The Emergence of Classical Dynamics in a Quantum World

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Ever since the advent of quantum mechanics in the mid 1920s, it has been clear that the atoms composing matter do not obey Newton's laws. Instead, their behavior is described by the Schrödinger equation. Surprisingly though, until recently, no clear explanation was given for why everyday objects, which are merely collections of atoms, are observed to obey Newton's laws. It seemed that, if quantum mechanics explains all the properties of atoms accurately, everyday objects should obey quantum mechanics. As noted in the box to the right, this reasoning led a few scientists to believe in a distinct macroscopic, or "big and complicated," world in which quantum mechanics fails and classical mechanics takes over although there has never been experimental evidence for such a failure. Even those who insisted that Newtonian mechanics would somehow emerge from the underlying quantum mechanics as the system became increasingly macroscopic were hindered by the lack of adequate experimental tools. In the last decade, however, this quantum-to-classical transition has become accessible to experimental study and quantitative description, and the resulting insights are the subject of this article.

A Historical Perspective

The demands imposed by quantum mechanics on the disciplines of epistemology and ontology have occupied the greatest minds. Unlike the theory of relativity, the other great idea that shaped physical notions at the same time, quantum mechanics does far more than modify Newton's equations of motion. Whereas relativity redefines the concepts of space and time in terms of the observer, quantum mechanics denies an aspect of reality to system properties (such as position and momentum) until they are measured. This apparent creation of reality upon measurement is so profound a change that it has engendered an uneasiness defying formal statement, not to mention a solution. The difficulties are often referred to as "the measurement problem." Carried to its logical extreme, the problem is that, if quantum mechanics were the ultimate theory, it could deny any reality to the measurement results themselves unless they were observed by yet another system, ad infinitum. Even the pioneers of quantum mechanics had great difficulty conceiving of it as a fundamental theory without relying on the existence of a classical world in which it is embedded (Landau and Lifshitz 1965).

Quantum mechanics challenges us on another front as well. From our intuitive understanding of Bayes' theorem for conditional probability, we constantly infer the behavior of systems that are observed incompletely. Quantum mechanics, although probabilistic, violates Bayes' theorem and thereby our intuition. Yet the very basis for our concepts of space and time and for our intuitive Bayesian view comes from observing the natural world. How come the world appears to be so classical when the fundamental theory describing it is manifestly not so? This is the problem of the quantum-to-classical transition treated in this article.

One of the reasons the quantum-to-classical transition took so long to come under serious investigation may be that it was confused with the measurement problem. In fact, the problem of assigning intrinsic reality to properties of individual quantum systems gave rise to a purely statistical interpretation of quantum mechanics. In this view, quantum laws apply only to ensembles of identically prepared systems.

The quantum-to-classical transition may also have been ignored in the early days because regular, rather than chaotic, systems were the subject of interest. In the former systems, individual trajectories carry little information, and quantization is straightforward. Even though Henri Poincaré (1992) had understood the key aspects of chaos and Albert Einstein (1917) had realized its consequences for the Bohr-Sommerfeld quantization schemes, which were popular at that time, this subject was never in the spotlight, and interest in it was not sustained until fairly recently.

As experimental technology progressed to the point at which single quanta could be measured with precision,

the façade of ensemble statistics could no longer hide the reality of the counterclassical nature of quantum mechanics. In particular, a vast array of quantum features, such as interference, came to be seen as everyday occurrences in these experiments.

Many interpretations of quantum mechanics developed. Some appealed to an anthropic principle, according to which life evolved to interpret the world classically, others imagined a manifold of universes, and yet others looked for a set of histories that were consistent enough for classical reasoning to proceed (Omnès 1994, Zurek in this issue). However, by themselves, these approaches do not offer a dynamical explanation for the suppression of interference in the classical world. The key realization that led to a partial understanding of the classical limit was that weak interactions of a system with its environment are universal (Landau and Lifshitz 1980) and remove the nonclassical terms in the quantum evolution (Zurek 1991). The folklore developed that this was the the only effect of a sufficiently weak interaction in almost any system. In fact, Wigner functions (the closest quantum analogues to classical probability distributions in phase space) did often become positive, but they failed to become localized along individual classical trajectories. In the heyday of ensemble interpretations, this was not a problem because classical ensembles would have been represented by exactly such distributions. When applied to a single quantum system in a single experiment, however, this delocalized positive distribution is distinctly dissatisfying.

