The Electromagnetic Basis for Nondestructive Testing of Cylindrical Conductors

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Abstract—**Using an idealized model, we deduce the impedance per unit length of long solenoid of many turns that contains a cylindrical sample. The sample with a specified conductivity and magnetic permeability need not be centrally located within the solenoid provided all transverse dimensions are small compared with the free-space wavelength. The derivation is relatively straightforward and it provides a justification for earlier use of the impedance formula. The dual problem, where the solenoid is replaced by a toroidal coil is also considered. It is shown that both excitation methods have merit in nondestructive testing procedures.**

Introduction

A COMMON METHOD [1] of nondestructive testing (NDT) of metal rods and tubes is to induce eddy currents by means of an encircling solenoid carrying an alternating current. The impedance of the solenoid is related to the cross-sectional area and the electrical properties of the sample. A formula for this impedance was obtained by Förster and Stambke [2] on the assumption that end effects could be ignored. Also, they assumed that the cylindrical sample was centrally located within the solenoid. The same derivation was essentially repeated by Hochschild [3] and Libby [1].

A feature of the Förster–Stambke derivation is that the effect of the air gap is introduced in a somewhat heuristic fashion wherein the field in this concentric region is assumed to be the same as the one for the empty solenoid. We feel it is worthwhile to provide a more general derivation of the impedance formula. We also show it applies to the case of a nonconcentric air gap. Finally, we mention the relevance of the current analysis to the dual problem where the cylindrical sample is excited by a toroidal coil.

Formulation

To simplify the discussion, we consider first the concentric air gap model with a homogeneous cylindrical sample of radius a with conductivity σ and magnetic permeability μ. The situation is indicated in Fig. 1 where the solenoid of radius b encloses the sample, both of which are assumed to be infinite in length. Our objective is to find an expression for the impedance of the solenoid per unit length since this is the basis of the NDT eddy current methods that are commonly used.

In terms of cylindrical coordinates (ρ, ϕ, z), the only component of the magnetic field is H_z since the exciting current in the solenoid is uniform in both the axial and in the azimuthal direction. Within the sample, H_z satisfies the Helmholtz equation

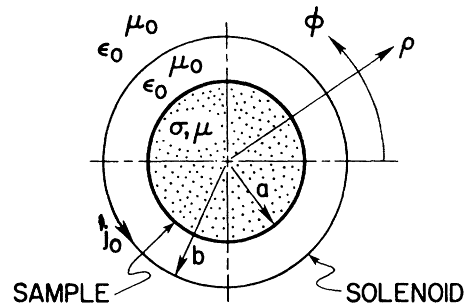

Fig. 1. Cross-sectional view of cylindrical sample located centrally within a solenoid of many turns.

$$(\nabla^2 - \gamma^2)H_z = 0 \tag{1}$$

where $\gamma^2 = i\sigma\mu\omega$ and where we have adopted a time factor exp $(i\omega t)$. Here ω is the angular frequency that is sufficiently low that displacement currents in the sample can be neglected. If not, we merely replace σ by $\sigma + i\varepsilon\omega$ where ε is the permittivity. Also, it goes without saying that the field amplitude is sufficiently small that nonlinear effects can be ignored.

Solution for Concentric Sample

For the highly idealized situation described, we can immediately write [4]

$$H_z = AI_0(\gamma\rho) \tag{2}$$

for $\rho < a$ where I_0 is a modified Bessel function of argument $\gamma\rho$ and where A is a constant. From Maxwell's equations the azimuthal component of the electric field is

$$E_\phi = -(1/\sigma)\,\partial H_z/\partial\rho = -A(\gamma/\sigma)I_1(\gamma\rho) \tag{3}$$

also for $\rho < a$. Now we can immediately form an expression for the "impedance" Z_c of the cylinder:

$$Z_c = [-E_\phi/H_z]_{\rho=a} = \eta I_1(\gamma a)/I_0(\gamma a) \tag{4}$$

where $\eta = \gamma/\sigma = (i\mu\omega/\sigma)^{1/2}$ is the intrinsic or wave impedance of the sample material.

