The elusive Heisenberg limit in quantum enhanced metrology

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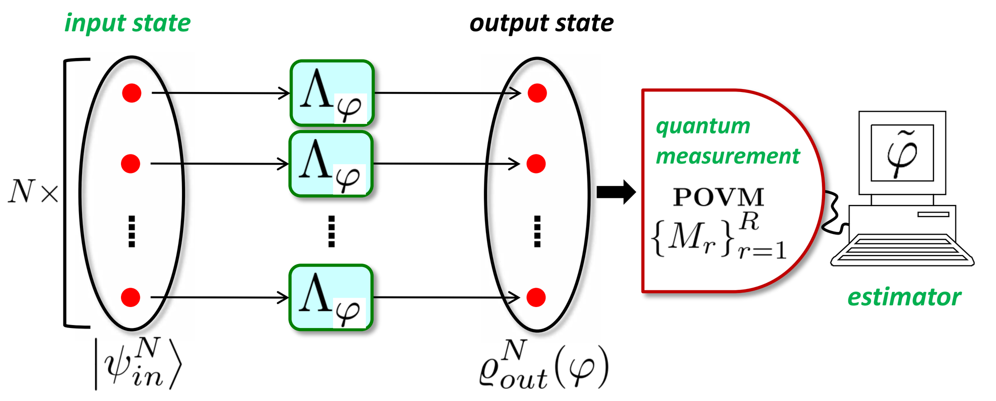
Abstract

Quantum precision enhancement is of fundamental importance for the development of advanced metrological optical experiments such as *gravitational wave detection* and *frequency calibration with atomic clocks*. Precision in these experiments is strongly limited by the **$1/N^{½}$ Shot Noise** factor with N being the number of probes (photons, atoms) employed in the experiment. Quantum theory provides tools to overcome this limit with use of *entangled* probes. While in an idealized scenario this gives rise to the **Heisenberg Scaling** of precision **$1/N$**, we show that when decoherence is taken into account, the maximal possible quantum enhancement in the asymptotic limit of $N\to\infty$, amounts generically to a constant factor rather than quadratic improvement. We provide efficient and intuitive tools for deriving the bounds based on the ***geometry of quantum channels*** and ***semi-definite programming***. We apply these tools to derive bounds for models of decoherence relevant for metrological applications including: *depolarization*, *dephasing*, *spontaneous emission* and *photonic loss in interferometry*.

General scheme of estimating φ



$$\varphi \longrightarrow \Lambda_\varphi \longrightarrow \varrho_{out}^N(\varphi)=\Lambda_\varphi^{\otimes N}[|\psi_{in}^N\rangle] \longrightarrow \tilde\varphi$$

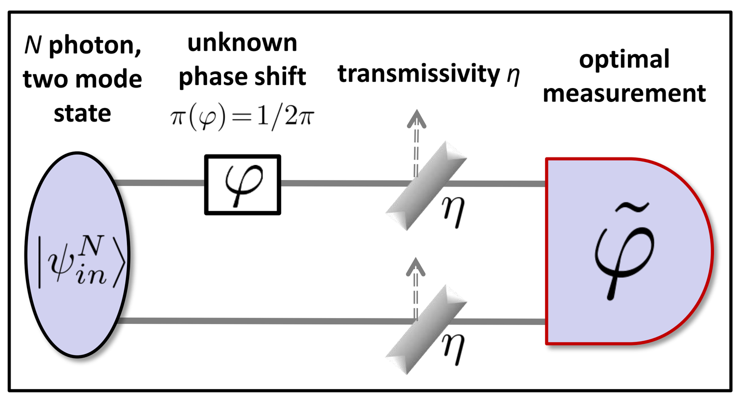
$\pi(\varphi)$ - ***prior probability distribution*** of the estimated parameter φ.