Furthermore, even when a state is describable by a positive distribution, it is not obvious that the dynamics can be interpreted as the dynamics of any classical ensemble without hypothesizing a multitude of "hidden" variables (Schack and Caves 1999). And finally, the original hope that a weak interaction merely erases interference turned out to be untenable, at least in some systems (Habib et al. 2000).

The underlying reason for environmental action to produce a delocalized probability distribution is that even if we take a single classical system with its initial (or subsequent) positions unknown, our state of knowledge can be encoded by that distribution. But in an actual experiment, we do know the position of the system because we continuously measure it. Without this continuous (or almost continuous) measurement, we would not have the concept of a classical trajectory. And without a classical trajectory, such remarkable signals of chaos as the Lyapunov exponent would be experimentally immeasurable. These developments brought us to our current view that continuous measurements provide the key to understanding the quantum-to-classical transition.

We will illustrate the problems involved in describing the quantum-to-classical transition by using the example of a baseball moving through the air. Most often, we describe how the ball moves through air, how it spins, or how it deforms. Regardless of which degree of freedom we might consider—whether it is the position of the center of mass, angular orientation, or deviation from sphericity—in the final analysis, those variables are merely a combination of the positions (or other properties) of the individual atoms. As all the properties of each of these atoms, including position, are described by quantum mechanics, how is it that the ball as a whole obeys Newton's equation instead of some averaged form of the Schrödinger equation?

Even more difficult to explain is how the chaotic behavior of classical, nonlinear systems emerges from the behavior of quantum systems. Classical, nonlinear, dynamical systems exhibit extreme sensitivity to initial conditions. This means that, if the initial states of two identical copies of a system (for example, particle positions and momenta) differ by some tiny amount, those differences magnify with time at an exponential rate. As a result, in a very short time, the two systems follow very different evolutionary paths. On the other hand, concepts such as precise position and momentum do not make sense according to quantum mechanics: We can describe the state of a system in terms of these variables only probabilistically. The Schrödinger equation governing the evolution of these probabilities typically makes the probability distributions diffuse over time. The final state of such systems is typically not very sensitive to the initial conditions, and the systems do not exhibit chaos in the classical sense.

The key to resolving these contradictions hinges on the following observation: While macroscopic mechanical systems may be described by single quantum degrees of freedom, those variables are subject to observation and interaction with their environment, which are continual influences. For example, a baseball's center-of-mass coordinate is continually affected by the numerous properties of the atoms composing the baseball, including thermal motion, the air that surrounds it, which is also in thermal motion, and the light that reflects off it. The process of observing the baseball's motion also involves interaction with the environment: Light reflected off the baseball and captured by the observer's eye creates a trace of the motion on the retina.

In the next section, we will show that, under conditions that refine the intuitive concept of what is macroscopic, the motion of a quantum system is basically indistinguishable from that of a classical system! In effect, observing a quantum system provides information about it and counteracts the inherent tendency of the probability distribution to diffuse over time although observation creates an irreducible disturbance. In other words, as we see the system continuously, we know where it is and do not have to rely upon the progressively imprecise theoretical predictions of where it could be. When one takes into account this "localization" of the probability distribution encoding our knowledge of the system, the equations governing the expected measurement results (that is, the equations telling us what we observe) become nonlinear in precisely the right way to recover an approximate form of classical dynamics—for example, Newton's laws in the baseball example.

What happens when no one observes the system? Does the baseball suddenly start behaving quantum mechanically if all observers close their eyes? The answer is hidden in a simple fact: Any interaction with a sufficiently complicated external world has the same effect as a series of measurements whose results are not recorded. In other words, the nature of the disturbance on the system due to the system's interactions with the external world is identical to that of the disturbance observed as an irreducible component of measurement. Naturally, questions about the path of the baseball can't be verified if there are no observers, but other aspects of its classical nature can, and do, survive.

