZ, AUSTRIA

Email address, Walter Zulehner: zulehner@numa.uni-linz.ac.at