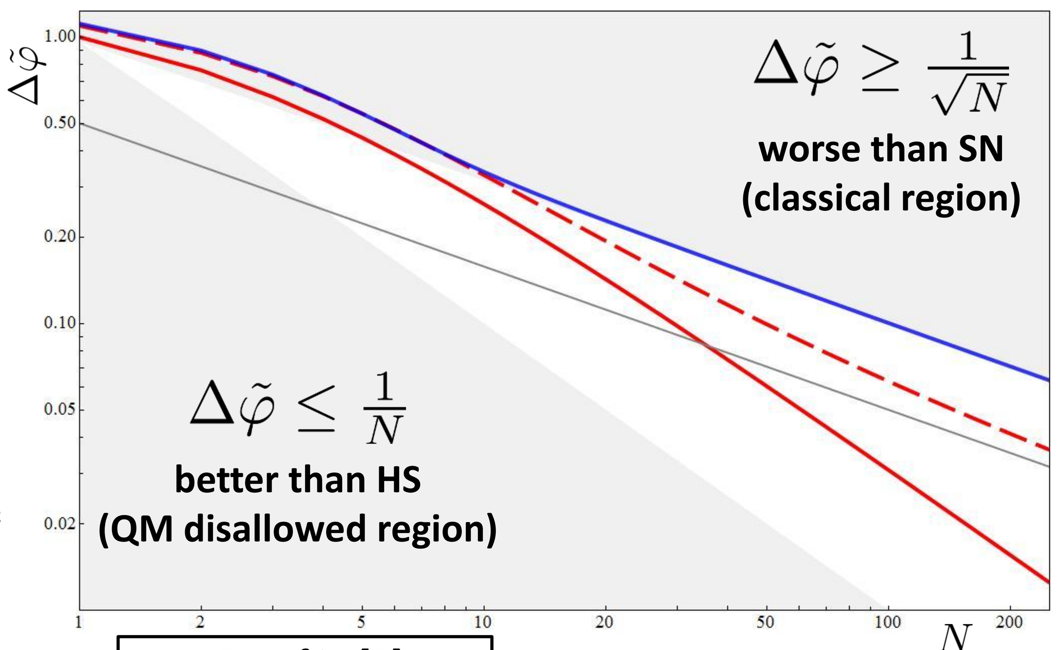
METROLOGICAL "GAME" [1]:

- Consider (*entangled*) **pure input states** of N atoms/photons: $|\psi_{in}^N\rangle$.
- Focus on the most destructive separable noise of each probe → total evolution modelled by N **independent channels**.
- Design a *strategy* of estimating $\tilde\varphi$ as close as possible to φ, which gives on average the *minimal error*: $\Delta\tilde\varphi=\sqrt{\langle(\tilde\varphi-\varphi)^2\rangle}$.
- Seek for the optimal: **input state**, **measurement scheme** and **the estimator**.

Heisenberg Scaling (HS) vs. Shot Noise (SN)

e.g. *PHASE ESTIMATION* IN AN **OPTICAL INTERFEROMETER** WITH *PHOTONIC LOSS* [2]:





Lossless (η = 1) *unitary* evolution:

- **Classical light** (split coherent beam) asymptotically, approaches the $\underset{N\to\infty}{\sim}\frac{1}{\sqrt{N}}$ **Shot Noise** scaling.
- **Optimal quantum** pure input state asymptotically, approaches the $\underset{N\to\infty}{\sim}\frac{1}{N}$ **Heisenberg Scaling**.

Lossy (η = 0.8) *non-unitary* evolution:

- **Optimal quantum** (*dashed*) pure input state asymptotically GOES BACK TO **SHOT NOISE** SCALING!!!

⇨ Proof in [2] (grey line for η=0.8): $\Delta\tilde\varphi\geq\sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$ ⇨ **INFINITESIMAL DECOHERENCE DESTROYS THE ASYMPTOTIC HEISENBERG SCALING!!!**

Ultimate bound on precision

The upper (lower) bound on average precision (error) is given by the ***quantum Cramer-Rao bound*** [3]:

$$\Delta\tilde\varphi\geq\frac{1}{\sqrt{\mathcal{F}_N}},\quad \mathcal{F}_N=\max_{\psi_{in}^N}F_Q\left[\Lambda_\varphi^{\otimes N}[|\psi_{in}^N\rangle]\right]$$

where F_Q is the **Quantum Fisher Information** (**QFI**)

Is this bound theoretically **saturable** (hence, possibly achievable in an experiment)? *YES*, but… :

- Only when the estimation is **local**, i.e. we estimate the deviations of φ from a known value φ_0, so that the *prior distribution* is fully localized → $\pi(\varphi)=\delta(\varphi-\varphi_0)$ (real *"priors"* can only do worse…).
- An optimal **POVM** exists, but may be very hard to find (and realize in an experiment…).
- An efficient estimator is proven always to exist (max. likelihood) only in the limit of **infinitely many repetitions** of the experiment (otherwise, we still need to seek for one…).

