# The electromagnetic field on a simplicial net*

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The "Regge calculus" approach is extended to the electromagnetic case. To this end an "affine" tensor formalism and associated exterior calculus are developed. The simplicial approach to linear field equations is illustrated by the two-dimensional scalar wave equation, on which also a discussion of the treacherous character of the continuum limit is based.

## I. INTRODUCTION

A previous paper[1] developed the simplicial approach to the purely metrical field ("Regge calculus"). Therein the formalism was brought to a point where full-blown computer calculations ought to be possible, but my own attempts succeeded only in developing programs to calculate all the basic quantities but not in solving efficiently or reliably the basic set of equations for the time evolution problem. (See Sec. IV B of Ref. 1).

This paper will extend the formalism of Ref. 1 to the case of the coupled metrical and electromagnetic thatches ("geometrodynamics"). Since the electromagnetic part of the equations is linear, the new calculational problem is probably not much harder than the old.

In working with tensors defined relative to an $n$-simplex, it is convenient to use a system of coordinates which reflects the $(n+1)$-fold character of the vertices. Section II elaborates such a formalism, and Sec. III develops the "exterior calculus" in those terms. The key result which allows one to formulate the electromagnetic action is furnished by condition (3) of the theorem, in which Sec. III culminates.

Section IV presents the equations for the electromagnetic thatch and verifies all the formal consequences of these equations that one has become used to in the continuum.

Finally, Secs. V and VI discuss how the simplicial approach works out in a particularly simple situation—a "massless" scalar thatch on a two-dimensional net. It appears that the simplicial approach will agree with the finite difference scheme only "on the average." In particular, the investigation of the continuum limit by Taylor expansion at a point is in general misleading.

The notations and terminology of this paper agree with those of Ref. 1.

## II. AFFINE COORDINATES

### A. Affine coordinates

By considering a point $P$ in the interior of an $n$-simplex as the centroid of $n+1$ masses $t^j$ placed at the vertices $v_0 \cdots v_n$ we can express it as an "affine sum"

$$P=\sum_{j=0}^{n} t^j v_j \Big/ \sum_{j=0}^{n} t^j$$

of the vertices $v_j$. Renormalizing the masses, we can write

$$P=\sum_{j=0}^{n} t^j v_j, \tag{1}$$

in which

$$\sum_{j=0}^{n} t^j = 1 \tag{2}$$

and with all the $t^j>0$. By relaxing this latter condition we can express any point in the affine space $S$ of the simplex in the form (1), (2).[2] We call $v_0, v_1, \ldots, v_n$ an affine point basis for $S$.

A vector of an affine space is the "difference" of two points which we write as $Q-P$ or $\overrightarrow{PQ}$. Let $V$ denote the space of all vectors of $S$. If

$$P=\sum p^j v_j, \quad Q=\sum q^j v_j,$$

then we take for coordinate of $\overrightarrow{PQ}$ the differences $x^j \equiv q^j - p^j$. Then (2) implies

$$\sum_{j=0}^{n} x^j = 0. \tag{3}$$

Another way to explain these coordinates is as components of $\overrightarrow{PQ}$ relative to the (redundant) "barycentric basis" comprising the $n+1$ vectors

$$\mathbf{e}_i = v_i - \frac{1}{n+1}\sum_{k=0}^{n} v_k. \tag{4}$$

A simple computation verfies this:

$$\begin{aligned}\sum_i x^i \mathbf{e}_i &= \sum_i x^i v_i - \Big(\sum_i x_i\Big)\Big(\frac{1}{n+1}\sum_k v_k\Big)\\ &= \sum_i (q^i - p^i) v_i \quad [\text{by } (3)]\\ &= \sum q^i v_i - \sum p^i v_i\\ &= Q - P = \overrightarrow{PQ}.\end{aligned}$$

Corresponding to the basis $(\mathbf{e}_j)$ for $V$, we introduce for the dual space $V^*$ a basis $(\mathbf{e}^i)$ defined by

$$\langle \mathbf{e}^j, \mathbf{e}_k\rangle = \tilde{\delta}^j_k \equiv \delta^j_k - \frac{n}{n+1} = \begin{cases} \dfrac{1}{n+1} & \text{if } j=k,\\[2ex] -\dfrac{1}{n+1} & \text{if } j\neq k.\end{cases} \tag{5}$$

Notice that

$$\sum_k \mathbf{e}_k = 0, \quad \sum_k \mathbf{e}^k = 0, \tag{6}$$

$$\sum_k \mathbf{e}_k \otimes \mathbf{e}^k = \mathbf{1}. \tag{7}$$

  

Let us check the last relation, for example, by applying its left-hand side to the vector $a=\sum a^i \mathbf{e}_i$. First, however, we point out the lemma:

*Lemma*: If for any quantities $Q_j$, $j=0,\ldots,n$, $\sum_j Q_j=0$, then

$$Q_j=\tilde{\delta}_j^k Q_k. \tag{8}$$

Continuing with the check, we have

$$\begin{aligned}\sum_k \mathbf{e}_k\otimes\mathbf{e}^k\cdot a &= \sum_k \mathbf{e}_k\left\langle \mathbf{e}^k, \sum_j a^j\mathbf{e}_j\right\rangle\\ &=\sum_{k,j}\mathbf{e}_k a^j\langle\mathbf{e}^k,\mathbf{e}_j\rangle\\ &=\sum_j a^j\sum_k \tilde{\delta}_j^k\mathbf{e}_k\\ &=\sum_j a^j\mathbf{e}_j \quad \text{[by the lemma and (6)]}\\ &=a. \qquad \text{QED}\end{aligned}$$

If $T$ is any sort of tensor relative to the vector space $V$, we define its *affine components* $T^{j\cdots k}_{l\cdots m}$ by contracting it with the relevant product of basis vectors $\mathbf{e}_j$, $\mathbf{e}^k$. Then (7) guarantees the expansion:

$$T=\sum_{\substack{j\cdots k\\ l\cdots m}} T^{j\cdots k}_{l\cdots m}\mathbf{e}_j\otimes\cdots\otimes\mathbf{e}_k\otimes\mathbf{e}^l\cdots\otimes\mathbf{e}^m, \tag{9}$$

from which follows, with the aid of (6),

$$\sum_j T^{j\cdots k}_{l\cdots m}=0 \tag{10}$$

for any index $j$, up or down. This last result is the distinguishing feature of affine components and, together with (5) and the lemma, it guarantees that contraction works as usual by summing on the contracted indices.

Finally, we derive the affine components of two "special tensors." The "Kronecker delta tensor" $\delta$ has components formed as follows (in a slightly cumbersome notation):

$$\tilde{\delta}_k^j=\delta^\mu_\nu(\mathbf{e}_k)^\nu(\mathbf{e}^j)_\mu=(\mathbf{e}_k)^\nu(\mathbf{e}^j)_\nu=\langle\mathbf{e}_k,\mathbf{e}^j\rangle=\tilde{\delta}_k^j,$$

which shows the consistency of our earlier definition (5).

