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# The Elasticity Theory of Dislocations in Real Earth Models and Changes in the Rotation of the Earth

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*Summary*

Steketee's Elasticity Theory of Dislocations is generalized to real Earth models. Taken into account are; (i) self-gravitation, (ii) radial variation of elastic properties, density and gravity, (iii) initial hydrostatic stress, (iv) the presence of the liquid core. Volterra's formula for the displacement field is found to hold in the more general circumstances for slip faults.

The dilemma, previously pointed out by Jeffreys and Vicente, which arises when the Adams and Williamson condition is assumed not to hold everywhere perfectly in the core, is resolved. This result also bears on the theory of Earth tides and tidal loading.

Changes in the inertia tensor are shown to arise only from spheroidal displacement fields of degree zero and two. These fields have virtually no attenuation with distance from the fault. In the one example in which a direct comparison can be made, the present theory gives a factor of 7·5 increase over a mapped half-space theory and a factor of 2·9 increase over the result for a uniform, spherical Earth, in the contribution to secular polar shift and excitation of Chandler wobble. Calculated and observed levels appear now to be in agreement.

## 1. Introduction

Models of the deformation fields caused by earthquake faulting, in which the fault surface is taken to be a displacement discontinuity in an elastic half-space, have been developed to the point where analytical expressions are now available for slip faults of arbitrary dip (Rochester 1956; Steketee 1958; Chinnery 1961; Maruyama 1964; Press 1965; Mansinha & Smylie 1971). The subject has been named the Elasticity Theory of Dislocations by Steketee to distinguish it from the theory of crystal dislocations in physics.

Recently, the theory has been extended by Ben–Menahem, Singh & Solomon (1969) to the case of a uniform, non-self-gravitating, spherical Earth. Interest in the theory for more realistic earth models has sprung from Press' demonstration (Press 1965) that the deformation fields predicted by the theory are very extensive.

The effect of the mass displacements given by the half-space theory on the rotation of the Earth has been calculated by Mansinha & Smylie (1967). Ben-Menahem & Israel (1970) have done the calculation for the displacements in a uniform, non-self-gravitating spherical Earth. For this application the change in the inertia tensor is required. The change in the inertia tensor depends only on those parts of the displacement field which are spherical harmonics of zero and second degree.

In this paper, we extend the general elasticity theory of dislocations to self-gravitating, radially inhomogeneous Earth models with liquid cores, which we call real Earth models (Fig. 1), and compute the inertia tensor changes for a number of earthquakes whose fault parameters have been obtained by other workers. Implications for the rotation of the Earth are discussed.

## 2. Equations of equilibrium

In a real Earth model one must allow for the possibility of an initial state of stress $T_{ij}$. The state of deformation under this initial stress is taken as the reference state. The additional stress $\tau_{ij}$ and the displacement field $u_i$ which it produces are measured from the reference state. They are taken to be related by the generalized Hooke's law for isotropic media,

$$\tau_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_j{}^i + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), \tag{1}$$

where $\lambda$, $\mu$ are the Lamé constants and $\delta_j{}^i$ is the Kronecker delta. The summation convention will be observed throughout.

The stresses given by (1) refer to points in the deformed medium, but in infinitesimal strain theory they may equally well be taken to refer to the co-ordinates the points had before deformation, for the corrections are of second order in the displacements. Because the initial stresses are independent of the displacement field $u_i$ and are likely

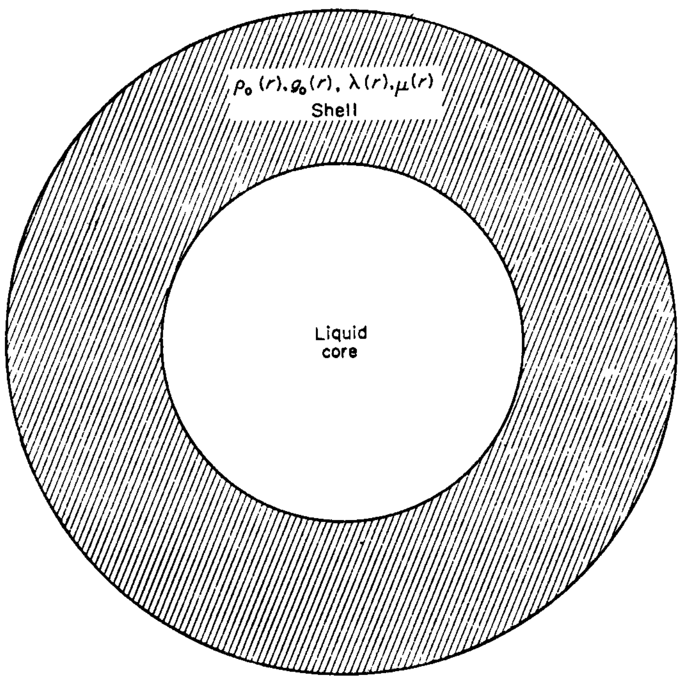


FIG. 1. Real Earth model. Density, gravity and elastic constants are assumed to depend on radius. The inner core is treated as a liquid with the same bulk modulus as in the real Earth. The effects of self-gravitation and initial hydrostatic stress are included.

to be large, the same assumption cannot be made for the $T_{ij}$. The total stress field to be used at the undeformed points is then

$$\sigma_{ij} = T_{ij} - u_k \frac{\partial T_{ij}}{\partial x_k} + \tau_{ij}, \tag{2}$$

correct to first order in the displacements.

To proceed beyond (2) we must specify the $T_{ij}$. We may follow Rayleigh (1906) and take them to be due entirely to hydrostatic equilibrium under self-gravitation. If the density and gravity in the reference state are respectively $\rho_0$ and $g_{0_i}$,

$$T_{ij} = -p_0 \delta_j^i, \quad \frac{\partial p_0}{\partial x_i} = \rho_0 g_{0_i}, \tag{3}$$

the hydrostatic pressure being denoted by $p_0$.

The conditions of equilibrium (3) in the reference state are to be supplemented by the equilibrium equation for the deformed state,

$$\frac{\partial \sigma_{ji}}{\partial x_j} = -\rho g_i - f_i \tag{4}$$

where $\rho$ is the density, $g_i$ is gravity and $f_i$ is the body force/unit volume.

We may write

$$\rho = \rho_0 + \rho_1, \tag{5}$$

$$g_i = g_{0_i} + g_{1i}, \tag{6}$$

with $\rho_1$, $g_{1i}$ representing, respectively, the changes in density and gravity associated with the displacement field $u_i$. The change in density can be described as

$$\rho_1 = -\rho_0 \frac{\partial u_i}{\partial x_i} - u_i \frac{\partial \rho_0}{\partial x_i} = -\frac{\partial}{\partial x_i}(\rho_0 u_i), \tag{7}$$

correct to first order in the displacements. To the same degree of approximation the change in gravity can be written

$$g_{1i} = \frac{\partial V_1}{\partial x_i}, \tag{8}$$

where

$$\frac{\partial^2 V_1}{\partial x_i \partial x_i} = 4\pi G \frac{\partial}{\partial x_i}(\rho_0 u_i). \tag{9}$$

$V_1$ is the decrease in gravitational potential and $G$ is the universal constant of gravitation.

Neglecting products of the small quantities $\rho_1$, $g_{1i}$ ($\rho_1$, $g_{1i}$ are of the same order as the displacements) and combining (2), (3), (4), (5) and (6), we obtain a modified equation of equilibrium (Hoskins 1920) with the form

$$\frac{\partial \tau_{ji}}{\partial x_j} = -f_i - \frac{\partial}{\partial x_i}(\rho_0 u_j g_{0_j}) - g_{0_i} \rho_1 - \rho_0 g_{1i}. \tag{10}$$

Equations (1) and (10) supply the necessary modification of the Navier equation of elasticity. In symbolic notation we have

$$(\lambda+2\mu)\,\nabla(\nabla.\mathbf{u})-\mu\nabla x(\nabla x\mathbf{u})+(\nabla\lambda+\nabla\mu)\,\nabla.\mathbf{u}+\nabla x(\mathbf{u}x\nabla\mu)+\nabla(\mathbf{u}.\nabla\mu)-\mathbf{u}\nabla^2\,\mu$$

$$= -\nabla(\rho_0\,\mathbf{u}.\mathbf{g}_0)-\mathbf{g}_0\,\rho_1-\rho_0\,\mathbf{g}_1-f. \quad (11)$$

Additional terms on the LHS of (11) arise from allowance for inhomogeneity of the elastic properties. The additional terms on the RHS are due, respectively, to displacement in the initial hydrostatic stress field, action of the initial gravity field on the changed density, and action of the changed gravity field on the initial density distribution.

## 3. Generalization of Volterra's formula

We begin with the generalization of the reciprocal theorem of Betti (Sokolnikoff 1956, pp. 390–391).