FURTHERMORE, **QFI** IS VERY **HARD TO MAXIMIZE** OVER THE INPUT FOR A **GENERAL MIXED OUTPUT**, BUT…

Quantum Fisher Information definition(s)

- **QFI** is normally defined ([3]) for a general mixed state ρ_φ dependent on the estimated parameter as

$$F_Q[\rho_\varphi]=\mathrm{Tr}\{\rho_\varphi\,\mathcal{L}_\varphi^2\}$$ with the *symmetric logarithmic derivative* implicitly specified by the relation: $$\partial_\varphi\rho_\varphi=\frac{1}{2}(\mathcal{L}_\varphi\rho_\varphi+\rho_\varphi\mathcal{L}_\varphi)$$

However, other definitions have been formulated, with which help one can directly establish **upper bounds** on the **QFI** that can be **computed analytically** when maximising over the input.

- Equivalent definitions of **QFI** via **minimization over purifications** of $\rho_\varphi=\mathrm{Tr}_E\{|\Psi(\varphi)\rangle\langle\Psi(\varphi)|\}$

[4] – $F_Q[\rho_\varphi]=\min_{\Psi(\varphi)}4\left\langle\dot\Psi(\varphi)\middle|\dot\Psi(\varphi)\right\rangle$ where $\left|\dot\Psi(\varphi)\right\rangle=\partial_\varphi|\Psi(\varphi)\rangle$

[5] – $F_Q[\rho_\varphi]=\min_{\Psi(\varphi)}4\left(\left\langle\dot\Psi(\varphi)\middle|\dot\Psi(\varphi)\right\rangle-\left|\left\langle\dot\Psi(\varphi)\middle|\Psi(\varphi)\right\rangle\right|^2\right)=F_Q[|\Psi(\varphi)\rangle]$

For the case of channel output, $\mathcal{E}_\varphi[|\psi_{in}\rangle]=\sum_i K_i(\varphi)\,|\psi_{in}\rangle\langle\psi_{in}|\,K_i^\dagger(\varphi)$, the definition [5] corresponds to

$$F_Q\left[\mathcal{E}_\varphi[|\psi_{in}\rangle]\right]=\min_K C_Q^{\mathrm{tot}}\quad\text{with}\quad C_Q^{\mathrm{tot}}=4\left(\langle\alpha_K\rangle_{in}-\langle\beta_K\rangle_{in}^2\right)$$

where $\alpha_K=\sum_i\dot K_i^\dagger(\varphi)\dot K_i(\varphi)$, $\beta_K=\mathrm{i}\sum_i\dot K_i^\dagger(\varphi)K_i(\varphi)$ and $\min_K$ stands for the **minimization over linearly independent Kraus representations**, i.e. all $K_i=\sum_j\mathbf{u}_{ij}\tilde K_j$.

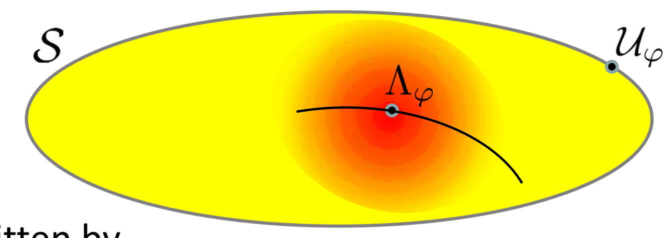
AS IN OUR MODEL $\mathcal{E}_\varphi=\Lambda_\varphi^{\otimes N}$, CAN WE CONSTRUCT AN INSTRUCTIVE UPPER BOUND ON THE PRECISION (AND THE QFI) THAT DEPENDS SOLELY ON THE SINGLE USE OF THE CHANNEL Λ_φ?