Now, for the air gap region $a < \rho < b$, we write corresponding field expressions

$$H_{0z} = BI_0(\gamma_0\rho) + CK_0(\gamma_0\rho) \tag{5}$$

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and

$$E_{0\phi} = -B\eta_0 I_1(\gamma_0\rho) + C\eta_0 K_1(\gamma_0\rho) \tag{6}$$

where B and C are constants and where $\eta_0 = \gamma_0/(i\varepsilon_0\omega) = (\mu_0/\varepsilon_0)^{1/2} \cong 120\pi$ in terms of the permittivity ε_0 and permeability μ_0 of the air region. Here and in the above, the Bessel function identities $\partial I_0(x)/\partial x = I_1(x)$ and $\partial K_0(x)/\partial x = -K_1(x)$ have been employed.

Compatible with the requirement that tangential fields must be continuous at $\rho = a$ we can write

$$[E_{0\phi} + Z_c H_{0z}]_{\rho=a} = 0. \tag{7}$$

This immediately tells us that

$$\frac{C}{B} = \frac{\eta_0 I_1(\gamma_0 a) - Z_c I_0(\gamma_0 a)}{\eta_0 K_1(\gamma_0 a) + Z_c K_0(\gamma_0 a)}. \tag{8}$$

In the external region $\rho > b$, the field expressions must clearly have the form

$$H_{0z} = DK_0(\gamma_0\rho) \tag{9}$$

$$E_{0\phi} = D\eta_0 K_1(\gamma_0\rho) \tag{10}$$

where D is another constant.

Now the solenoid current is idealized as a continuous current distribution j_0 A/m in the azimuthal direction defined such that

$$\lim_{\Delta\to 0}\begin{cases} H_{0z}(\rho = b+\Delta) - H_{0z}(\rho = b-\Delta) = -j_0 & (11)\\ E_{0\phi}(\rho = b+\Delta) - E_{0\phi}(\rho = b-\Delta) = 0 & (12)\end{cases}$$

Application of these conditions immediately leads to

$$D = C - I_1(\gamma_0 b)B/K_1(\gamma_0 b) \tag{13}$$

and

$$B = j_0\gamma_0 b K_1(\gamma_0 b). \tag{14}$$

Among other things, this tells us that the magnetic field external to the solenoid (i.e., $\rho > b$) has the form

$$H_{0z} = \{-B[I_1(\gamma_0 b)/K_1(\gamma_0 b)] + C\}K_0(\gamma_0\rho). \tag{15}$$

The quantity of immediate interest is the impedance Z of the solenoid itself. Clearly, within the limits of our basic assumptions,

$$Z = \text{constant} \times E_{0\phi}(\rho = b)/j_0. \tag{16}$$

The corresponding impedance of the empty solenoid is denoted Z_0. Thus it follows that

$$\frac{Z}{Z_0} = 1 - \frac{C}{B}\frac{K_1(\gamma_0 b)}{I_1(\gamma_0 b)} \tag{17}$$

which is explicit since C/B is given by (8).

Quasi-Static Form

We now can simplify the impedance ratio formula if we invoke the small argument approximations for Bessel functions of order $\gamma_0 a$ and $\gamma_0 b$. That is, we use $I_0(x) \simeq 1$, $I_1(x) \simeq x/2$, $K_0(x) \simeq -\log x$ and $K_1(x) \simeq 1/x$. This exercise leads to

$$\frac{Z}{Z_0} \cong \left[1 - \frac{a^2}{b^2} + \frac{\mu}{\mu_0}\frac{a^2}{b^2}\frac{2}{\gamma a}\frac{I_1(\gamma a)}{I_0(\gamma a)}\right] = \frac{R + iX}{Z_0} \tag{18}$$

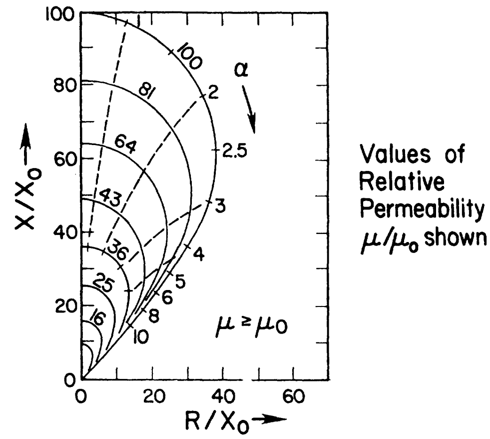

Fig. 2. Argand plot of the impedance $Z = R + iX$ normalized to the reactance X_0 of the empty solenoid, for $a = b$.