The other "special" tensor we will need is the epsilon symbol, which strictly speaking is not a tensor but a tensor density and thus defined *a priori* only up to an overall factor. We fix this factor by setting

$$\tilde{\epsilon}^{12\cdots n}=+1,$$

from which it is easy to evaluate the other components using the antisymmetry of $\tilde{\epsilon}^{i\cdots j}$ and the sum rule (10). Thus, for example,

$$\tilde{\epsilon}^{213\cdots n}=-\tilde{\epsilon}^{12\cdots n}=-1$$

and

$$\tilde{\epsilon}^{023\cdots n}+\tilde{\epsilon}^{123\cdots n}+\tilde{\epsilon}^{223\cdots n}+\cdots+\tilde{\epsilon}^{n23\cdots n}=0,$$

$$\tilde{\epsilon}^{023\cdots n}+1+0+\cdots+0=0, \text{ so that } \tilde{\epsilon}^{023\cdots n}=-1.$$

Let $j_0 j_1\cdots j_n$ be any permutation of the indices $01\cdots n$. Then

$$\tilde{\epsilon}^{j_1\cdots j_n}=\begin{cases}+1 & \text{if the permutation } j_0j_1\cdots j_n \text{ is even,}\\ -1 & \text{if the permutation } j_0j_1\cdots j_n \text{ is odd.}\end{cases} \tag{11}$$

We note without proof that our definition is equivalent (for the contravariant $\epsilon$) to

$$\epsilon=\overrightarrow{v_0v_1}\wedge\overrightarrow{v_0v_2}\wedge\cdots\wedge\overrightarrow{v_0v_n}.$$

A final subtlety needs mention. Let $n=3$ for definiteness. Then under the usual definitions

$$\epsilon^{\sigma\mu\nu}\epsilon_{\sigma\alpha\beta}=\delta^\mu_\alpha\delta^\nu_\beta-\delta^\mu_\beta\delta^\nu_\alpha.$$

That the analogous formula apply to $\tilde{\epsilon}^{ijk}$ and $\tilde{\epsilon}_{ilm}$ requires, as is easily checked, that $\tilde{\epsilon}^{123}\tilde{\epsilon}_{123}=\frac{1}{4}=1/(n+1)$. Accordingly, we define the covariant $\epsilon$ with components of magnitude $(n+1)^{-1}$:

$$\tilde{\epsilon}_{j_1\cdots j_n}=[1/(n+1)]\tilde{\epsilon}^{j_1\cdots j_n}. \tag{12}$$

With these definitions all the expected formulas obtain.

### B. The metric tensor of a simplex

As pointed out in Ref. 1, to specify the $n(n+1)/2$ edge lengths of an $n$-simplex is equivalent to specifying a flat metric for the interior of that simplex. In this section we calculate the affine components $\tilde{g}_{ij}$ of this metric. Let $l^2_{ij}$ be the length squared of the edge joining $v_i$ to $v_j$. Then, since, plainly, $\overrightarrow{v_iv_j}=\mathbf{e}_j-\mathbf{e}_i$,

$$\begin{aligned}l^2_{ij}&=\langle g,\overrightarrow{v_iv_j}\otimes\overrightarrow{v_iv_j}\rangle\\ &=\langle g,(\mathbf{e}_j-\mathbf{e}_i)\otimes(\mathbf{e}_j-\mathbf{e}_i)\rangle\\ &=\langle g,\mathbf{e}_j\otimes\mathbf{e}_j\rangle-2\langle g,\mathbf{e}_i\otimes\mathbf{e}_j\rangle+\langle g,\mathbf{e}_i\otimes\mathbf{e}_i\rangle\\ &=\tilde{g}_{jj}-2\tilde{g}_{ij}+\tilde{g}_{ii}\\ &\equiv A_{ij}.\end{aligned}$$

By forming the combination $\tilde{\delta}^k_i\tilde{\delta}^l_jA_{kl}$, we can, in view of (8) and (10) as applied to $\tilde{\delta}$, recover $\tilde{g}_{ij}$:

$$\tilde{\delta}^k_i\tilde{\delta}^l_j l^2_{kl}=0-2\tilde{g}_{ij}+0,\quad \tilde{g}_{ij}=-\tfrac{1}{2}l^2_{kl}\tilde{\delta}^k_i\tilde{\delta}^l_j, \tag{13}$$

which says that $\tilde{g}_{ij}$ is just $-\frac{1}{2}l^2_{ij}$ "rendered affine" or "projected into the affine tensors."

Now suppose that $\tilde{A}^{jk}$ are the affine components of some tensor. Then according to (13)

$$\begin{aligned}\tilde{A}^{ij}\tilde{g}_{ij}&=-\tfrac{1}{2}(\tilde{A}^{ij}\tilde{\delta}^k_i\tilde{\delta}^l_j)l^2_{kl}\\ &=-\tfrac{1}{2}\tilde{A}^{kl}l^2_{kl};\end{aligned}$$

in other words, one has the general

*Replacement rule*: If $\tilde{g}_{jk}$ occurs with both indices contracted against affine indices, then it can be replaced by $-\frac{1}{2}l^2_{jk}$.

### C. "Geometrical" tensors

In this subsection we fix some normalizations and derive a useful expression for the volume of a simplex.

Let the wedge product be defined in the usual way and normalized so that, for instance, the wedge product $a\wedge b\wedge c$ of three vectors consists of six terms each with coefficient $\pm 1$. Then we take the product $a\wedge b$ to represent the parallelogram determined by $a$ and $b$, and $\frac{1}{2}a\wedge b$ the triangle or 2-simplex spanned by them. In general, the normalized product

$$\omega=(1/m!)a_1\wedge\cdots\wedge a_m \tag{14}$$

will represent the $m$-simplex spanned by vectors $a_1\cdots a_m$. We also introduce the more conveniently normalized contraction

$$\langle\omega|\phi\rangle \equiv \langle\omega,\phi\rangle/m!, \tag{15}$$

where $m$ is the rank of the forms $\omega, \phi$. For example, if $m=2$, then

$$\langle\omega|\phi\rangle = \tfrac{1}{2}\omega^{\mu\nu}\phi_{\mu\nu}.$$

With these definitions the volume represented by any rank $m$ totally antisymmetric tensor is

$$\mathrm{vol}(\omega) = |\langle\omega|\omega\rangle|^{1/2} \equiv \|\omega\|. \tag{16}$$