***YES*, AND IT CAN BE SUFFICIENT FOR ANALYSIS OF THE ASYMPTOTIC SCALING !!!**

Classical simulation of a quantum channel

The φ-parameterised family of channels $\{\Lambda_\varphi\}_\varphi$ forms a curve in the *convex* set of all **Completely Positive Trace Preserving** (CPTP) maps, $\mathcal{S}=\{\Lambda_X:\mathcal{B}(\mathcal{H}_{in})\to\mathcal{B}(\mathcal{H}_{out})\}_X$, of common input and output spaces.
If for every value of $\varphi=\varphi_0$, the channel is *non-extremal*, i.e. $\exists\Lambda_1,\Lambda_2\in\mathcal{S}:\Lambda_{\varphi_0}=\mu\Lambda_1+(1-\mu)\Lambda_2$, then we can interpret each channel Λ_φ as **classically simulable** by a probability distribution over all channels belonging to $\mathcal{S}$ with mean such that [6]

$$\Lambda_\varphi=\langle\Lambda_X\rangle_\varphi=\int\mathrm{d}x\,p_\varphi(x)\,\Lambda_x$$



Hence, the metrological "game" can be rewritten by means of *N independent random variables X* that are used to generate the action of N parallel channels.

$$\varphi\to p_\varphi\to X^N\to\otimes_{i=1}^N\Lambda_{X_i}[|\psi_{in}^N\rangle]\to\varrho_{out}^N(\varphi)\to\tilde\varphi$$

As then, the **classical** (sampling) scenario can only do better… ⟹ $\varphi\to p_\varphi\to X^N\to\tilde\varphi$

… we obtain an **upper bound** on the **QFI**, by the **Classical Fisher Information**, F_{cl}, of distribution p_φ, i.e.

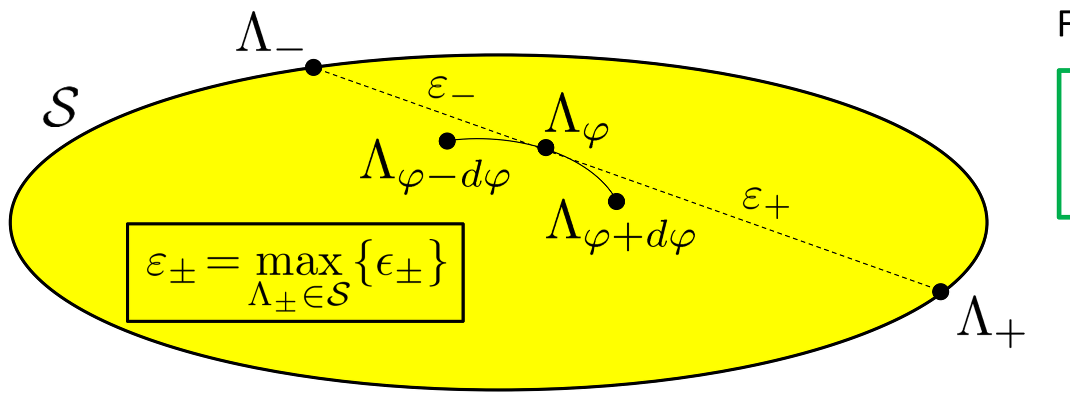
$$F_Q\left[\Lambda_\varphi^{\otimes N}[|\psi_{in}^N\rangle]\right]\leq N\,F_{\mathrm{cl}}[p_\varphi],\quad F_{\mathrm{cl}}[p_\varphi]=\int dx\frac{[\partial_\varphi p_\varphi(x)]^2}{p_\varphi(x)}.$$

BUT, WHAT p_φ IS OPTIMAL AND GIVES THE TIGHTEST BOUND?