Let there be two systems of surface and body forces $t_i$, $f_i$ and $t_i'$, $f_i'$ producing, respectively, displacement fields $u_i$ and $u_i'$ which satisfy the equation of equilibrium (10), augmented by the relations (7), (8) and (9). Let these force systems act on material contained in a volume $V$ by a surface $S$ which is such as to allow the divergence theorem of Gauss to be applied. The work done by the unprimed forces acting through the primed displacements is

$$\int_S t_i\,u_i'\,dS+\int_V f_i\,u_i'\,dV = \int_S \tau_{ji}\,\nu_j\,u_i'\,dS+\int_V f_i\,u_i'\,dV, \quad (12)$$

where $\nu_i$ is the unit outward normal vector to $S$.

Transforming the surface integral with the aid of Gauss' theorem and substituting for $f_i$ from (10) and (7), we can rearrange (12) to read

$$\int_V \left[\tau_{ji}\frac{\partial u_i'}{\partial x_j}-g_{0_j}\frac{\partial \rho_0}{\partial x_i}\,u_j\,u_i'-\rho_0\,u_j\,\frac{\partial}{\partial x_j}(u_i'\,g_{0_i})-\rho_0\,u_j'\,\frac{\partial}{\partial x_j}(u_i\,g_{0_i})\right.$$

$$\left.+\frac{\partial}{\partial x_j}(\rho_0\,g_{0_i}\,u_j\,u_i')-\rho_0\,g_{1i}\,u_i'\right] dV. \quad (13)$$

The second of equations (3) may be written symbolically as

$$\nabla p_0 = \rho_0\,\mathbf{g}_0.$$

On taking the curl we have

$$\nabla x(\rho_0\,\mathbf{g}_0) = -\mathbf{g}_0\,x\nabla\rho_0+\rho_0\,\nabla x\mathbf{g}_0$$

$$= \nabla x(\nabla p_0) = 0.$$

But $\mathbf{g}_0$ is a lamellar vector field and hence

$$\mathbf{g}_0\,x\nabla\rho_0 = 0. \quad (14)$$

This implies that $\mathbf{g}_0$ and $\nabla\rho_0$ are parallel. With $r$ as the co-ordinate in the direction opposite to $\mathbf{g}_0$, (13) becomes

$$\int_V \left[\tau_{ji}\frac{\partial u_i'}{\partial x_j} + g_0\frac{\partial\rho_0}{\partial r}u_r u_r' + \rho_0 u_i\frac{\partial}{\partial x_i}(u_r' g_0) + \rho_0 u_i'\frac{\partial}{\partial x_i}(u_r g_0) - \frac{\partial}{\partial x_i}(\rho_0 g_0 u_i u_r') - \rho_0 g_{1i} u_i'\right] dV. \quad (15)$$

On substituting the expression (1) for $\tau_{ij}$ in the first term of the integrand of (15) it becomes

$$\lambda\frac{\partial u_i}{\partial x_i}\frac{\partial u_j'}{\partial x_j} + \mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i'}{\partial x_j} + \mu\begin{Bmatrix}\dfrac{\partial u_j}{\partial x_i}\dfrac{\partial u_i'}{\partial x_j}\\[2ex] \dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_j'}{\partial x_i}\end{Bmatrix},$$

where the quantity in braces is shown in two equivalent forms.

The identity

$$\frac{1}{4\pi G}\left\{\frac{\partial}{\partial x_i}\left[V_1\left(\frac{\partial V_1}{\partial x_i} - 4\pi G\rho_0 u_i'\right)\right] - \frac{\partial V_1}{\partial x_i}\frac{\partial V_1'}{\partial x_i}\right\} = -\rho_0 g_{1i} u_i'$$

is easily established from (8) and (9), providing an alternative form for the last term in the integrand of (15).

We are now able to write, with the help of Gauss' theorem,

$$\begin{aligned}\int_S t_i u_i'\,dS + \int_V f_i u_i'\,dV = \int_V &\left[\lambda\frac{\partial u_i}{\partial x_i}\frac{\partial u_j'}{\partial x_j} + \mu\frac{\partial u_i}{\partial x_j}\frac{\partial u_i'}{\partial x_j} + \mu\begin{Bmatrix}\dfrac{\partial u_j}{\partial x_i}\dfrac{\partial u_i'}{\partial x_j}\\[2ex] \dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_j'}{\partial x_i}\end{Bmatrix}\right.\\ &+ g_0\frac{\partial\rho_0}{\partial r}u_r u_r' + \rho_0 u_i\frac{\partial}{\partial x_i}(u_r' g_0)\\ &\left.+ \rho_0 u_i'\frac{\partial}{\partial x_i}(u_r g_0) - \frac{1}{4\pi G}\frac{\partial V_1}{\partial x_i}\frac{\partial V_1'}{\partial x_i}\right] dV\\ &- \int_S \rho_0 g_0 u_i u_r' \nu_i\,dS\\ &+ \frac{1}{4\pi G}\int_S V_1\left(\frac{\partial V_1'}{\partial x_i} - 4\pi G\rho_0 u_i'\right)\nu_i\,dS.\end{aligned} \quad (16)$$

On the interchange of the primed and unprimed systems of forces and displacements only the surface integrals change of all the terms on the RHS of (16). We therefore find, finally, that

$$\int_S (t_i - \rho_0 g_0 u_r \nu_i) u_i' \, dS + \int_V f_i u_i' \, dV = \int_S (t_i' - \rho_0 g_0 u_r' \nu_i) u_i \, dS + \int_V f_i' u_i \, dV$$

$$+ \frac{1}{4\pi G} \int_S \left[ \left( V_1 \frac{\partial V_1'}{\partial x_i} - 4\pi G \rho_0 u_i' \right) - V_1' \left( \frac{\partial V_1}{\partial x_i} - 4\pi G \rho_0 u_i \right) \right] \nu_i \, dS. \quad (17)$$

This is the required generalization of the reciprocal theorem of Betti. The surface tractions are modified by the contribution from displacement in the initial hydrostatic stress field and a new term arises to take into account the effect of self-gravitation.

We may now use Betti's theorem in the form (17) to obtain the generalization of Volterra's formula (Volterra 1907).

Let the primed system of forces and displacements represent the solution for a point force of unit strength at **r** in the shell (crust plus mantle) and the unprimed system of forces and displacements represent the solution of the dislocation problem. We take the surface of the Earth to be force free and assume that the Earth is in hydrostatic equilibrium in the reference state. The derivation of Volterra's formula then closely follows Maruyama (1964, pp. 357–358). There are some exceptions; the treatment of the surface integrals in (17) over the core–mantle boundary, the Earth's surface and the faces of the dislocation.

Equation (9) can be rewritten

$$\frac{\partial}{\partial x_i} \left( \frac{\partial V_1}{\partial x_i} - 4\pi G \rho_0 u_i \right) = 0.$$

The normal component of

$$\frac{\partial V_1}{\partial x_i} - 4\pi G \rho_0 u_i \quad (18)$$

can then be expected to be continuous across boundaries. Together with continuity of gravitational potential and the switch in sign of total surface traction, this is sufficient to ensure that when (17) is applied to the core, the surface integrals over the core–mantle boundary arising from its application to the shell, will be found to vanish.

On account of continuity of the gravitational potential and the normal component of (18), the integral

$$\frac{1}{4\pi G} \int_S \left[ V_1 \left( \frac{\partial V_1'}{\partial x_i} - 4\pi G \rho_0 u_i' \right) - V_1' \left( \frac{\partial V_1}{\partial x_i} - 4\pi G \rho_0 u_i \right) \right] \nu_i \, dS \quad (19)$$

over the surface of the Earth is equivalent to

$$\frac{1}{4\pi G} \int_S \left[ V_1 \frac{\partial V_1}{\partial x_i} - V_1' \frac{\partial V_1}{\partial x_i} \right] \nu_i \, dS \quad (20)$$

evaluated just outside the Earth's surface. There, the gravitational potential satisfies

Laplace's equation and (20) may be transformed using Gauss' theorem to

$$-\frac{1}{4\pi G}\int_V \left[\frac{\partial}{\partial x_i}\left(V_1\frac{\partial V_1'}{\partial x_i}-V_1'\frac{\partial V_1}{\partial x_i}\right)\right] dV$$

$$=-\frac{1}{4\pi G}\int_V \left[\frac{\partial V_1}{\partial x_i}\frac{\partial V_1'}{\partial x_i}-\frac{\partial V_1'}{\partial x_i}\frac{\partial V_1}{\partial x_i}\right] dV=0,$$

where $V$ is all of space outside the Earth. The vanishing of the tractions $t_i$, $t_i'$ over the Earth's surface leaves only the integral

$$\int_S \rho_0\, g_0(u_r'\, u_i-u_r\, u_i')\, \nu_i\, dS$$

to be evaluated there. It clearly is zero for an Earth which is taken to be in hydrostatic equilibrium in the reference state.