Classical vs Quantum Trajectories

Let us now turn to some significant details. To describe the motion of a single classical particle, all we need to do is specify a spatially dependent, and possibly time-dependent, force that acts on the particle and substitute it into Newton's equations. The resulting set of two coupled differential equations, one for the position x of the particle and the other for the momentum p, predicts the evolution of the particle's state. If the force on the particle is denoted by $F(x,t)$, the equations of motion are

$$\dot{x} = \frac{p}{m} \ , \tag{1}$$

and

$$\dot{p} = F(x,t) = -\partial_x V(x,t) \ , \tag{2}$$

where V is the potential.

To visualize the motion, one can plot the particle's position and momentum as they change in time. The resulting curve is called a trajectory in phase space (see Figure 1). The axes of phase space delineate the possible spatial and momentum coordinates that the single particle can take. A classical particle's state is given at any time by a point in phase space, and its motion therefore traces out a curve, or trajectory, in phase space.

By contrast, the state of a quantum particle is not described by a single point in phase space. Because of the Heisenberg uncertainty principle, the position and momentum cannot simultaneously be known with arbitrary precision, and the state of the system must therefore be described by a kind of probability density in phase space. This pseudoprobability function is called the Wigner function and is denoted by $f_W(x,p)$. As expected for a true probability density, the integral of the Wigner function over position gives the probability density for p, and the integral over p gives the probability density for x. However, because the Wigner function may be negative in places, we should not try to interpret it too literally. Be that as it may, when we specify the force on the parti-cle, $F(x,t)$, the evolution of the Wigner function is given by the quantum Liouville equa-tion, which is

$$\dot{f}_W(x,p) = -\left[\frac{p}{m}\,\partial_x + F(x,t)\partial_p\right] f_W(x,p)$$

$$+ \sum_{\lambda=1}^{\infty} \frac{1}{(2\lambda+1)!}\left(\frac{\hbar}{2i}\right)^{2\lambda} \partial_x^{2\lambda+1} V(x,t)\partial_p^{2\lambda+1} f_W(x,p) \ . \tag{3}$$

Clearly, in order for a quantum particle to behave as a classical particle, we must be able to assign it a position and momentum, even if only approximately. For example, if the Wigner function stays localized in phase space throughout its evolution, then the centroid of the Wigner function[1] could be interpreted at each time as the location of the particle in phase space.

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[1] The centroid of the Wigner function is the point in phase space consisting of the mean values of x and p, that is $(\langle x \rangle, \langle p \rangle)$.

Figure 1. Potentials and Phase-Space Trajectories for Single-Particle Systems
The figure shows four systems in which a single particle is constrained to move in a one-dimensional potential. The four systems are (a) a harmonic oscillator, (b) a double well, (c) a driven harmonic oscillator, and (d) a driven double well, also known as a Duffing oscillator. As the potentials increase in complexity from (a) to (d), so do the phase-space trajectories. In (c) and (d), the potential is time dependent, oscillating back and forth between the solid and dashed curves during each period. In (d), the force is nonlinear, and the trajectory covers increasingly more of the phase space as time passes.