N.B. For the bound to be practical, $F_{\mathrm{cl}}<\infty$, p_φ must be *regular* in the "direction of $d\varphi$" for all $\varphi=\varphi_0$, that is $\exists\epsilon_\pm>0:\Lambda_{\varphi_0}=\Lambda_++\Lambda_-$, where $\Lambda_\pm=\Lambda_{\varphi_0}\pm\epsilon_\pm\partial_\varphi\Lambda_\varphi|_{\varphi_0}$ and $\Lambda_\pm\in\mathcal{S}$. ⟺ Λ_{φ_0} is *φ-non-extremal*

Optimal *local* Classical Simulation (CS)

- The **QFI** of the output state $\varrho_{out}^N(\varphi)$ at given φ_0 depends only on single channel's Λ_{φ_0} and $\partial_\varphi\Lambda_\varphi|_{\varphi_0}$. Hence, the **QFI** calculated for channel Λ_{φ_0} coincides with the QFIs of all channels $\tilde\Lambda_{\varphi_0}$ that are *locally* equivalent, i.e. such that for $d\varphi=\varphi-\varphi_0$: $\Lambda_{\varphi\approx\varphi_0}=\tilde\Lambda_{\varphi_0}+d\varphi\;\partial_\varphi\tilde\Lambda_\varphi|_{\varphi_0}+O(d\varphi^2)$.
- Using this fact and the convexity of space, one can prove that the **optimal simulation** corresponds to a two-point p_φ^{opt} comprising of channels $\Lambda_\pm$ lying at the two outermost points of the set $\mathcal{S}$:



Finally, $F_Q\left[\varrho_{out}^N(\varphi)\right]\leq C_Q^{\mathrm{CS}}$ with

$$C_Q^{\mathrm{CS}}=N\,F_{\mathrm{cl}}[p_\varphi^{\mathrm{opt}}]=\frac{N}{\varepsilon_+\varepsilon_-}$$

IMPLYING ASYMPTOTIC SN SCALING!!

$$\Delta\tilde\varphi\geq\sqrt{\frac{\varepsilon_+\varepsilon_-}{N}}$$

REMARKS:

- ***φ-non-extremal*** channels can be classically simulated, hence asymptotically attain the **SN** scaling.
- Those include **full rank** channels lying strictly (not at boundaries) within the set of all CPTP maps.
- All ***extremal*** channels cannot be classically simulated, e.g. *spontaneous emission channel*.
- There exist ***φ-extremal*** channels, which are not ***extremal*** (lie on a flat boundary of $\mathcal{S}$), e.g. the *lossy interferometer channel*.

HOWEVER, FOR THOSE… → **the CE method**

The Channel Extension (CE) method

By allowing the channel to act in a trivial way on an ***extended*** input space, one can only improve the precision of estimation. ⇨ $\max_{\psi_{in}}F_Q[\mathcal{E}_\varphi[|\psi_{in}\rangle]]\leq\max_{\psi_{in}^{\mathrm{ext}}}F_Q\left[\mathcal{E}_\varphi\otimes\mathbb{I}\left[|\psi_{in}^{\mathrm{ext}}\rangle\right]\right]$

This leads to an upper bound on **QFI** that goes around the input state optimization, defined via the **minimization over Kraus representations** [4]: $\mathcal{F}_N\leq4\min_K\left\{N\,\|\alpha_K\|+N\,(N-1)\,\|\beta_K\|^2\right\}$ $\|\cdot\|$ denotes the operator norm.

Importantly, any channel that admits a Kraus representation, for which the second term vanishes, asymptotically scales like SN !!!