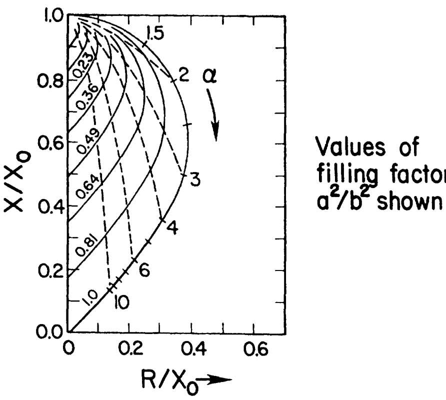

Fig. 3. Argand plot of the impedance $Z = R + iX$ normalized to the reactance X_0 of the empty solenoid, for $\mu = \mu_0$.

where no restriction has been placed on the magnitude of γa. Here R and X denote the resistance and reactance, respectively.

The formula for Z/Z_0 given by (18) is in agreement with Förster and Stambke [2] (if one remembers they used the old German designations J_0 and J_1 for modified Bessel functions). Förster and Stambke [2], Hochschild [3] and Libby [1] present extensive numerical data for this quasi-static approximation to Z/Z_0 in Argand diagrams in the complex plane for various values of $|\gamma a|$ and μ/μ_0. Two examples, using dimensionless parameters, are shown in Figs. 2 and 3 when the ordinates and abscissas are normalized by X_0 which is the reactance of the empty solenoid. That is, we assume $Z_0 \simeq iX_0$ corresponding to negligible ohmic losses in the solenoid itself. The real parameter α is defined by $\alpha = \gamma a \exp(-i\pi/4) = (\sigma\mu\omega)^{1/2}a$. In Fig. 2 the sample radius b is assumed to be the same as the sample radius a (i.e., no air gap). Different values of the magnetic permeability are shown. Not surprisingly, when α is small, R vanishes, and X/X_0 tends to μ/μ_0. However, in general, the eddy currents have the effect of reducing X/X_0, which is the effective flux, and to introduce a resistive portion R/X_0. In Fig. 3, the relative permeability of the sample $\mu/\mu_0 = 1$ but the filling factor a^2/b^2 assumes different values. The results indicate that the presence of the air gap reduces the sensitivity of the

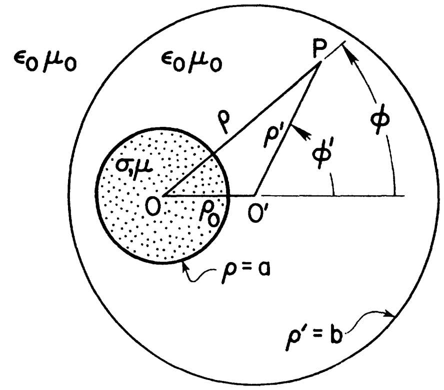

Fig. 4. Cross-section view of the noncentrally located sample.

device for probing the conductivity but the effect is predictable.

Actually, if α is sufficiently small (i.e., $|\gamma a| \ll 1$), (18) reduces to

$$Z/Z_0 \simeq X/X_0 \simeq 1 + (a^2/b^2)[\mu/\mu_0 - 1] \tag{18'}$$

which is consistent with the curves in Figs. 2 and 3. In this dc limit the results only depend on the magnetic permeability of the sample.

Analysis for Nonconcentric Sample

We now consider the formal extension of the theory to the case where the exciting solenoid is no longer concentric with the cylindrical sample. The situation is indicated in Fig. 4. As before, cylindrical coordinates (ρ, ϕ, z) are chosen coaxial with the sample. But now, the shifted coordinates (ρ', ϕ', z) are chosen to be coaxial with the exciting solenoid. The shift is ρ_0 as indicated in Fig. 4 where we do impose the rather obvious physical restriction that $b > \rho_0 + a$.