The integrals (19) over the faces of the dislocation cancel each other for slip faults, since for slip faults there is no discontinuity in the displacement component normal to the face of the dislocation. The integrals

$$\int_S (t_i-\rho_0\, g_0\, u_r\, \nu_i)\, u_i'\, dS$$

over the faces of the dislocation cancel each other since the tractions

$$t_i-\rho_0\, g_0\, u_r\, \nu_i$$

required to maintain the dislocation are equal and opposite on the two faces. Since $u_r'$ is continuous across the dislocation, for slip faults the integrals

$$\int_S (t_i'-\rho_0\, g_0\, u_r'\, \nu_i)\, u_i\, dS$$

contribute only

$$-\int_\Sigma \Delta u_i\, t_i'\, dS,$$

where $\Sigma$ is the negative face of the dislocation and where

$$\Delta u_i=u_i^+-u_i^-,$$

$u_i^+$ being the displacement of the positive side of the dislocation and $u_i^-$ being the displacement of the negative side of the dislocation.

If the unit point force at $\mathbf{r}$ is in the $i$-direction, we find that

$$u_i(\mathbf{r})=\int_\Sigma \Delta u_j\, t_j'\, dS.$$

Equation (1) gives

$$t_j'=\tau_{kj}'\, \nu_k=\lambda\frac{\partial u_i'}{\partial x_i}\nu_j+\mu\left(\frac{\partial u_j'}{\partial x_k}+\frac{\partial u_k'}{\partial x_j}\right)\nu_k,$$

where $\nu_k$ is the unit outward normal vector to $\Sigma$. For slip faults $\Delta u_j$ is in the plane of $\Sigma$ and

$$u_i(\mathbf{r}) = \int_\Sigma \mu \Delta u_j \left( \frac{\partial u_j'}{\partial x_k} + \frac{\partial u_k'}{\partial x_j} \right) \nu_k \, dS. \tag{21}$$

This is Volterra's formula which we find to be valid in the more general circumstances at hand.

In (21) $u_k'$ represents the $k$-component of the displacement field due to a point force of unit strength applied at $\mathbf{r}$ in the $i$-direction. To be more specific, let us write it as

$$u_k^i\,(\mathbf{r}'', \mathbf{r})$$

where $\mathbf{r}''$ is the point of observation. Consider a second displacement field

$$u_i^k(\mathbf{r}'', \mathbf{r}').$$

We observe this as the $i$-component of displacement at $\mathbf{r}''$ caused by a unit point force acting in the $k$-direction at $\mathbf{r}'$. Let us now apply (17) to the shell and core with the first point force system as the unprimed system and the second as the primed system. Using the same arguments that were used in the derivation of Volterra's formula, the result is

$$u_k^i(\mathbf{r}', \mathbf{r}) = u_i^k(\mathbf{r}, \mathbf{r}').$$

If $\mathbf{r}'$ is taken as the position on the dislocation, (21) then transforms to

$$u_i(\mathbf{r}) = \int_\Sigma \mu \Delta u_j \left( \frac{\partial u_i^j(\mathbf{r}, \mathbf{r}')}{\partial x_k'} + \frac{\partial u_i^k(\mathbf{r}, \mathbf{r}')}{\partial x_j'} \right) \nu_k \, dS. \tag{22}$$

This form of Volterra's formula allows the displacement field to be interpreted as being due to the superposition of the displacement fields of a continuous distribution of dipole forces over the fault surface.

The fault geometry and force systems are illustrated in Fig. 2. For strike-slip faults we have

$$u_i(\mathbf{r}) = \int_\Sigma \mu \Delta u_1 \left\{ \left( \frac{\partial u_i^1(\mathbf{r}, \mathbf{r}')}{\partial x_2'} + \frac{\partial u_i^2(\mathbf{r}, \mathbf{r}')}{\partial x_1'} \right) \sin\alpha - \left( \frac{\partial u_i^1(\mathbf{r}, \mathbf{r}')}{\partial x_3'} + \frac{\partial u_i^3(\mathbf{r}, \mathbf{r}')}{\partial x_1'} \right) \cos\alpha \right\} dS$$

and for dip-slip faults we have

$$u_i(\mathbf{r}) = \int_\Sigma \mu \Delta u \left\{ \left( \frac{\partial u_i^2(\mathbf{r}, \mathbf{r}')}{\partial x_2'} - \frac{\partial u_i^3(\mathbf{r}, \mathbf{r}')}{\partial x_3'} \right) \sin 2\alpha - \left( \frac{\partial u_i^2(\mathbf{r}, \mathbf{r}')}{\partial x_3'} + \frac{\partial u_i^3(\mathbf{r}, \mathbf{r}')}{\partial x_2'} \right) \cos 2\alpha \right\} dS,$$

where $\Delta u_2 = \Delta u \cos\alpha$, $\Delta u_3 = \Delta u \sin\alpha$.

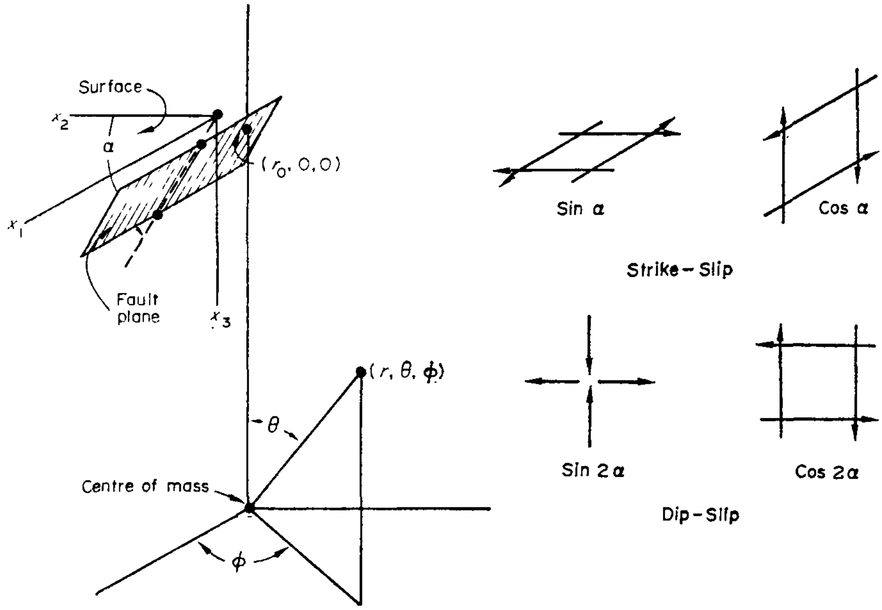


FIG. 2. Fault geometry and focal force systems.

## 4. Displacement fields

A vector field in spherical geometry may be expanded in spheroidal and torsional parts with vector elements

$$\left.\begin{aligned}(\mathbf{S}_n^m)_r &= u_n^m(r)\, P_n^m(\cos\theta)\exp(im\phi),\\ (\mathbf{S}_n^m)_\theta &= v_n^m(r)\frac{dP_n^m(\cos\theta)}{d\theta}\exp(im\phi),\\ (\mathbf{S}_n^m)_\phi &= \frac{im}{\sin\theta}v_n^m(r)\, P_n^m(\cos\theta)\exp(im\phi),\end{aligned}\right\}\qquad(23)$$

and

$$\left.\begin{aligned}(T_n^m)_r &= 0,\\ (\mathbf{T}_n^m)_r &= \frac{im}{\sin\theta}\, t_n^m(r)\, P_n^m(\cos\theta)\exp(\mathrm{im}\phi),\\ (\mathbf{T}_n^m)_\phi &= -t_n^m(r)\frac{dP_n^m(\cos\theta)}{d\theta}\exp(\mathrm{im}\phi),\end{aligned}\right\}\qquad(24)$$

respectively, where $r$, $\theta$, $\phi$ are the usual spherical polar co-ordinates and the $P_n^m(\cos\theta)$ are the associated Legendre functions (Copson 1935). The complete vector field is obtained by summing over $m$ from $-n$ to $n$ and over $n$ from zero to infinity (Smylie 1965). The justification of such an expansion for solenoidal fields has been given by Backus (1958). The lamellar part of the field simply gives an additional contribution to the spheroidal mode as may be shown from potential theory.

The vector elements (23), (24) have the orthogonality properties (Bullard & Gellman 1954)

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{S}_n^m . \mathbf{S}'^k_l \sin\theta \, d\theta \, d\phi$$

$$= (-1)^m \frac{4\pi}{2n+1} [u_n^m(r)\, u'^k_l(r) + n(n+1)\, v_n^m(r)\, v'^k_l(r)]\, \delta_l^n \, \delta_{-k}^m, \quad (25)$$

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{S}_n^m . \mathbf{T}'^k_l \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} \mathbf{T}_n^m . \mathbf{S}'^k_l \sin\theta \, d\theta \, d\phi = 0, \quad (26)$$

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{T}_n^m . \mathbf{T}'^k_l \sin\theta \, d\theta \, d\phi = (-1)^m \frac{4\pi n(n+1)}{2n+1} t_n^m(r)\, t'^k_l(r)\, \delta_l^n \, \delta_{-k}^m, \quad (27)$$

where $\mathbf{S}'^k_l$, $\mathbf{T}'^k_l$ have radial coefficients $u'^k_l(r)$, $v'^k_l(r)$, $t'^k_l(r)$ which are at our disposal. By choosing each of these in turn to be unity, we can use the properties (25), (26), (27) to extract successively the spheroidal radial, spheroidal transverse and torsional parts of a vector field.