(The absolute value is needed because of the indefinite metric, i.e., the volume is defined to be a real number.) Thus, for example, the bone [012] of some 4-simplex $\sigma$ corresponds to the tensor

$$\begin{aligned}\omega &= (1/2!)\overrightarrow{v_0v_1}\wedge\overrightarrow{v_0v_2}\\ &= (1/2!)(\mathbf{e}_1-\mathbf{e}_0)\wedge(\mathbf{e}_2-\mathbf{e}_0)\\ &= (1/2!)(\mathbf{e}_1\wedge\mathbf{e}_2+\mathbf{e}_2\wedge\mathbf{e}_0+\mathbf{e}_0\wedge\mathbf{e}_1)\\ &= (1/2!)\tilde{\epsilon}^{ij}_{(012)}\mathbf{e}_i\otimes\mathbf{e}_j,\end{aligned}$$

where $\tilde{\epsilon}^{ij}_{(012)}$ is of course formed from the indices 0, 1, 2 in the manner of (11). In general the $m$ subsimplex with vertices $k_0\cdots k_m$ corresponds via (14) to the tensor with affine components:

$$\tilde{\omega}^{j_1\cdots j_m} = (1/m!)\epsilon^{j_1\cdots j_m}_{(k_0\cdots k_m)}. \tag{17}$$

According to (16) the volume $V$ of this simplex is given by

$$\pm V^2 = \langle\omega|\omega\rangle = (m!)^{-1}\tilde{\omega}^{i\cdots j}\tilde{g}_{ia}\cdots\tilde{g}_{jb}\tilde{\omega}^{a\cdots b}.$$

By the replacement $\tilde{g}^2_{ij} \to -\frac{1}{2}l^2_{ij}$ discovered in the previous subsection we convert this result to one expressed directly in terms of edge lengths:

$$\pm\mathrm{vol}^2 = (-\tfrac{1}{2})^m(1/m!)\tilde{\omega}^{j_1\cdots j_m}\tilde{\omega}^{k_1\cdots k_m}\prod_{a=1}^{m} l^2_{j_a k_a}. \tag{18}$$

To find the volume of any $m$-simplex of the net, we can work within the $m$-dimensional affine space spanned by that simplex and (calling its vertices $0\cdots m$) write

$$\tilde{\omega}^{j_1\cdots j_m} = (1/m!)\tilde{\epsilon}^{j_1\cdots j_m}.$$

Then

$$\pm\mathrm{vol}^2 = (-\tfrac{1}{2})^m(m!)^{-3}\tilde{\epsilon}^{j_1\cdots j_m}\tilde{\epsilon}^{k_1\cdots k_m}l^2_{j_1k_1}\cdots l^2_{j_mk_m}. \tag{19}$$

To facilitate numerical evaluation of such expressions, we introduce the concept of a "bordered determinant" which has the form (with $A$ representing any $m\times m$ matrix)

$$B(A) \equiv \begin{vmatrix} 0 & 1 & 1 & \cdots & 1\\ 1 & & & & \\ 1 & & & & \\ \cdot & & A & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & & & & \end{vmatrix}$$

Then the expression

$$\tilde{\epsilon}^{j_1\cdots j_m}\tilde{\epsilon}^{k_1\cdots k_m}A_{j_1k_1}\cdots A_{j_mk_m}$$

can be evaluated as $-m!B(A)$, the proof being left to the interested reader. Thus we get the expression for volume in terms of edges, as

$$\pm\mathrm{vol}^2 = -(-\tfrac{1}{2})^m(m!)^{-2}\begin{vmatrix} 0 & 1 & 1 & \cdots & 1\\ 1 & & & & \\ 1 & & l^2_{ij} & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & & & & \end{vmatrix}, \tag{20}$$

a result which appears in Ref. 3.

For a triangle we find (setting $m=2$, $x=l^2_{01}$, $y=l^2_{02}$, $z=l^2_{12}$)

$$\begin{aligned}\pm A^2 &= -(-\tfrac{1}{2})^2(2!)^{-2}\begin{vmatrix} 0 & 1 & 1 & 1\\ 1 & 0 & x & y\\ 1 & x & 0 & z\\ 1 & y & z & 0\end{vmatrix}\\ &= -\tfrac{1}{16}[x^2+y^2+z^2-2(xy+yz+zx)].\end{aligned}$$

## III. SIMPLICIAL EXTERIOR CALCULUS

Let

$\Sigma_0$ = set of all 0-simplexes (vertices) of the net,

$\Sigma_1$ = set of all oriented 1-simplexes (legs) of the net,

$\Sigma_2$ = set of all oriented 2-simplexes of the net, etc.,

and represent a typical oriented 2-simplex, e.g., as $[xyz]$, where $x, y, z \in \Sigma_0$. Then we define[4]

a 0-*form* (scalar thatch) as a map $\phi: \Sigma_0 \to R$,

a 1-*form* (co-vector thatch) as a map $A: \Sigma_1 \to R$ such that $A(xy) = -A(yx)$,

a 2-*form* as a map $F: \Sigma_2 \to R$ such that $F(xyz) = -F(yxz) = F(yzx)$, etc.

To understand these definitions, one could think of $A(xy)$, e.g., as the line integral $\int_x^y A_\mu\,dx^\mu$ of some field $A_\mu$ along the leg $[xy]$. If, then, $F = dA$, then Stokes' theorem becomes

$$\begin{aligned}F(xyz) &\leftrightarrow \int_{[xyz]} F_{\mu\nu}\,da^{\mu\nu} = \oint_{[xy]+[yz]+[zx]} A_\mu\,dx^\mu\\ &\leftrightarrow A(xy)+A(yz)+A(zx).\end{aligned}$$

Generalizing this relation to arbitrary dimension we define the operator "$d$" from $m$- to $(m+1)$-forms as follows:

$$d\omega(k_0k_1\cdots k_m) = \sum_{j=0}^{m}(-1)^j\omega(k_0\cdots\hat{k}_j\cdots k_m), \tag{21}$$

where the "hat" indicates omission. It is easy to check that

$$d\,d\omega = 0: \tag{22}$$

$$\begin{aligned} d\,d\omega(k_0\cdots k_{m+1}) &= \sum_{j=0}^{m+1}(-1)^j\,d\omega(k_0\cdots\hat{k}_j\cdots k_{m+1})\\ &= \sum_{j=0}^{m+1}(-1)^j\sum_{l=0}^{m+1}(-1)^l\,\mathrm{sgn}(l-j)\\ &\quad\times\omega(k_0\cdots\hat{k}_a\cdots\hat{k}_b\cdots k_{m+1})\\ &\qquad[\text{where } a=\min(j,l),\ b=\max(j,l)]\\ &= \sum_{j,l=0}^{m+1}(-1)^{j+l}\,\mathrm{sgn}(l-j)\\ &\quad\times\omega(k_0\cdots\hat{k}_a\cdots\hat{k}_b\cdots k_{m+1})\\ &= 0\end{aligned}$$

since $\mathrm{sgn}(l-j)$ alone in the penultimate expression is antisymmetric in the exchange of $j$ with $l$. QED

The main theorem of this section is in part a partial converse to (22).