The field scattered from the solenoid now is no longer azimuthally symmetric. But, in analogy to (15), we can write

$$H_{0z} = BI_0(\gamma_0 \rho') + \sum_{m=-\infty}^{+\infty} C_m K_m(\gamma_0 \rho) e^{-im\phi} \tag{19}$$

for the region $\rho > a$ and $\rho' < b$, while

$$H_{0z} = -B[I_1(\gamma_0 b)/K_1(\gamma_0 b)]K_0(\gamma_0 \rho') + \sum_{m=-\infty}^{+\infty} C_m K_m(\gamma_0 \rho) e^{-im\phi} \tag{20}$$

for the region $\rho' > b$. As before, B is given by (14) in terms of the source current j_0 in the solenoid. Here C_m is to be determined.

To proceed further, we now note that the field inside the sample (i.e., $\rho < a$) must have the form

$$H_z = \sum_{m=-\infty}^{+\infty} H_{mz} \tag{21}$$

where

$$H_{mz} = A_m I_m(\gamma\rho) e^{-im\phi}. \tag{22}$$

Similarly, for the same region,

$$E_\phi = \sum_{m=-\infty}^{+\infty} E_{m\phi} \tag{23}$$

where

$$E_{m\phi} = -(\gamma/\sigma) A_m I'_m(\gamma\rho) e^{-im\phi} \tag{24}$$

where the prime indicates differentiation with respect to $\gamma\rho$. Now we define the cylinder impedance parameter Z_{mc} for harmonic waves of order m by

$$Z_{mc} = [-E_{m\phi}/H_{mz}]_{\rho=a} = \eta I'_m(\gamma a)/I_m(\gamma a) \tag{25}$$

in analogy to (4). In fact, $Z_{0c} \equiv Z_c$.

A known addition theorem [5] for modified Bessel functions $I_0(\gamma_0 \rho')$ allows us to write (19) in the form

$$H_{0z} = \sum_{m=-\infty}^{+\infty} H_{0mz} \tag{26}$$

where

$$H_{0mz} = \{B(-1)^m I_m(\gamma_0 \rho_0) I_m(\gamma_0 \rho) + C_m K_m(\gamma_0 \rho)\} e^{-im\phi}. \tag{27}$$

Similarly, for the same region,

$$E_{0\phi} = \sum_{m=-\infty}^{+\infty} E_{0mz} \tag{28}$$

where

$$E_{0mz} = -\eta_0 \, \partial H_{0mz}/\partial(\gamma_0 \rho). \tag{29}$$

Application of the condition

$$[E_{0m\phi} + Z_{mc} H_{0mz}]_{\rho=a} = 0 \tag{30}$$

now leads to the relation

$$\frac{C_m}{B} = \frac{(-1)^m I_m(\gamma_0 \rho_0)[\eta_0 I'_m(\gamma_0 a) - Z_{mc} I_m(\gamma_0 a)]}{-\eta_0 K'_m(\gamma_0 a) + Z_{mc} K_m(\gamma_0 a)}. \tag{31}$$

Inserting this result into (19) or (20) yields explicit expressions for the fields external to the sample. However, in order to deduce the resultant impedance of the solenoid, it is desirable to reexpress H_{0z} in terms of the (ρ', ϕ', z) coordinates. Here we use a known addition theorem [6] for $K_m(\gamma_0 \rho) \exp(-im\phi)$. The rather horrendous result is

$$H_{0z} = BI_0(\gamma_0 \rho') + \sum_{m=-\infty}^{+\infty} C_m \sum_{n=-\infty}^{+\infty} K_{m+n}(\gamma_0 \rho') I_n(\gamma_0 \rho_0)(-1)^n e^{-i(n+m)\phi'}. \tag{32}$$