We first apply this expansion method to the unit force systems illustrated in Fig. 2. Taking them to be situated at the spherical polar co-ordinates $(r_0, \theta_0, 0)$ and oriented so that $x_1$ is in the direction of increasing $\theta$, $x_2$ in the direction of decreasing $\phi$ and $x_3$ in the direction of decreasing $r$, we have the strike-slip system

$$\left.\begin{aligned} &\frac{1}{r^2 \sin\theta}\Bigg[\frac{\hat{r}}{r}\,\delta(r-r_0)\,\delta'(\theta-\theta_0)\,\delta(\phi)\cos\alpha \\ &+\hat{\theta}\left\{\frac{1}{r\sin\theta}\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta'(\phi)\sin\alpha\right. \\ &\left.-\delta'(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi)\cos\alpha\right\} \\ &+\frac{\hat{\phi}}{r}\,\delta(r-r_0)\,\delta'(\theta-\theta_0)\,\delta(\phi)\sin\alpha\Bigg], \end{aligned}\right\} \quad (28)$$

and the dip-slip system

$$\left.\begin{aligned} &\frac{1}{r^2 \sin\theta}\Bigg[\hat{r}\left\{\frac{1}{r\sin\theta}\,\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta'(\phi)\cos 2\alpha\right. \\ &\left.+\delta'(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi)\sin 2\alpha\right\} \\ &+\hat{\phi}\left\{-\frac{1}{r\sin\theta}\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta'(\phi)\sin 2\alpha\right. \\ &\left.+\delta'(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi)\cos 2\alpha\right\}\Bigg], \end{aligned}\right\} \quad (29)$$

where $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ are the unit vectors in the co-ordinate directions and $\delta$, $\delta'$ are the Dirac function and its derivative. The angle $\alpha$ denotes the dip of the fault surface. After taking the limit as $\theta_0$ approaches zero (this may be done by using suitable recurrence relations among the Legendre functions and the property $P_n^m(1) = \delta_0^m$, the strike-slip system is found to have the non-vanishing radial coefficients

$$\left.\begin{aligned}
u_n^{-1}(r) &= \frac{2n+1}{8\pi r^3} n(n+1)\,\delta(r-r_0)\cos\alpha,\\
u_n^{1}(r) &= -\frac{2n+1}{8\pi r^3}\delta(r-r_0)\cos\alpha,\\
v_n^{-1}(r) &= -\frac{2n+1}{8\pi r^2}\delta'(r-r_0)\cos\alpha,\\
v_n^{1}(r) &= \frac{2n+1}{8\pi r^2\, n(n+1)}\delta'(r-r_0)\cos\alpha,\\
v_n^{-2}(r) &= -\frac{3i(2n+1)}{16\pi r^3}(n-1)(n+2)\,\delta(r-r_0)\sin\alpha,\\
v_n^{2}(r) &= \frac{3i(2n+1)}{16\pi r^3\, n(n+1)}\delta(r-r_0)\sin\alpha,\\
t_n^{0}(r) &= -\frac{2n+1}{8\pi r^3}\delta(r-r_0)\sin\alpha,\\
t_n^{-1}(r) &= -\frac{i(2n+1)}{8\pi r^2}\delta'(r-r_0)\cos\alpha,\\
t_n^{1}(r) &= -\frac{i(2n+1)}{8\pi r^2\, n(n+1)}\delta'(r-r_0)\cos\alpha,\\
t_n^{-2}(r) &= \frac{3(2n+1)}{16\pi r^3}(n-1)(n+2)\,\delta(r-r_0)\sin\alpha,\\
t_n^{2}(r) &= \frac{3(2n+1)}{16\pi r^3\, n(n+1)}\delta(r-r_0)\sin\alpha,
\end{aligned}\right\} \quad (30)$$

while the dip-slip system has the non-vanishing radial coefficients

$$\left.\begin{aligned}
u_n^{0}(r) &= \frac{2n+1}{4\pi r^2}\,\delta'(r-r_0)\sin 2\alpha,\\
u_n^{-1}(r) &= -\frac{i(2n+1)}{8\pi r^3}\,n(n+1)\,\delta(r-r_0)\cos 2\alpha,\\
u_n^{1}(r) &= -\frac{i(2n+1)}{8\pi r^3}\,\delta(r-r_0)\cos 2\alpha,\\
v_n^{-1}(r) &= \frac{i(2n+1)}{8\pi r^2}\,\delta'(r-r_0)\cos 2\alpha,\\
v_n^{1}(r) &= \frac{i(2n+1)}{8\pi r^2\,n(n+1)}\,\delta'(r-r_0)\cos 2\alpha,\\
v_n^{-2}(r) &= -\frac{2n+1}{8\pi r^3}\,(n-1)(n+2)\,\delta(r-r_0)\sin 2\alpha,\\
v_n^{2}(r) &= -\frac{2n+1}{8\pi r^3\,n(n+1)}\,\delta(r-r_0)\sin 2\alpha,\\
t_n^{-1}(r) &= -\frac{2n+1}{8\pi r^2}\,\delta'(r-r_0)\cos 2\alpha,\\
t_n^{1}(r) &= \frac{2n+1}{8\pi r^2\,n(n+1)}\,\delta'(r-r_0)\cos 2\alpha,\\
t_n^{-2}(r) &= -\frac{i(2n+1)}{8\pi r^3}\,(n-1)(n+2)\,\delta(r-r_0)\sin 2\alpha,\\
t_n^{2}(r) &= \frac{i(2n+1)}{8\pi r^3\,n(n+1)}\,\delta(r-r_0)\sin 2\alpha.
\end{aligned}\right\} \quad (31)$$

The expressions (30), (31) are for force systems situated at $r = r_0$ on the polar axis with the particular orientations shown in Fig. 2.

The expansion of the modified Navier equation (11) and the gravitational relations (8) and (9) in spheroidal and torsional parts is well known from free oscillations theory (Alterman, Jarosch & Pekeris 1959). To obtain the static equations for the shell it is only necessary to let the angular frequency go to zero and to introduce a body force.

For $n \geqslant 1$, we then obtain the sixth-order system of linear, non-homogeneous differential equations for the spheroidal part,

$$\left.\begin{aligned}
\frac{dy_1}{dr} &= -\frac{2\lambda}{(\lambda+2\mu)\,r}y_1+\frac{1}{\lambda+2\mu}\,y_2+\frac{n(n+1)\,\lambda}{(\lambda+2\mu)\,r}y_3,\\
\frac{dy_2}{dr} &= \left[-\frac{4\rho_0 g_0}{r}+4\mu\frac{(3\lambda+2\mu)}{(\lambda+2\mu)r^2}\right]y_1-\frac{4\mu}{(\lambda+2\mu)\,r}y_2\\
&\quad+\left[\frac{n(n+1)}{r}\rho_0 g_0-\frac{2n(n+1)}{r^2}\,\mu\,\frac{(3\lambda+2\mu)}{(\lambda+2\mu)}\right]y_3\\
&\quad+\frac{n(n+1)}{r}y_4-\rho_0\,y_6-u_{n_f}^m,\\
\frac{dy_3}{dr} &= -\frac{1}{r}y_1+\frac{1}{r}\,y_3+\frac{1}{\mu}\,y_4,\\
\frac{dy_4}{dr} &= \left[\frac{\rho_0 g_0}{r}-2\mu\frac{(3\lambda+2\mu)}{(\lambda+2\mu)\,r^2}\right]y_1-\frac{\lambda}{(\lambda+2\mu)\,r}y_2\\
&\quad+\frac{2\mu}{(\lambda+2\mu)\,r^2}\,[(2n^2+2n-1)\,\lambda+2(n^2+n-1)\mu]\,y_3\\
&\quad-\frac{3}{r}y_4-\frac{\rho_0}{r}\,y_5-v_{n_f}^m,\\
\frac{dy_5}{dr} &= 4\pi G\rho_0\,y_1+y_6,\\
\frac{dy_6}{dr} &= -4\pi G\rho_0\frac{n(n+1)}{r}y_3+\frac{n(n+1)}{r^2}y_5-\frac{2}{r}\,y_6,
\end{aligned}\right\}\qquad(32)$$

where

$$y_1 = u_n^m,$$

$$y_2 = \lambda\left[\frac{du_n^m}{dr}+\frac{2u_n^m-n(n+1)\,v_n^m}{r}\right]+2\mu\frac{du_n^m}{dr},$$

$$y_3 = v_n^m,$$

$$y_4 = \mu\left[\frac{dv_n^m}{dr}-\frac{v_n^m-u_n^m}{r}\right],$$

$$y_5 = \Phi_n^m,$$

$$y_6 = \frac{d\Phi_n^m}{dr}-4\pi G\rho_0\,u_n^m$$

with

$$V_1 = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Phi_n^m(r)\,P_n^m(\cos\theta)\exp{(im\phi)}.$$

The $y$'s represent, respectively, radial displacement, change in normal stress, transverse displacement, transverse shear stress, decrease in gravitational potential and change in radial gravitational flux density. $u_{n_f}{}^m$, $v_{n_f}{}^m$ are the radial coefficients of the spheroidal part of the body force.