Consider now a particular $m$-form $\omega$ and a particular simplex $\sigma\in\Sigma_4$ and ask whether $\omega$ "extends to the interior of $\sigma$," i.e., whether there is defined in the space of $\sigma$ an $m$-form $\omega(\sigma)$ which agrees with $\omega$ on all the $m$-subsimplexes of $\sigma$. By definition $\omega(\sigma)$ "agrees with" $\omega$ on $[x_0\cdots x_m]$ iff

$$\langle\omega(\sigma),[x_0\cdots x_m]\rangle=\omega(x_0\cdots x_m). \tag{23}$$

If $\omega$ does extend to $\sigma$, then we can form a simple expression [analogous to (13)] for the affine components of $\omega(\sigma)$:

From above we know that, calling $\sigma=[01\cdots n]$,

$$m![k_0\cdots k_m]=\overrightarrow{k_0k_1}\wedge\overrightarrow{k_0k_2}\wedge\cdots\wedge\overrightarrow{k_0k_m}$$
$$=(\mathbf{e}_{k_1}-\mathbf{e}_{k_0})\wedge(\mathbf{e}_{k_2}-\mathbf{e}_{k_0})\wedge\cdots(\mathbf{e}_{k_m}-\mathbf{e}_{k_0}).$$

In the expansion of the right-hand side, only terms lacking or linear in $\mathbf{e}_{k_0}$ survive since $\mathbf{e}_{k_0}\ \ \mathbf{e}_{k_0}=0$, and one obtains, in a hopefully clear notation,

$$m![k_0\cdots k_m]=\mathbf{e}_{k_1}\wedge\cdots\wedge\mathbf{e}_{k_m}-\sum_{j=1}^{m}\mathbf{e}_{k_1}\wedge\cdots\wedge\underset{(j)}{\mathbf{e}_{k_0}}\wedge\cdots\wedge\mathbf{e}_{k_m}. \tag{24}$$

Because of (6) we can isolate $\mathbf{e}_{k_1}\wedge\cdots\wedge\mathbf{e}_{k_m}$ by summing on $k_0$:

$$m!\sum_{k_0=0}^{n}[k_0\cdots k_m]=(n+1)\mathbf{e}_{k_1}\wedge\cdots\wedge\mathbf{e}_{k_m},$$
$$\mathbf{e}_{k_1}\wedge\cdots\wedge\mathbf{e}_{k_m}=\frac{m!}{1+n}\sum_{k_0=0}^{n}[k_0k_1\cdots k_m]. \tag{25}$$

Applying $\omega(\sigma)$ to both sides, we have

$$m!\,\tilde{\omega}(\sigma)_{k_1\cdots k_m}=\frac{m!}{1+n}\sum_{k_0=0}^{n}\langle\omega(\sigma),[k_0\cdots k_m]\rangle,$$
$$\tilde{\omega}(\sigma)_{k_1\cdots k_m}=\frac{1}{1+n}\sum_{k_0=0}^{n}\omega(k_0k_1\cdots k_m). \tag{26}$$

In order to study this condition more closely, we make the definition (relative to the simplex $\sigma$)

$$S\omega(k_1\cdots k_m)=\frac{1}{1+n}\sum_{k_0=0}^{n}\omega(k_0k_1\cdots k_n) \tag{27}$$

so that (26) can be expressed in the droll form

$$\tilde{\omega}(\sigma)_{k_1\cdots k_m}=S\omega(k_1\cdots k_m). \tag{28}$$

It is easy to see that $S\omega$ is an $(m-1)$-form (on $\sigma$) when $\omega$ is an $m$-form, and that

$$S^2=0. \tag{29}$$

We can also verify the important relation (relative to $\sigma$ as always)

$$Sd+dS=1. \tag{30}$$

*Proof*:

$$dS\omega(k_0\cdots k_m)=\sum_{j=0}^{m}(-1)^jS\omega(k_0\cdots\hat{k}_j\cdots k_m)$$
$$=\sum_{j=0}^{m}(-1)^j\frac{1}{1+n}\sum_{l=0}^{n}\omega(lk_0\cdots\hat{k}_j\cdots k_m)$$
$$=\frac{1}{1+n}\sum_{j=0}^{m}\sum_{l=0}^{n}\omega(k_0\cdots\underset{(j)}{l}\cdots k_m),$$

$$S\,d\omega(k_0\cdots k_m)=\frac{1}{1+n}\sum_{l=0}^{n}d\omega(lk_0\cdots k_m)$$
$$=\frac{1}{1+n}\sum_{l=0}^{n}\omega(k_0\cdots k_m)$$
$$-\sum_{j=0}^{m}(-1)^j\omega(lk_0\cdots\hat{k}_j\cdots k_m)$$
$$=\omega(k_0\cdots k_m)-\frac{1}{1+n}\sum_{l=0}^{n}$$
$$\times\sum_{j=0}^{m}\omega(k_0\cdots\underset{(j)}{l}\cdots k_m).$$

Comparing the two expressions completes the proof.

Returning to the question whether $\omega$ extends to $\sigma$, we note that the formula for $\omega(\sigma)$ given in (28) or (26) will define a form in the space of $\sigma$ whether or not $\omega$ extends to $\sigma$. If we call this form $\rho$, then the condition that $\rho$ agree with $\omega$ on $\sigma$ (which is just that that $\omega$ extend to $\sigma$) becomes

$$\langle\rho,[k_0\cdots k_m]\rangle=\omega(k_0\cdots k_m).$$

But by (24)

$$\langle\rho,[k_0\cdots k_m]\rangle$$
$$=\frac{1}{m!}\langle\rho,\mathbf{e}_{k_1}\wedge\cdots\wedge\mathbf{e}_{k_m}\rangle$$
$$-\frac{1}{m!}\sum_{j=1}^{m}\langle\rho,\mathbf{e}_{k_1}\wedge\cdots\underset{(j)}{\mathbf{e}_{k_0}}\wedge\cdots\mathbf{e}_{k_m}\rangle$$
$$=\tilde{\rho}_{k_1\cdots k_m}-\sum_{j=1}^{m}\tilde{\rho}_{k_1\cdots\underset{(j)}{k_0}\cdots k_m}$$
$$=\frac{1}{1+n}\sum_{l=0}^{n}\omega(lk_1\cdots k_m)-\sum_{j=1}^{m}\frac{1}{1+n}$$
$$\times\sum_{l=0}^{n}\omega(lk_1\cdots\underset{(j)}{k_0}\cdots k_m)$$
$$=\frac{1}{1+n}\sum_{l=0}^{n}\omega(lk_1\cdots k_m)+\frac{1}{1+n}\sum_{j=1}^{m}\sum_{l=0}^{n}\omega(k_0k_1\cdots\underset{(j)}{l}\cdots k_m)$$
$$=\frac{1}{1+n}\sum_{l=0}^{n}\sum_{j=0}^{m}\omega(k_0\cdots\underset{(j)}{l}\cdots k_m).$$

Comparing this with the proof of (30) furnishes the condition for $\omega$ to extend to $\sigma$ in the form

$$dS\omega=\omega. \tag{31}$$

We can now prove the following fundamental theorem which has been the goal of this subsection:

*Theorem*: Let $\omega$ be any form defined on a net including the simplex $\sigma$ and set $\Omega=S\omega$, as defined in (27). Then the following three conditions are equivalent:

(1) $\omega=d\Omega$,

(2) $d\omega=0$,

(3) $\omega$ extends to $\sigma$, the extension being furnished by (26).