This is valid in the nonconcentric air gap region (i.e., $\rho > a$ and $\rho_0 < \rho' < b$). The needed azimuthal component is obtained from

$$E_{0\phi'} = -\eta_0 \, \partial H_{0z}/\partial(\gamma_0 \rho'). \tag{33}$$

The relevant quantity for the impedance calculation is the "average" field $\bar{E}_{0\phi'}$ at the solenoid. Clearly, this is given by

$$\bar{E}_{0\phi'} = \frac{1}{2\pi} \int_0^{2\pi} [E_{0\phi'}]_{\rho'=b} \, d\phi' = -\eta_0 B I_1(\gamma_0 b) - \eta_0 \sum_{m=-\infty}^{+\infty} C_m K'_0(\gamma_0 b) I_m(\gamma_0 \rho_0)(-1)^m. \tag{34}$$

Thus it follows that the impedance Z per unit length of the solenoid with the sample divided by the impedance Z_0 of the empty solenoid is given by

$$\frac{Z}{Z_0} = 1 - \frac{K_1(\gamma_0 b)}{I_1(\gamma_0 b)} \sum_{m=0}^{\infty} \varepsilon_m \frac{C_m}{B} I_m(\gamma_0 \rho_0)(-1)^m \quad (35)$$

where $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m \neq 0$ and where C_m/B is given explicitly by (31). Not surprisingly, (35) reduces to (17) for the centrally located sample, i.e., $I_m(\gamma_0 \rho_0) = 0$ for $\rho_0 \to 0$ when $m \neq 0$.

We again may invoke the small argument approximations for Bessel functions of order $\gamma_0 a$, $\gamma_0 b$, and $\gamma_0 \rho_0$. That is, for $m = 1, 2, 3, \cdots$, we use

$$I_m(x) \simeq x^m/(m!\ 2^m),\ I'_m(x) \simeq x^{m-1}/[(m-1)!\ 2^m]$$

$$K_m(x) \simeq (m-1)!\ 2^{m-1}/x^m \text{ and } K'_m(x) \simeq -m!\ 2^{m-1}/x^{m+1}.$$

Lo and behold, these show that Z/Z_0 reduces again to the formula given by (18). This confirms the conjecture of Förster and Stambke who seemed to be gifted with keen physical insight into such problems. Of course, we do not expect the result to hold in any sense when the dimensions of the solenoid become comparable with the free-space wavelength. In that case, many other complications arise such as the assumed uniformity of the solenoid current.

The Dual Problem and Concluding Remarks

There is an extremely interesting duality to the problem we have discussed. That is, rather than exciting the cylindrical sample with an *azimuthal electric current*, we employ an *azimuthal magnetic current*. This is an idealized representation for a thin toroidal coil but, again, it effectively is of infinite length in the z or axial direction. The assumed source discontinuity is now in the electric field at $\rho = b$ which has only a z component. The much more complicated case of the toroidal coil of finite axial extent was analyzed recently [7].

Under the present assumption of axial uniformity, the admittance Y per unit length of the toroid is the dual of the impedance Z of the solenoid discussed above. Thus all the earlier equations apply if we make the following transformations: $i\mu\omega \to \sigma$, $\mu_0 \to \varepsilon_0$, $\eta \to \eta^{-1}$, $\eta_0 \to \eta_0^{-1}$, $H_z \to E_z$, $E_\phi \to -H_\phi$, $H_{0z} \to E_{0z}$, and $E_{0\phi} \to -H_{0\phi}$. Then the dual of (18) is the ratio of the admittance Y of the toroidal coil with the sample to the admittance Y_0 without the sample. It is written explicitly

$$\frac{Y}{Y_0} = \left[1 - \frac{a^2}{b^2} + \frac{\sigma}{i\varepsilon_0\omega}\frac{a^2}{b^2}\frac{2}{\gamma a}\frac{I_1(\gamma a)}{I_0(\gamma a)}\right]$$

for the case where $|\gamma_0 b| \ll 1$. That is, the radius of the toroidal coil should be much smaller than the free-space wavelength. Also, in full analogy to the earlier discussion, the quasi-static result holds for any location of the cylindrical sample within the toroid. Furthermore, in the low frequency limit where $|\gamma a| = \alpha \ll 1$, we see that

$$Y \simeq Y_0[1 + (a^2/b^2)[(\sigma/i\varepsilon_0\omega) - 1]]$$

which depends only on the conductivity of the sample. Thus this type of excitation should be preferred with probing the effective conductivity in the axial direction in the sample.

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