In the liquid core $u_{n_f}{}^m$, $v_{n_f}{}^m$ are zero and the conditions of hydrostatic equilibrium (3), (14) prevail, even in the deformed state. We have

$$\sigma_{ij} = -\mathrm{p}\delta_j{}^i, \; \frac{\partial p}{\partial x_i} = \rho g_i,$$

and

$$\mathbf{g} x \nabla \rho = 0,$$

where $p$ is the hydrostatic pressure. Hence, equipotential surfaces, isobaric surfaces and surfaces of equal density remain parallel after the deformation. Thus, an individual fluid particle is able to move about force free on such surfaces in the absence of viscosity. $y_3$, therefore, becomes indeterminate and we can no longer identify individual fluid particles. On eliminating $y_3$ and taking the limit of zero rigidity, the system (32) is found to degenerate to

$$\left.\begin{aligned} y_1 &= \frac{1}{g_0} y_5, \\ y_2 &= 0, \\ y_4 &= 0, \\ \frac{dy_5}{dr} &= \frac{4\pi G\rho_0}{g_0} y_5 + y_6, \\ \frac{dy_6}{dr} &= \left[-\frac{16\pi G\rho_0}{g_0 r} + \frac{n(n+1)}{r^2}\right] y_5 - \left[\frac{4\pi G\rho_0}{g_0} + \frac{2}{r}\right] y_6. \end{aligned}\right\} \quad (33)$$

The second equation of this set implies zero dilatation, in agreement with the condition that there is zero change in core volume for displacement fields with $n \geqslant 1$. Zero dilatation is implied only if

$$\frac{d}{dr}(ln\rho_0) \neq -\frac{\rho_0 g_0}{\lambda}. \quad (34)$$

Equality in the relation (34) would give the condition of Adams & Williamson (1923). It has been assumed to hold in treatments of the theory of Earth tides and tidal loading (Takeuchi 1950; Longman 1963). As Jeffreys & Vicente (1966) point out, the Adams & Williamson condition requires both chemical homogeneity and an adiabatic temperature distribution. These requirements are unlikely to be met exactly everywhere in the core.

The last two equations of the set (33) are decoupled from the others, reflecting the fact that for $n \geqslant 1$ the only interaction of the shell and core is gravitational. These equations have only one solution regular at $r = 0$. Together with the first of the equations (33), this would appear to determine all of the variables at the core–mantle boundary except $y_3$ in terms of a single free constant. Thus, only two free constants would be available to satisfy, as we shall see, three conditions at the Earth's surface. This led Jeffreys & Vicente (1966) to conclude that a solution may be impossible when the inequality (34) holds. However, we cannot insist on continuity of $y_1$ at the core–mantle boundary. The mantle may be allowed to project into the liquid core provided the prevailing total hydrostatic pressure of the core is assumed at its base. The resulting discontinuity in $y_1$ gives rise to the required third free constant. Fig. 3 shows the conditions which exist at the core–mantle interface. Table 1 compares the present treatment of the core–mantle boundary conditions with the treatments of Longman (1963) and Jeffreys & Vicente (1966).

The state of deformation in the core is then specified by the radial displacement of the gravitational equipotentials given by the first equation of the set (33) and by the second equation of this set which implies that the isobaric surfaces and surfaces of equal density experience the same radial displacement.

Near $r = 0$, the last two equations of the system (33) have the regular solutions

$$y_5 = Cr^n + \dots,$$

$$y_6 = (n-3)\, Cr^{n-1} + \dots,$$

where

$$\frac{4\pi G\rho_0 r}{g_0} = 3 + Dr + \dots$$

**Table 1**

*Comparisons of various treatments of core–mantle boundary conditions for spheroidal deformations of degree $n \geqslant 1$*

| | Longman (1963) | Jeffreys and Vicente (1966) | Present Paper |
|---|---|---|---|
| Dilatation | $\Delta \neq 0$, undetermined in the core but related to $y_1$ there | $\Delta = 0$ is possible, but a solution based on this assumption may be impossible | $\Delta = 0$ |
| Adams and Williamson Condition | Holds for elastic equilibrium under gravity | If assumed to hold a solution is possible | Does not hold |
| $y_1$ | Continuous but undetermined in the core | Continuous | Discontinuous, but total core volume unchanged, known throughout the core |
| $y_2$ | Continuous, not identically zero in the core | Continuous | Discontinuous but equal to zero in the core, value at base of mantle determined by discontinuity in $y_1$ |
| $y_3$ | Discontinuous, undetermined in the core and on its boundary | Discontinuous | Discontinuous, undetermined in the core and on its boundary |
| $y_4$ | Continuous and equal to zero in the core and on its boundary | | |
| $y_5$ | Continuous | Continuous, a given value of $y_5(0)$ may supply the missing condition | Continuous |
| $y_6$ | Continuous | | |

and $C$ and $D$ are constants. With the variable change

$$z_5 = \frac{y_5}{r^{n-1}} = Cr + \ldots,$$

$$z_6 = \frac{y_6}{r^{n-2}} = (n-3)\,Cr + \ldots,$$

we then integrate

$$\frac{dz_5}{dr} = \left[\frac{4\pi G\rho_0}{g_0} - \frac{(n-1)}{r}\right] z_5 + \frac{1}{r} z_6,$$

$$\frac{dz_6}{dr} = \left[-\frac{16\pi G\rho_0}{g_0} + \frac{n(n+1)}{r}\right] z_5 - \left[\frac{4\pi G\rho_0}{g_0} + \frac{n}{r}\right] z_6,$$

subject to

$$z_5 = 0, \ \frac{dz_5}{dr} = C,$$

$$z_6 = 0, \ \frac{dz_6}{dr} = (n-3)\,C,$$

at $r = 0$. When the integration is carried forward to the core–mantle boundary ($r = b$), we obtain the conditions

$$y_1 = A, \qquad y_4 = 0,$$

$$y_2 = \rho_0(b^-)[Ag_0(b) - C\alpha], \qquad y_5 = C\alpha,$$

$$y_3 = B, \qquad y_6 = C\beta,$$

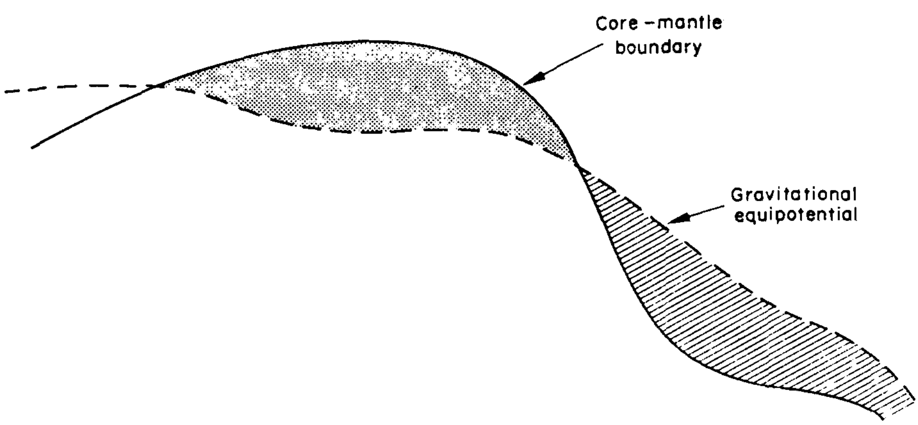


FIG. 3. Conditions at the deformed core–mantle boundary for spheroidal displacements of degree $n \geqslant 1$. The deformation in the liquid core is defined by the gravitational equipotentials, isobaric surfaces and surfaces of equal density, all of which are material surfaces. The mantle may project into the core provided the prevailing hydrostatic stress in the core is assumed at its base.

to be applied at the base of the shell in the integration of the set of equations (32). $\alpha$, $\beta$ are determined by the core integration and the free constants $A$, $B$, $C$ are fixed by the three conditions which prevail at the Earth's surface ($r = d$),

$$y_2 = 0, \qquad y_4 = 0,$$

$$y_5 + \frac{d}{n+1} y_6 = 0.$$

The first two conditions ensure that the Earth's surface is force free, while the third reflects the fact that the change in gravitational potential is of internal origin and becomes harmonic outside the Earth.