*Proof*: We just saw that we can replace (3) by the condition

(3′) $dS\omega=\omega$;

we already know by (22) that (1)$\Rightarrow$(2), and (3′)$\Rightarrow$(1) is obvious. To complete the circle of implication, we need only (2)$\Rightarrow$(3′) which follows immediately from (30) applied to $\omega$. QED

By the way, $\Omega$ becomes unique through the condition $S\Omega=0$, which follows from (29).

## IV. THE ELECTROMAGNETIC THATCH

(All components in this section are affine components—but the tilde (˜) will usually be omitted.)

### A. The source free thatch equations

In this section we assume a net $\Sigma$ with fixed metric thatch $l_{ij}^2$ and the associated metric tensors $g(\sigma)$ for each $\sigma\in\Sigma_4$.

The "vector potential" $A$ is a 1-form on $\Sigma$, as defined in Sec. III, and $F=dA$ is the electromagnetic thatch. By the theorem of the previous chapter $F$ extends in each cell $\sigma\in\Sigma_4$ to a tensor $F_{ij}(\sigma)$ given by (26). Calling $V(\sigma)$ the volume of $\sigma$, we take for the action

$$\begin{aligned} S_e&=-\tfrac{1}{2}\sum_{\sigma\in\Sigma_4}V(\sigma)\langle F(\sigma)\,|\,F(\sigma)\rangle_\sigma\\ &=-\tfrac{1}{4}\sum_\sigma V(\sigma)F^{ij}(\sigma)F_{ij}(\sigma)\\ &=-\tfrac{1}{4}\sum_\sigma V(\sigma)g(\sigma)^{ia}g(\sigma)^{jb}F(\sigma)_{ij}F(\sigma)_{ab}. \end{aligned}\tag{32}$$

The thatch equations equate to zero the variation of $S$ with respect to the thatch $A$:

$$\frac{\partial S}{\partial A(ij)}=0\quad\text{for all legs }[ij]\in\Sigma_1.\tag{33}$$

Well,

$$\delta S_e=-\tfrac{1}{2}\sum_\sigma V(\sigma)F^{ij}(\sigma)\delta F_{ij}(\sigma).$$

But according to (26)

$$\begin{aligned} F_{ij}(\sigma)&=\tfrac{1}{5}\sum_{k\in\sigma}F(kij)\\ &=\tfrac{1}{5}\sum_k[A(ij)+A(jk)+A(ki)]\\ &=A(ij)+\tfrac{1}{5}\sum_k[A(jk)+A(ki)] \end{aligned}\tag{34}$$

Because $F_{ij}$ are affine components we can also write

$$\begin{aligned} F_{ij}(\sigma)&=\tilde\delta_i^a\tilde\delta_j^bF_{ab}(\sigma)\\ &=\tilde\delta_i^a\tilde\delta_j^bA(ab), \end{aligned}\tag{35}$$

where the remaining terms vanish because of (10) applied to $\tilde\delta_k^j$. Just as for $\tilde g_{jk}$ one discovers from (35) the *replacement rule*

$$\tilde F_{jk}\to A(jk)\tag{36}$$

whenever $\tilde F_{jk}$ occurs with both $j$ and $k$ contracted against affine indices.

This allows us to express $\delta S_e$ directly in terms of $\delta A$:

$$\delta S_e=-\tfrac{1}{2}\sum_\sigma V(\sigma)F^{ij}(\sigma)\delta A(ij)\tag{37}$$

There is thus one thatch equation for each leg of the net:

$$\sum_\sigma V(\sigma)F^{ij}(\sigma)=0,\tag{38}$$

where, of course, $\sigma$ ranges only over those 4-simplexes for which the expression has sense, i.e., for those of which $[i]$ and $[j]$ are vertices.

We can also express $F^{ij}(\sigma)$, and thereby the thatch equations, directly in terms of $A$:

$$\begin{aligned} F^{ij}(\sigma)&=g^{ia}(\sigma)g^{jb}(\sigma)F_{ab}=g^{ia}(\sigma)g^{jb}(\sigma)A(ab)\\ F^{ij}(\sigma)&=\tfrac{1}{2}h^{ijab}(\sigma)A(ab), \end{aligned}\tag{39}$$

where

$$h^{ijab}=g^{ia}g^{jb}-g^{ib}g^{ja}.$$

### B. The equations with a source—Charge conservation

If there is prescribed a source $J$, then the action has an additional term

$$S_i=\sum_{[ij]\in\Sigma_1}A(ij)J(ij),\tag{40}$$

in which $J(ij)$ should be considered, not as a 1-form, but rather as a "vector density" or "current." In place of (37) stands (half of)

$$-\sum_\sigma V(\sigma)F^{ij}(\sigma)\delta A(ij)+J(ij)\delta A(ij)$$

so that the thatch equations become

$$\sum_\sigma V(\sigma)F^{ij}(\sigma)=J(ij).\tag{41}$$

The natural interpretation of $J$ regards $J(ij)$ as the charge flowing "along" leg $[ij]$ of the net. It is as if $\Sigma_1$ were an electrical network, $A$ the potential drop, and $J$ the current. Then the conservation of charge (like one of Kirchhoff's laws) reads

$$\sum_j J(ij)=0\tag{42}$$

and follows from (41) because of the rule (10).

We can also cast the conservation law in an "integral" form as opposed to its "local" statement (42): Let $\Omega\subset\Sigma_0$ be all the vertices in some region of the net and form the two expressions

$$\sum_{i\in\Omega}\sum_{k\in\Sigma_0}J(ik)\quad\text{and}\quad\sum_{i,k\in\Omega}J(ik).$$

The first vanishes by the equation of conservation (42) and the second by the antisymmetry of $J$. Then

$$\begin{aligned} 0&=\sum_{i\in\Omega}\sum_{k\in\Sigma_0}J(ik)\\ &=\sum_{i\in\Omega}\Big(\sum_{k\in\Omega}+\sum_{k\notin\Omega}\Big)J(ik)\\ &=0+\sum_{i\in\Omega}\sum_{k\notin\Omega}J(ik),\\ &\sum_{i\in\Omega}\sum_{k\notin\Omega}J(ik)=0. \end{aligned}\tag{43}$$

In words: "The total charge leaving the region $\Omega$ vanishes."

### C. Gauge invariance

As usual, $F=dA$ determines $A$ only up to an addition of the form $d\theta$, for arbitrary 0-form $\theta$. Since $A$ does not occur explicitly in $S_e$, we are free to require invariance under the "gauge transformation"

$$A\to A+d\theta\tag{44}$$

as long as the interaction term (40) is unaffected. But under (44) $S_i$ acquires an additional term

$$\frac{1}{2}\sum_{i,j} d\theta(ij)J(ij)\ ^5$$
$$=\frac{1}{2}\sum [\theta(j)-\theta(i)]J(ij)$$
$$=-\sum_i \theta(i)\sum_j J(ij)$$

whence gauge invariance requires

$$\sum_j J(ij)=0 \tag{45}$$

since $\theta$ is arbitrary. This is exactly the familiar connection between gauge invariance and charge conservation.