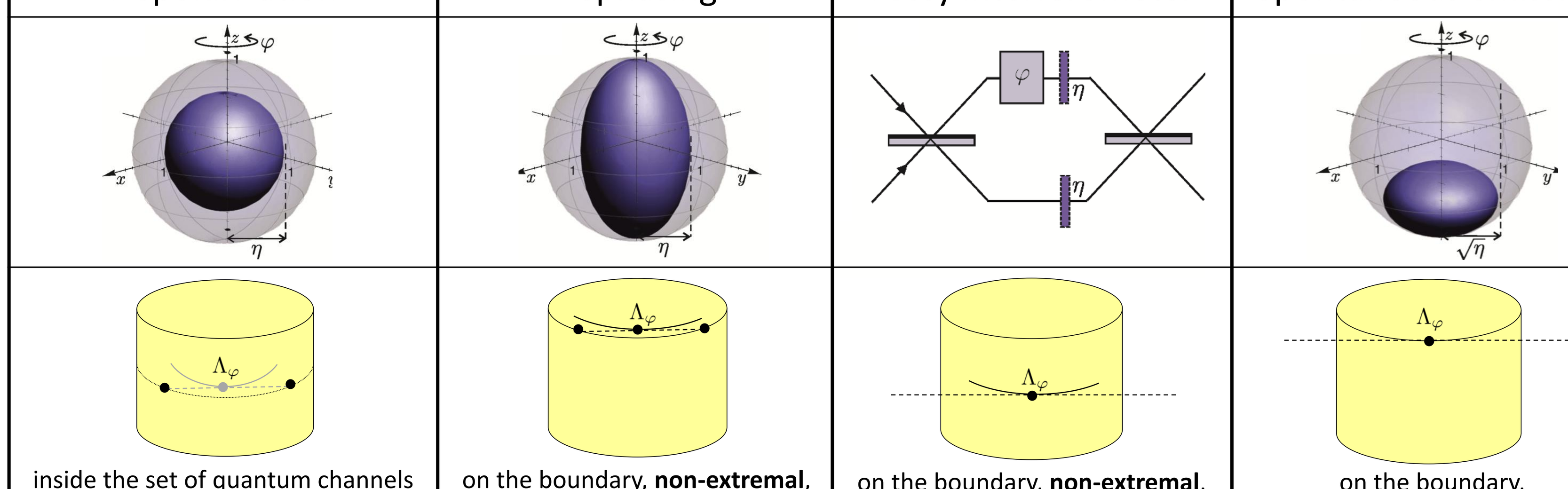
For the asymptotic regime of $N\to\infty$, the **optimal bound** then reads: ⟹ $C_Q^{\mathrm{CE}}=N\,4\min_{\mathbf{h}}\|\alpha_{\tilde K}\|$

where the Hermitian matrices $\mathbf{h}$ are any generators of unitary Kraus operators rotations $\mathbf{u}$, $\tilde K_i=\sum_j\mathbf{u}_{ij}K_j$ that satisfy the **necessary condition**: $\beta_K=\mathbf{0}$ ⟺ $\sum_{i,j}\mathbf{h}_{ij}K_i^\dagger K_j=\mathrm{i}\sum_q\dot K_q^\dagger K_q$

REMARKS:

- A numerical minimization over $\mathbf{h}$ may be efficiently performed by recasting the problem into a **semi-definite programming** optimization task.
- One may prove that for any **classically simulable** channel its optimal **CS** bound can be achieved by a special choice Kraus operator within the **CE** method (i.e. "**CS** $\subseteq$ **CE**").
- However, an open question on saturabilities and conditions for equivalence of methods remains: $F_Q=C_Q^{\mathrm{tot}}\underset{?}{\leq}C_Q^{\mathrm{CE}}\underset{?}{\leq}C_Q^{\mathrm{CS}}$

Examples and Results

Depolarization	Dephasing	Lossy interferometer		Spontaneous emission	
					
inside the set of quantum channels **full rank → φ-non-extremal**	on the boundary, **non-extremal**, **φ-non-extremal**	on the boundary, **non-extremal**, but **φ-extremal**		on the boundary, **extremal**	
C_Q^{CS} $\vee$ $\sqrt{\frac{1+3\eta}{4\eta}}\cdot\frac{1-\eta}{\eta}\frac{1}{\sqrt{N}}$	C_Q^{CS} $\Vert$ $\frac{\sqrt{1-\eta^2}}{\eta}\frac{1}{\sqrt{N}}$	C_Q^{CS}	N/A	C_Q^{CS}	N/A
C_Q^{CE} $\sqrt{\frac{1+2\eta}{2\eta}}\cdot\frac{1-\eta}{\eta}\frac{1}{\sqrt{N}}$	$C_Q^{\mathrm{CE}}=F_Q$	$C_Q^{\mathrm{CE}}=F_Q$	$\sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$	C_Q^{CE}	$\frac{1}{2}\sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$

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