For $n = 0$, there can be no transverse spheroidal components and the third and fourth equations of the system (32), obtained by equating them, no longer hold. We are left with

$$\frac{dy_1}{dr} = -\frac{2\lambda}{(\lambda+2\mu)\,r} y_1 + \frac{1}{\lambda+2\mu} y_2,$$

$$\frac{dy_2}{dr} = \left[-\frac{4\rho_0 g_0}{r} + 4\mu \frac{(3\lambda+2\mu)}{(\lambda+2\mu)\,r^2}\right] y_1 - \frac{4\mu}{(\lambda+2\mu)\,r} y_2$$

$$-\rho_0 y_6 - u^0_{0,r}, \tag{35}$$

$$\frac{dy_5}{dr} = 4\pi G\rho_0 y_1 + y_6,$$

$$\frac{dy_6}{dr} = -\frac{2}{r} y_6.$$

In the core for $n = 0$, we have

$$\frac{dy_1}{dr} = -\frac{2}{r} y_1 + \frac{1}{\lambda} y_2,$$

$$\frac{dy_2}{dr} = -\frac{4\rho_0 g_0}{r} y_1 - \rho_0 y_6,$$

$$\frac{dy_5}{dr} = 4\pi G\rho_0 y_1 + y_6,$$

$$\frac{dy_6}{dr} = -\frac{2}{r} y_6.$$

These have the regular solutions

$$y_1 = Ar + \ldots,$$

$$y_2 = A[3\lambda(0) - \tfrac{8}{3}\pi G\rho_0(0)^2 r^2 + \ldots],$$

$$y_5 = B + 2\pi G\rho_0(0)\,Ar^2 + \ldots,$$

$$y_6 = 0,$$

near $r = 0$. We, therefore, require the integration through the core of

$$\frac{dy_1}{dr} = -\frac{2}{r}y_1 + \frac{1}{\lambda}y_2,$$

$$\frac{dy_2}{dr} = -\frac{4\rho_0 g_0}{r}y_1,$$

$$\frac{dy_5}{dr} = 4\pi G\rho_0 y_1,$$

subject to the conditions

$$y_1 = 0, \qquad \frac{dy_1}{dr} = A,$$

$$y_2 = 3A\lambda(0), \qquad \frac{dy_2}{dr} = 0,$$

$$y_5 = B, \qquad \frac{dy_5}{dr} = 0,$$

at $r = 0$. This integration leads to the conditions

$$y_1 = A\alpha, \qquad y_5 = B\gamma,$$

$$y_2 = A\beta, \qquad y_6 = 0,$$

to be applied at the base of the shell in the integration of the set of equations (35). $\alpha$, $\beta$, $\gamma$ are determined by the core integration and the free constants $A$, $B$ are fixed by the conditions at the surface,

$$y_2 = 0, \qquad y_5 = 0.$$

For $n = 0$, $y_6$ is seen to vanish throughout the Earth and the system (35) becomes

$$\frac{dy_1}{dr} = -\frac{2\lambda}{(\lambda+2\mu)r}y_1 + \frac{1}{\lambda+2\mu}y_2,$$

$$\frac{dy_2}{dr} = \left[-\frac{4\rho_0 g_0}{r} + 4\mu\frac{(3\lambda+2\mu)}{(\lambda+2\mu)r^2}\right]y_1 - \frac{4\mu}{(\lambda+2\mu)r}y_2 - u_{0_r}{}^0,$$

$$\frac{dy_5}{dr} = 4\pi G\rho_0 y_1.$$

For $n \geqslant 1$, the torsional part of the displacement field in the shell is described by

$$\frac{dy_1}{dr} = \frac{1}{r}y_1 + \frac{1}{\mu}y_2,$$

$$\frac{dy_2}{dr} = \frac{\mu}{r^2}[n(n+1)-2]y_1 - \frac{3}{r}y_2 - t_{n_r}{}^m, \tag{36}$$

where

$$y_1 = t_n^m,$$

$$y_2 = \mu\left(\frac{dt_n^m}{dr} - \frac{1}{r}t_n^m\right).$$

The $y$'s now represent transverse displacement and transverse shear stress, respectively. There is no torsional field in the core, and for $n = 0$ it vanishes in the shell as well.

The conditions to be applied in the integration of (36) through the shell are

$$y_2 = 0, \text{ at } r = b,$$

and

$$y_2 = 0, \text{ at } r = d.$$

Calculation of the displacement fields is now reduced to the integration of systems of linear differential equations with singular non-homogeneities of the forms given by (30) and (31). In this situation, the propagator matrix formalism can be usefully applied (Gilbert & Backus 1966).

The propagator matrix $P(r, \rho)$ is the solution of the homogeneous system of equations

$$\frac{d}{dr}P(r, \rho) = A(r)\, P(r, \rho) \tag{37}$$

with the initial condition

$$P(\rho, \rho) = I,$$

where $I$ is the unit matrix.

It is easily verified that the solution of the system of non-homogeneous equations

$$\frac{d}{dr}\mathbf{y}(r) = A(r)\,\mathbf{y}(r) + \mathbf{g}(r)$$

is then given by

$$\mathbf{y}(r) = \int_b^r P(r, \rho)\,\mathbf{g}(\rho)\,d\rho + P(r, b)\,\mathbf{y}(b).$$

If $\mathbf{g}(r) = \mathbf{G}\delta(r - r_0)$, where $\mathbf{G}$ is a constant vector,

$$\mathbf{y}(r) = \begin{cases} P(r, r_0)\,\mathbf{G} + P(r, b)\,\mathbf{y}(b), & r > r_0 \\ P(r, b)\,\mathbf{y}(b), & r < r_0. \end{cases} \tag{38}$$

If $\mathbf{g}(r) = \mathbf{G}\delta'(r - r_0)$,

$$\mathbf{y}(r) = \begin{cases} -\dfrac{d}{dr_0}P(r, r_0)\,\mathbf{G} + P(r, b)\,\mathbf{y}(b), & r > r_0 \\ P(r, b)\,\mathbf{y}(b), & r < r_0. \end{cases} \tag{39}$$

Integration from $r_0$ to $r$ and then back to $r_0$ gives

$$P(r, r_0)\,P(r_0, r) = I.$$

Differentiating and using (37) we get

$$\frac{d}{dr_0} P(r, r_0) P(r_0, r) = -P(r, r_0) A(r_0) P(r_0, r). \tag{40}$$

Right multiplication of (40) by $P(r, r_0)$ allows us to write (39) as

$$\mathbf{y}(r) = \begin{cases} P(r, r_0) A(r_0) \mathbf{G} + P(r, b) \mathbf{y}(b), & r > r_0 \\ P(r, b) \mathbf{y}(b), & r < r_0. \end{cases} \tag{41}$$

In calculations involving (38), (41), it is convenient to use the propagator matrix property

$$P(r, c) = P(r, r_0) P(r_0, c),$$

which on right multiplication by $P^{-1}(r_0, c)$ becomes $P(r, r_0) = P(r, c) P^{-1}(r_0, c)$, where $c$ may be chosen arbitrarily.

Summation of the solutions provided by (38), (41) over all $n$, $m$ yields the displacement fields for the slip fault point force systems (28), (29). By (22) these are