Since the gauge freedom of $A$ introduces a free number for each vertex of the net, one can remove this freedom by imposing one condition at each vertex. One which suggests itself is

$$\sum_j A(ij)=0 \quad \text{at all } i\in\Sigma_0. \tag{46}$$

This looks something like the "Lorentz gauge," but it is not, since $A$ is a 1-form rather than a current.

### D. Coupling to the metric thatch—The energy-momentum tensor

Equation (38) already includes the effects of an arbitrary background metric. To find the reciprocal influence of the electromagnetic thatch on the metric, we must evaluate

$$T(ij)=-\frac{\partial S_e}{\partial l_{ij}^2}. \tag{47}$$

Writing (32) in the form

$$S_e=\sum_\sigma L(\sigma)$$

$$L(\sigma)\equiv-\tfrac{1}{4}V(\sigma)g^{\mu\mu}g^{\nu\nu}F_{\mu\nu}(\sigma)F_{\mu\nu}(\sigma), \tag{48}$$

and, varying the metric $g(\sigma)$ interior to $\sigma$, one finds

$$2\delta L=-\tfrac{1}{2}\delta V\langle F,F\rangle-Vg^{\mu\mu}\delta g^{\nu\nu}F_{\mu\nu}F_{\mu\nu}$$
$$=-\tfrac{1}{2}\delta V\langle F,F\rangle+Vg_{\mu\mu}\delta g_{\nu\nu}F^{\mu\nu}F^{\mu\nu}.$$

If we express this in affine components, then $\delta V$ assumes a simple form which follows readily from the method of Sec. II C:

$$\delta V=\tfrac{1}{2}V\tilde{g}^{ij}\delta\tilde{g}_{ij} \tag{49}$$

whence

$$2\delta L=V\tilde{g}_{ii}\delta\tilde{g}_{jj}\tilde{F}^{ij}\tilde{F}^{ij}-\tfrac{1}{4}V\tilde{F}^{ab}\tilde{F}_{ab}\tilde{g}^{jj}\delta\tilde{g}_{jj}$$
$$=V\delta\tilde{g}_{jj}(\tilde{F}^{ij}\tilde{g}_{ii}\tilde{F}^{ij}-\tfrac{1}{4}\tilde{F}^{ab}\tilde{F}_{ab}\tilde{g}^{jj})$$
$$=V(\sigma)\delta\tilde{g}_{jk}(\sigma)\tilde{T}^{jk}(\sigma), \tag{50}$$

in which

$$\tilde{T}^j{}_k(\sigma)\equiv\tilde{F}^{ja}(\sigma)\tilde{F}_{ka}(\sigma)-\tfrac{1}{4}\tilde{F}^{ab}(\sigma)\tilde{F}_{ab}(\sigma)\tilde{\delta}^j{}_k \tag{51}$$

is the well-known formation in terms of $\tilde{F}_{ij},\tilde{g}_{ij}$. Applying the replacement rule of Sec. II B converts (50) into

$$\delta L=-\tfrac{1}{4}\delta l_{jk}^2V\tilde{T}^{jk}$$

so that, finally,

$$\delta S=\sum_\sigma \delta L(\sigma)$$
$$=-\tfrac{1}{4}\sum_\sigma\sum_{j,k}\delta l_{jk}^2V(\sigma)\tilde{T}^{jk}(\sigma)$$
$$=-\tfrac{1}{4}\sum_{j,k}\delta l_{jk}^2\sum_\sigma V(\sigma)\tilde{T}^{jk}(\sigma)$$
$$=-\sum_{[j,k]\in\Sigma_1}\delta l_{jk}^2T_e(jk),$$
$$\text{where}\quad T_e(jk)=\tfrac{1}{2}\sum_\sigma V(\sigma)\tilde{T}^{jk}(\sigma). \tag{52}$$

Thus the thatch equation (4) of Ref. 1 becomes for the present case

$$G(jk)=T_e(jk). \tag{53}$$

## V. EXAMPLE: THE WAVE EQUATION IN TWO DIMENSIONS

To develop some feeling for the behavior of simplicial equations, we can study a particularly simple, linear case: the two-dimensional wave equation.

If $\phi$ is the basic scalar thatch then, in analogy with the continuum theory, we choose for the action

$$S=\sum_{\sigma\in\Sigma_n}L(\sigma)V(\sigma) \tag{54}$$

where

$$L(\sigma)=\langle d\phi(\sigma)\mid d\phi(\sigma)\rangle$$
$$=\tfrac{1}{2}\tilde{g}^{ij}(\sigma)d\tilde{\phi}_i(\sigma)d\tilde{\phi}_j(\sigma). \tag{55}$$

Here, of course, $g(\sigma)$ and $d\phi(\sigma)$ are defined as in Secs. II B and III, respectively.

There is also a replacement rule for $d\tilde{\phi}_j$. Explicitly

$$d\tilde{\phi}_i(\sigma)=\frac{1}{1+n}\sum_{k\in\sigma}d\phi(ki)$$
$$=\frac{1}{1+n}\sum_k\phi(i)-\phi(k),$$
$$d\tilde{\phi}_i(\sigma)=\phi(i)-\langle\phi\rangle_\sigma, \tag{56}$$

where $\langle\phi\rangle_\sigma$ is the average value of $\phi$ in the simplex $\phi$. Then, since $\sum_j\tilde{g}^{ij}=0$,

$$\tilde{g}^{ij}(\sigma)d\tilde{\phi}_j(\sigma)=\sum_{j\in\sigma}\tilde{g}^{ij}(\sigma)\phi(j)$$

and

$$L(\sigma)=\sum_{i,j\in\sigma}\tfrac{1}{2}\tilde{g}^{ij}(\sigma)\phi(i)\phi(j), \tag{57}$$

whence

$$S=\tfrac{1}{2}\sum_{\substack{i,j\in\Sigma_0\\ \sigma\in\Sigma_n}}\tilde{g}^{ij}(\sigma)\phi(i)\phi(j)V(\sigma). \tag{58}$$

[The sum is naturally over only those values which make sense—those for which $i\in\sigma$ and $j\in\sigma$. Equivalently one can define $\tilde{g}^{ij}(\sigma)\equiv(0)$ for all the nonsensical values.]

Varying $\phi(i)$:

$$\frac{\partial S}{\partial\phi(i)}=\sum_{j,\sigma}\tilde{g}^{ij}(\sigma)\phi(j)V(\sigma), \tag{59}$$

the vanishing of which constitutes the thatch equation for vertex $i$.