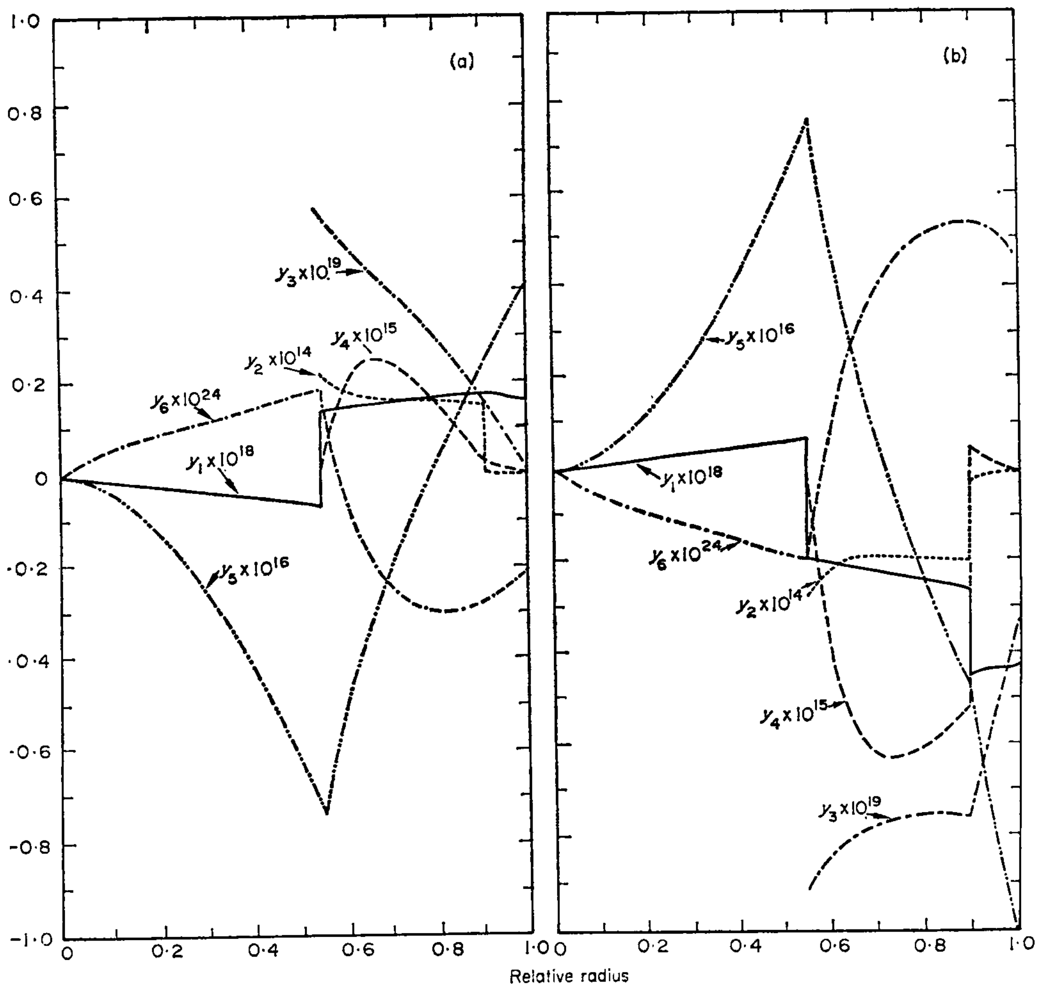


Fig. 4 (*a*) and (*b*)

then to be integrated over the fault surface to give the complete displacement fields. In carrying out the fault integration, changes in angular co-ordinates can be handled by co-ordinate rotations while changes in radius can be handled by operations with the propagator matrix.

The four spheroidal mode solutions of degree two, used in calculating the excitation of wobble and secular polar shift (see Section 5), are shown in Fig. 4(a), (b), (c) and (d). The integrations were performed using the standard fourth order Runge–Kutta method and cubic interpolation of the Earth model $B_1$ of Bullen & Haddon (1967).

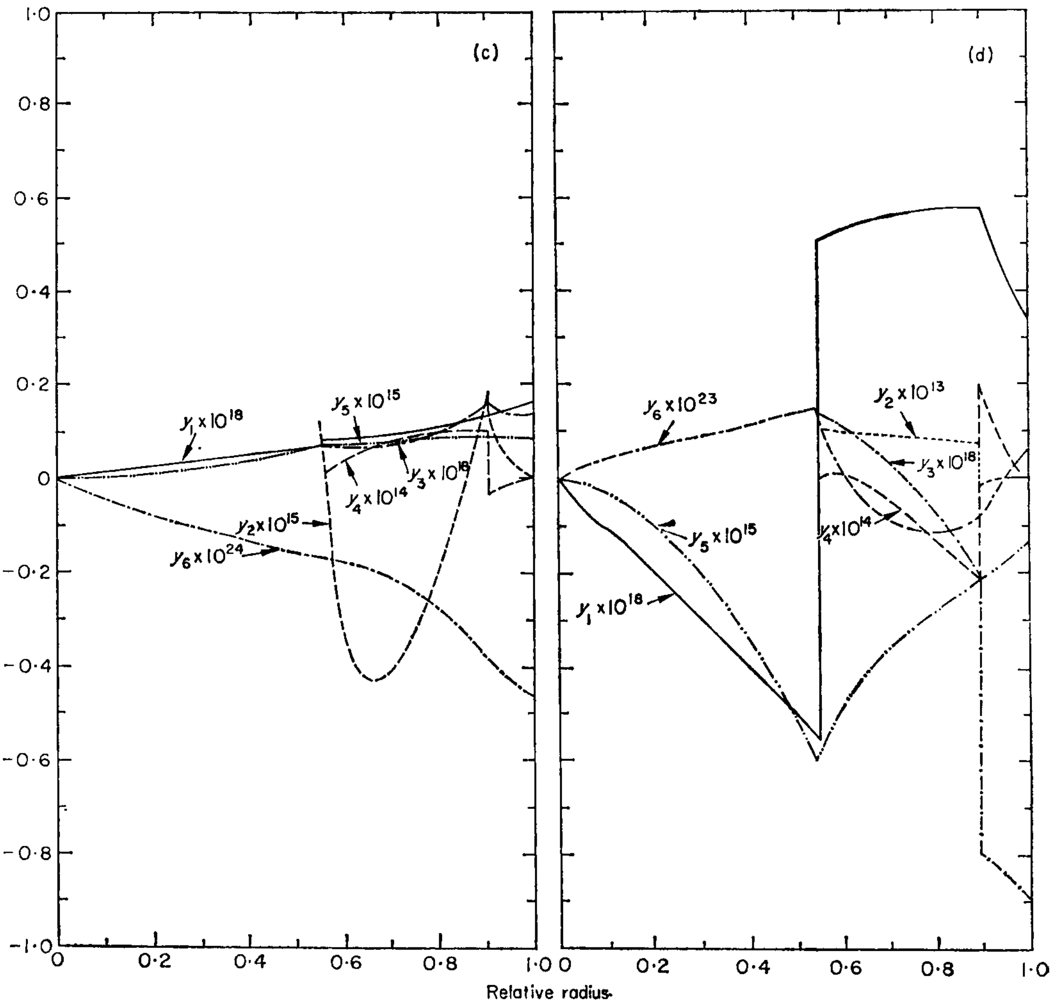


FIG. 4 (*a*), (*b*), (*c*) and (*d*). Spheroidal deformation fields of degree two which determine secular polar shift and excited Chandler wobble. They represent solutions for sources

$$u_{2_f} = \frac{5}{8\pi r^3}\,\mu\delta(r-r_0),\quad u_{2_f} = \frac{5}{8\pi r^2}\,\mu\delta'(r-r_0),$$

$$v_{2_f} = \frac{5}{8\pi r^3}\,\mu\delta(r-r_0) \text{ and } v_{2_f} = \frac{5}{8\pi r^2}\,\mu\delta'(r-r_0),$$

respectively. Source depths are at 0·1 d.

The step size used in the core was 0·05 d. In the shell, two step sizes were used, 0·01 d for the complimentary function integration and 0·0004 d for the particular integral, the latter step size allowing accurate location of the source functions. The core–mantle boundary was taken to be at $b = 0{\cdot}55$ d for convenience in carrying out the integrations.

## 5. Change in the inertia tensor

The effect the displacement fields described in the previous section have on the rotation of the Earth can be evaluated from the changes in the components of the inertia tensor (Mansinha & Smylie 1967). To first order in the small quantities $u_r/r$, $u_\theta/r$, $u_\phi/(r \sin\theta)$, they are

$$\begin{aligned}
\Delta I_{11} &= 2\int r[u_r(1-\sin^2\theta\cos^2\phi)-u_\theta\cos\theta\sin\theta\cos^2\phi+u_\phi\sin\theta\cos\phi\sin\phi]\,dm,\\
\Delta I_{22} &= 2\int r[u_r(1-\sin^2\theta\sin^2\phi)-u_\theta\cos\theta\sin\theta\sin^2\phi-u_\phi\sin\theta\cos\phi\sin\phi]\,dm,\\
\Delta I_{33} &= 2\int r[u_r\sin^2\theta+u_\theta\cos\theta\sin\theta]\,dm, \qquad (42)\\
\Delta I_{12} &= -\int r[2u_r\sin^2\theta\cos\phi\sin\phi+2u_\theta\cos\theta\sin\theta\cos\phi\sin\phi\\
&\qquad +u_\phi\sin\theta(\cos^2\phi-\sin^2\phi)]\,dm,\\
\Delta I_{13} &= -\int r[2u_r\cos\theta\sin\theta\cos\phi+u_\theta(\cos^2\theta-\sin^2\theta)\cos\phi-u_\phi\cos\theta\sin\phi]\,dm,\\
\Delta I_{23} &= -\int r[2u_r\cos\theta\sin\theta\sin\phi+u_\theta(\cos^2\theta-\sin^2\theta)\sin\phi+u_\phi\cos\theta\cos\phi]\,dm,
\end{aligned}$$

where the integrations are to be carried over all mass elements $dm$ of the Earth.

The integrands in (42) can all be cast in the form of scalar products of the displacement field with particular spheroidal vectors of degree zero and two. For example, we may write

$$\Delta I_{11} = 2\int r\mathbf{u}.\mathbf{S}'\,dm$$

where S′ is the spheroidal vector with radial coefficients

$$\begin{aligned}
&u_0^{0\prime} = \tfrac{2}{3}, \\
&u_2^{0\prime} = \tfrac{1}{3} \qquad && v_2^{0\prime} = \tfrac{1}{6},\\
&u_2^{2\prime} = -\tfrac{1}{12}, && v_2^{2\prime} = -\tfrac{1}{24},\\
&u_2^{-2\prime} = -2, && v_2^{-2\prime} = -1.
\end{aligned}$$

The orthogonality properties (25), (26) then allow the expressions (42) to be reduced to

$$\begin{aligned}
\Delta I_{11} &= 4\pi\int_0^d r^3\rho_0(r)[\tfrac{4}{3}u_0{}^0(r)+\tfrac{2}{15}u_2{}^0(r)-\tfrac{4}{5}u_2{}^2(r)-\tfrac{1}{30}u_2{}^{-2}(r)\\
&\qquad +\tfrac{2}{5}v_2{}^0(r)-\tfrac{12}{5}v_2{}^2(r)-\tfrac{1}{10}v_2{}^{-2}(r)]\,dr,\\
\Delta I_{22} &= 4\pi\int_0^d r^3\rho_0(r)[\tfrac{4}{3}u_0{}^0(r)+\tfrac{2}{15}u_2{}^0(r)+\tfrac{4}{5}u_2{}^2(r)+\tfrac{1}{30}u_2{}^{-2}(r)\\
&\qquad +\tfrac{2}{5}v_2{}^0(r)+\tfrac{12}{5}v_2{}^2(r)+\tfrac{1}{10}v_2{}^{-2}(r)]\,dr, \qquad (43)
\end{aligned}$$

$$\Delta I_{33} = 4\pi \int_0^d r^3 \rho_0(r)[\tfrac{4}{3}u_0{}^0(r) - \tfrac{4}{15}u_2{}^0(r) - \tfrac{4}{5}v_2{}^0(r)]\,dr,$$

$$\Delta I_{12} = 4\pi i \int_0^d r^3 \rho_0(r)[-\tfrac{4}{5}u_2{}^2(r) + \tfrac{1}{30}u_2{}^{-2}(r) - \tfrac{12}{5}v_2{}^2(r) + \tfrac{1}{10}v_2{}^{-2}(r)]\,dr,$$

$$\Delta I_{13} = 4\pi \int_0^d r^3 \rho_0(r)[\tfrac{2}{5}u_2{}^1(r) - \tfrac{1}{15}u_2{}^{-1}(r) + \tfrac{6}{5}v_2{}^1(r) - \tfrac{1}{5}v_2{}^{-1}(r)]\,dr,$$

$$\Delta I_{23} = 4\pi i \int_0^d r^3 \rho_0(r)[\tfrac{2}{5}u_2{}^1(r) + \tfrac{1}{15}u_2{}^{-1}(r) + \tfrac{6}{5}v_2{}^1(r) + \tfrac{1}{5}v_2{}^{-1}(r)]\,dr.$$

At the core–mantle boundary there is a discontinuity in the radial displacement (see Fig. 3). In the core, the radial displacement is defined by the distortion of the gravitational equipotentials, but the deformed core–mantle boundary does not coincide with an equipotential surface. There is, therefore, a redistribution of superficial core material (but no net change in core volume) necessary to fit the deformed core into the deformed shell. This redistributed core material contributes

$$\begin{aligned}
\Delta J_{11} &= 2\pi b^4 \rho_0(b^-)[\tfrac{4}{3}\Delta u_0{}^0 + \tfrac{2}{15}\Delta u_2{}^0 - \tfrac{4}{5}\Delta u_2{}^2 - \tfrac{1}{30}\Delta u_2{}^{-2}],\\
\Delta J_{22} &= 2\pi b^4 \rho_0(b^-)[\tfrac{4}{3}\Delta u_0{}^0 + \tfrac{2}{15}\Delta u_2{}^0 + \tfrac{4}{5}\Delta u_2{}^2 + \tfrac{1}{30}\Delta u_2{}^{-2}],\\
\Delta J_{33} &= 2\pi b^4 \rho_0(b^-)[\tfrac{4}{3}\Delta u_0{}^0 - \tfrac{4}{15}\Delta u_2{}^0],\\
\Delta J_{12} &= 2\pi i b^4 \rho_0(b^-)[-\tfrac{4}{5}\Delta u_2{}^2 + \tfrac{1}{30}\Delta u_2{}^{-2}],\\
\Delta J_{13} &= 2\pi b^4 \rho_0(b^-)[\tfrac{2}{5}\Delta u_2{}^1 - \tfrac{1}{15}\Delta u_2{}^{-1}],\\
\Delta J_{23} &= 2\pi i b^4 \rho_0(b^-)[\tfrac{2}{5}\Delta u_2{}^1 + \tfrac{1}{15}\Delta u_2{}^{-1}],
\end{aligned} \qquad (44)$$

to the changes in the components of the inertia tensor, where $\Delta u_n{}^m$ is the excess of the radial displacement of the base of the shell over that of the gravitational equipotential coincident with the core–mantle boundary in the undeformed state.

An examination of the expressions (43), (44) shows that the spheroidal mode of degree zero contributes only an equal amount to each of the diagonal components of the inertia tensor. This contribution will, therefore, be the same in all centre of mass coordinate systems. Since the excitation of wobble and secular polar shift depend on two of the off-diagonal components of the inertia tensor, we can ignore the zero degree spheroidal mode in these calculations. We are concerned only with the four fundamental spheroidal modes of degree two shown in Fig. 4.

The integrations in (43) were carried out using the trapezoidal rule in the core and for the particular integral in the shell. Simpson's $\tfrac{3}{8}$ rule was used for the complementary functions in the shell.

The scale of the displacement fields in Fig. 4 is evidently large compared to possible fault geometries. Finite faults have therefore been approximated by point sources. This approximation would appear to be justified by the work of Ben-Menahem & Israel (1970) on uniform, non-self-gravitating, spherical Earth models where the finite and point source solutions are found to differ only slightly except in the case of very deep dip-slip faults.

Results for several earthquake fault models and comparisons with previous calculations are summarized in Table 2.

**Table 2**

*Effect of major earthquakes on the motion of the rotation axis*

| Earthquake | | Alaska March 28 1964 | Alaska March 28 1964 | Alaska March 28 1964 | Alaska March 28 1964 | Alaska March 28 1964 | Chile May 22 1960 |
|---|---|---|---|---|---|---|---|
| Investigator | | Press (1965) | Hastie & Savage (1969) | Hastie & Savage (1969) | Hastie & Savage (1969) | Hastie & Savage (1969) | Plafker & Savage (1970) |
| Epicentre location | Locolatitude (deg) | 29·0 | 29·0 | 29·0 | 29·0 | 29·0 | 131·5 |
| | Longitude (°E) | 213·0 | 213·0 | 213·0 | 213·0 | 213·0 | 287·0 |
| Fault azimuth (deg) | | 222·0 | 222·0 | 222·0 | 222·0 | 222·0 | 10·0 |
| Fault width (km) | | 184 | 128 | 128 | 128 | 128 | 60 |
| Focal length (km) | | 800 | 600 | 600 | 600 | 600 | 1000 |
| Focal depth (km)* | | 108 | 18 | 637 | 18 | 637 | 18 |
| Fault dip (deg) | | 90·0 | 8·0 | 8·0 | 8·0 | 8·0 | 35·5 |
| Strike-slip (m) | | — | 14·0 | 14·0 | — | — | — |
| Dip-slip (m) | | 9·0 | — | — | −20·0 | −20·0 | −19·5 |
| Polar shift (0″·10) | | 1·64† | 0·59 | 4·45 | 1·15 | 7·89 | 1·03 |
| Longitude of polar shift (°E) | | 94 | 341 | 340 | 79 | 84 | 269 |

* Focal depths were adjusted to nearest 0·0004 d step in numerical integration.

† Previous result using mapped half-space theory by Mansinha & Smylie (1967) was 0·21. Calculation for uniform, non-self-gravitating Earth by Ben-Menahem & Israel (1970) for finite fault gave 0·56.

## 6. Discussion

Generalization of the Elasticity Theory of Dislocations to real Earth models has shown Volterra's formula for the displacements to hold for slip faults. The visualization of the dislocation as giving rise to a displacement field, which is the superposition of the displacement fields due to a system of point forces on the fault surface, is retained.

The conditions prevailing at the core–mantle boundary are shown not to require the assumption of the Adams & Williamson condition in the core. We believe this is the first time they have been treated correctly in a static problem. The theory of Earth tides and tidal loading are also affected.

Secular polar shift and excited Chandler wobble depend on the spheroidal displacements of second degree. These fields have virtually no fall off with distance from the fault. Observations on the position of the Earth's rotation axis should reflect a global integration of the strain field.

There is a very strong increase of the effect on the Earth's rotation with focal depth. As noted in Table 2 there is an increase of nearly an order of magnitude in going from shallow to deep events with the same fault parameters.

There now appears to be no difficulty in accounting theoretically for both the secular polar shift and Chandler wobble excitation as being due to earthquakes. Improved pole path measurements should provide a better understanding of earthquake mechanism.

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