So far everything was general. We now specialize to various two-dimensional nets with flat metric. To evaluate $\tilde{g}^{ij}(\sigma)$, the following formula, which can be proved by the methods of Sec. II C, will prove very convenient:

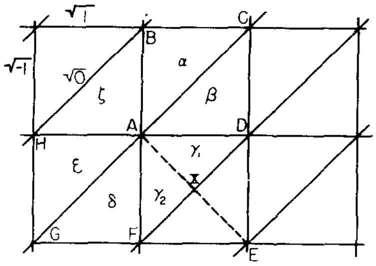


FIG. 1. A rectangular net for a two-dimensional flat space—time. The diagonal lines are lightlike.

$$\langle F(i), F(j)\rangle = [\tilde{g}/(n-1)!\,]\tilde{g}^{ij} = \pm nn!\, V^2 \tilde{g}^{ij}. \tag{60}$$

Here everything relates to a particular $n$-simplex; $F(i)$ is the oriented face opposite to the vertex $i$, $V$ the volume of the simplex, and

$$n!\,\tilde{g} = \tilde{\epsilon}^{i\cdots j}\tilde{g}_{ia}\cdots\tilde{g}_{jb}\tilde{\epsilon}^{a\cdots b}, \tag{61}$$

of course.

Work first with the net of Fig. 1 (without the dotted line), and consider the equation of vertex $A$. Because all the cells have the same volume $V$, Eq. (59) becomes

$$\sum_{\sigma}\sum_{j\in\sigma}\tilde{g}^{ij}(\sigma)\phi(j) = 0$$

or, in view of (60),

$$\sum_{\sigma}\sum_{j\in\sigma}\langle F(i), F(j)\rangle\phi(j) = 0. \tag{62}$$

There are two types of cell in the net, of which $\alpha$ and $\beta$ are exemplars. For $\alpha$ one finds from (60) (order: $A\ B\ C$)

$$\tilde{g}^{ij}(\alpha) = \begin{matrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{matrix}\,, \tag{63}$$

and from this $\tilde{g}^{ij}(\beta)$ must be (order: $A\ D\ C$)

$$\tilde{g}^{ij}(\beta) = \begin{matrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{matrix}\,. \tag{64}$$

The equation of $A$ is then

$$[\tilde{g}^{AA}(\alpha)+\tilde{g}^{AA}(\beta)+\tilde{g}^{AA}(\gamma)+\tilde{g}^{AA}(\delta)+\tilde{g}^{AA}(\epsilon)+\tilde{g}^{AA}(\zeta)]\phi(A)$$

$$+[\tilde{g}^{AB}(\zeta)+\tilde{g}^{AB}(\alpha)]\phi(B)+[\tilde{g}^{AC}(\alpha)+\tilde{g}^{AC}(\beta)]\phi(C)$$

$$+\cdots+[\tilde{g}^{AH}(\epsilon)+\tilde{g}^{AH}(\zeta)]\phi(H) = 0$$

or

$$(-1+1+0-1+1+0)\phi(A)+(1+1)\phi(B)+(0+0)\phi(C)$$

$$+(-1-1)\phi(D)+\cdots+(-1-1)\phi(H) = 0, \tag{65}$$

$$\phi(B)-\phi(D)+\phi(F)-\phi(H) = 0, \quad \phi(B)+\phi(F) = \phi(H)+\phi(D),$$

which is exactly the equation used by the method of finite differences, in place of $\Box^2\phi = 0$ (in two dimensions).

It is remarkable that $\phi(A)$, $\phi(C)$, $\phi(G)$ drop out of the equation completely. It is also odd that the vertices of the net fall into two variationally unrelated subsets, but there seems to be no way to set up a net which avoids this and still has basic equations of the type (65). The net indicated in Fig. 2, for example, relates every point to every other, but through the typical equations

$$\phi(a)+2\phi(h)+\phi(d) = \phi(b)+\phi(c)+\phi(e)+\phi(f), \tag{66}$$

which could be thought of as the sum of the two equations

$$\phi(a)+\phi(h) = \phi(f)+\phi(b) \quad\text{and}\quad \phi(d)+\phi(h) = \phi(c)+\phi(e).$$

The most disconcerting phenomenon implied by (62) is that of the totally unrelated vertex as illustrated by Fig. 3. The subnet pictured, which might be the refinement indicated by the dotted line in Fig. 1, consists of four cells (triangles). According to (58) their contribution to the action is a sum of terms in $\phi(X)\phi(X)$, $\phi(X)\phi(A), \ldots, \phi(A)\phi(F), \cdots$. From (63) and (64) the coefficient of $\phi(X)\cdot\phi(X)$ is

$$\tfrac{1}{8}\sum_{j=1}^{4}\tilde{g}^{XX}(j) = \tfrac{1}{8}(-4+4-4+4) = 0,$$

while that of $\phi(X)\phi(A)$, e.g., is,

$$\tfrac{1}{4}\sum_{j=1}^{2}\tilde{g}^{XA}(j) = \tfrac{1}{4}(2-2) = 0.$$

In other words $\phi(X)$ drops out of the action completely! In fact the expression for $S$ is the same for both nets: The dotted line makes no difference.

Lest all these surprises give the impression that the simplicial approach is especially productive of anomalies, we should add that for any other than the 1—1 ratio of sides, the net of Fig. 1 reproduces exactly the equation of the usual finite difference approximation. And, though we have stuck to flat space—time, the simplectic scheme comes into its own only with a curved background metric—which it handles with no extra trouble.

As a final example we take the two-dimensional potential equation. Using a "square" net with the topology

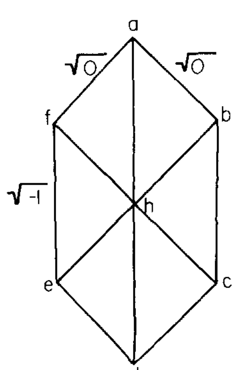


FIG. 2. A simple net in two dimensions.

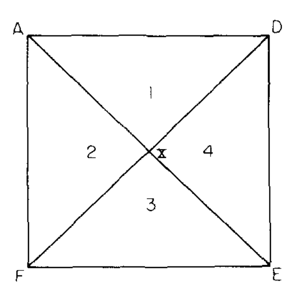


FIG. 3. A refinement of the net of Fig. 1. $\phi(x)$ makes no contribution to the action.

pictured in Fig. 1 one finds for vertex $A$, e.g., the equation

$$4\phi(A)=\phi(H)+\phi(D)+\phi(B)+\phi(F), \qquad (67)$$

which just says that $\phi(A)$ is the average of the neighboring values. And, of course, this is the well-known characteristic feature of a solution of $\nabla^2\phi=0$.

## VI. THE CHARACTER OF THE CONTINUUM LIMIT

### A. General considerations

All the thatch equations discussed above or in Ref. 1 share this feature: The thatch represents a true field but one of a very simple (possibly singular) type. Thus $A(jk)$ corresponds to a piecewise linear electromagnetic potential, while $l^2(jk)$ defines a piecewise flat manifold. In terms of these fields one defines the action in the usual way; then the thatch equations just assert the stationarity of the action—but only for variations which maintain the correspondence of the field to some thatch. By using ever finer nets one allows for ever more delicate variations of the field so that, in the limit of an infinitely fine net, one expects the solution thatch to correspond exactly to the true field.[6]

On the other hand, as we will see below in particular examples, it is in general *false* that the limit of a particular thatch equation is the correct field equation at that point. In other words, the discretization of an exact continuum solution will not produce a solution of the thatch equations, even in the continuum limit!

To understand this better, remember that $A(jk)$, for example, corresponds to a piecewise linear field. At any given point this can agree with $A_\mu(x)$ only to terms of the first order in $dx$ (precisely those involved in the definition of the action!); it can reflect the second derivatives of $A_\mu$ only *on the average* over a small region. Thus one can expect $A_\mu$ and even $F_{\mu\nu}$, but not $\partial_\mu F_{\mu\nu}$ to become exact in the continuum limit. The field equations $\partial_\mu F_{\mu\nu}=0$ can become exact only after averaging.

We can arrive at this conclusion again by a somewhat different argument. Let $\Omega$ be a region of spacetime and consider for simplicity the scalar thatch $\phi(j)$. In the continuum limit $\delta S$ must vanish for any smooth variation $\delta\phi(x)$ of the field $\phi(x)$. In particular, it must vanish for the variation $\delta\phi=$ const within $\Omega$, $\delta\phi=0$ outside. But for such a variation $\delta S$ is just the sum

$$\delta\phi \sum_{j\in\Omega} \frac{\partial S}{\partial\phi(j)} .$$

(The boundary terms are negligible if the net is sufficiently fine.) We conclude that even though the thatch equations $\partial s/\partial\phi(j)$ may fail individually, their sum

$$\sum_{j\in\Omega} \frac{\partial S}{\partial\phi(j)}=0 \qquad (68)$$

over any finite region $\Omega$ will be valid.

For the thatches $A$, $l^2$ the same argument applies except that, in place of $\delta\phi=$ const, one must put a variation of the $l^2(ij)$ [respectively $\delta A(ij)$] which corresponds to $\delta g_{\mu\nu}=$ const [resp. $\delta A=$ const]. There will be ten [resp. six] linearly independent such variations.

### B. Illustration

The simplest example of these considerations is the flat scalar wave equation in two dimensions. We will examine the thatch equation (59) for various nets and show that while the continuum limit of (59) is always a homogeneous second order differential equation, it is sometimes the wrong one. As expected, however, an appropriate sum of these equations (over a "unit cell" of the lattice) always reduces to the correct equation in the continuum limit.

Let us write (59) for the vertex $[0]\in\Sigma_0$ in the form

$$\sum_k \mu(k)\phi(k)=0, \qquad (69)$$

where

$$\mu(k)\equiv\sum_\sigma \tilde{g}^{0k}(\sigma)V(\sigma) \qquad (70)$$

and $k$ ranges over $\mathfrak{S}_0(\mathfrak{S}_1([0]))$. If we expand $\phi(x)$ about $[0]$ (assuming flat space—time recall), then

$$\phi(j)=\phi(0)+\phi'(0)\cdot\overrightarrow{0j}+\tfrac{1}{2}\phi''(0)\cdot\overrightarrow{0j}\otimes\overrightarrow{0j}+\cdots, \qquad (71)$$

and (69) becomes

$$\phi(0)\sum_k \mu(k)+\phi'(0)\cdot\sum_k \mu(k)\overrightarrow{0k} + \tfrac{1}{2}\phi''(0)\cdot\sum_k \mu(k)\overrightarrow{0k}\otimes\overrightarrow{0k}+\cdots=0. \qquad (72)$$

Since one can prove in general (flat space) that

$$\sum_k \mu(k)=0, \quad \sum_k \mu(k)[k]=0, \qquad (73)$$

(72) becomes, to second order in $\overrightarrow{0k}$,

$$\tfrac{1}{2}\phi''(0)\cdot\sum_k \mu(k)[k]\otimes[k]=0. \qquad (74)$$

[The notation of the second equation of (73) makes sense because of the first, just as that of (74) in turn makes sense because of (73).] In the continuum limit this has the form

$$a^{\mu\nu}\partial_\mu\partial_\nu\phi(0)=0. \qquad (75)$$

Unfortunately, $a^{\mu\nu}\neq g^{\mu\nu}$ in general.

Consider, for example, the star shown in Fig. 4(a). The corresponding equation (75) works out as

$$-\frac{1}{2}(1+q)\frac{\partial^2\phi}{\partial x^2}-\frac{1}{2}\left(1+\frac{1}{q}\right)\frac{\partial^2\phi}{\partial y^2}=0, \qquad (76)$$

in which

$$q=\alpha(1-\alpha)/\beta(1-\beta).$$

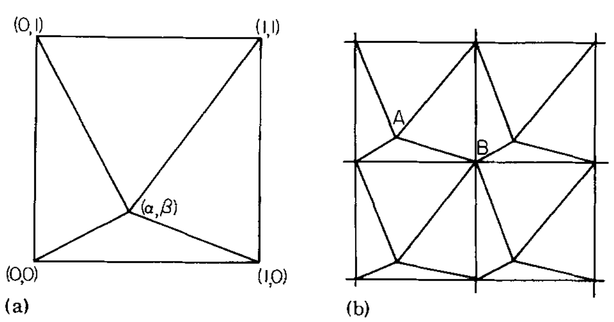


FIG. 4. (a) A star in flat two-dimensional space—time. The vertices are labelled by their rectangular coordinates. (b) A complete net made up of repetitions of this star.

(For convenience we deal with a positive definite metric.) This differs from the correct $\nabla^2\phi=0$ unless $q=1$, i.e., unless $(\alpha,\beta)$ lies on one of the diagonals of the square.

On the other hand, for a vertex such as $B$ in Fig. 4(b), (75) becomes, with the same normalization,

$$-\frac{1}{2}(3-q)\frac{\partial^2\phi}{\partial x^2}-\frac{1}{2}\left(3-\frac{1}{q}\right)\frac{\partial^2\phi}{\partial y^2}=0. \tag{77}$$

Since on the average there are equal numbers of vertices of types $A$ and $B$, the average thatch equation is the average of (76) and (77):

$$-\nabla^2\phi=0,$$

which is the correct continuum equation.

Notice that, in forming the average thatch equation, it was enough to consider one equation for each type of net vertex. In a net where all vertices were equivalent, each individual equation would already be completely typical. This explains why the nets of Part V produced the correct continuum limit without any averaging process.

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[1]R. Sorkin, "The Time Evolution Problem in Regge Calculus," Phys. Rev. D **12**, (1975), to appear.

[2]An "affine space" is just a vector space in which no point is distinguished as the "origin." Its symmetry group thus includes translations as well as the linear maps appropriate to a vector space.

[3]J. A. Wheeler, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[4]$k$-forms are also called $k$-cochains in combinatorial topology.

[5]The factor $\frac{1}{2}$ avoids double counting of legs in (40).

[6]Unfortunately, the action is not positive definite (hyperbolic equation); so there will also be some requirement for overall stability. That is not discussed at all here.