(a) Harmonic Oscillator

(b) Double Well

(c) Driven Harmonic Oscillator

(d) Duffing Oscillator
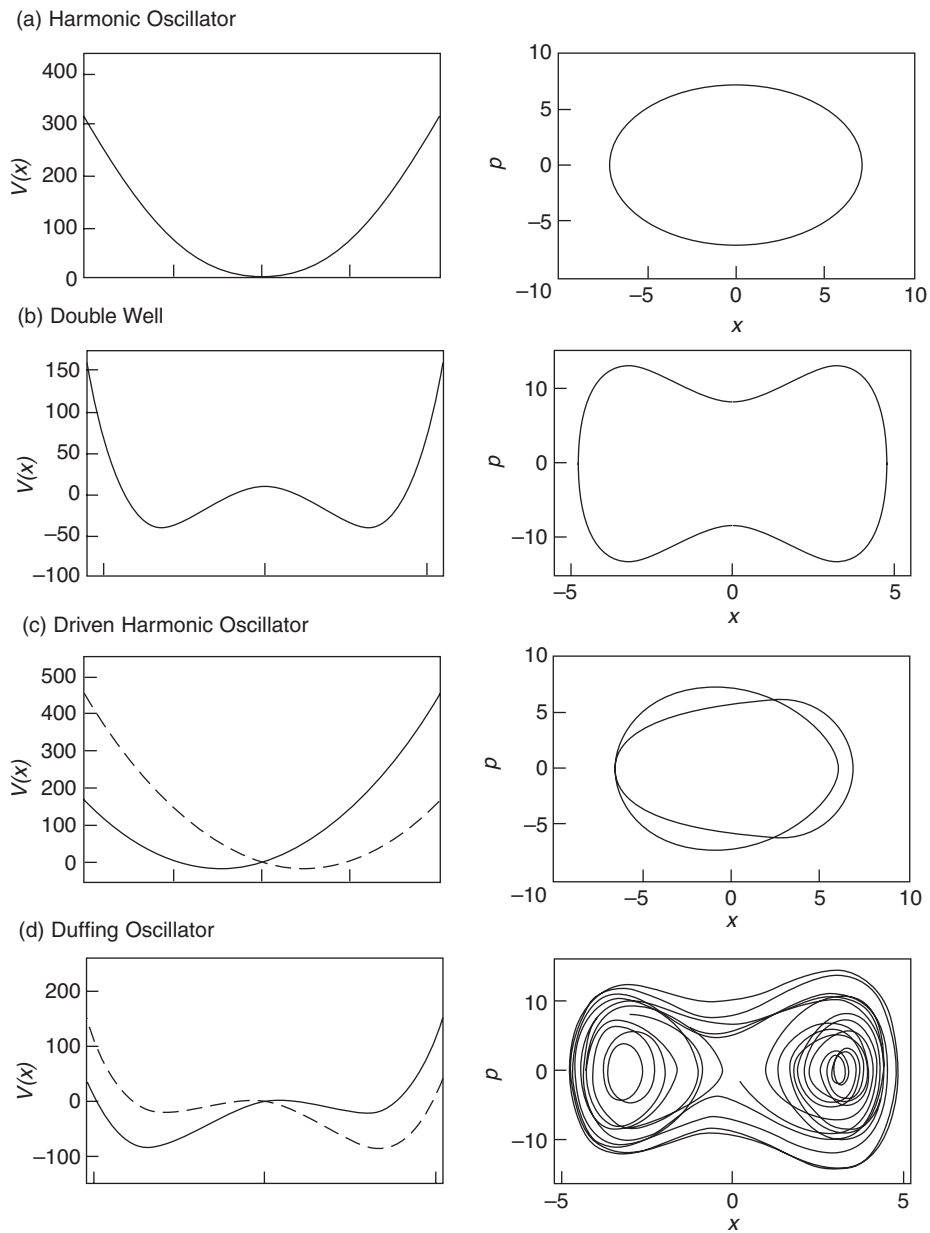
Moreover, the Liouville equation yields the following equations of motion for the centroid:

$$\langle \dot{x} \rangle = \frac{\langle p \rangle}{m} \ ,$$

(4)

and

$$\langle \dot{p} \rangle = \langle F(x,t) \rangle \ ,$$

(5)

where m is the mass of the particle. This result, referred to as Ehrenfest's theorem,[2] says that the equations of motion for the centroid formally resemble the classical ones but differ from classical dynamics in that the force F has been replaced with the average value of F over the Wigner function. Suppose again that the Wigner function is sharply peaked about $\langle x \rangle$ and $\langle p \rangle$. In that case, we can approximate $\langle F(x) \rangle$ as a Taylor series expansion about $\langle x \rangle$:

$$\langle F(x) \rangle = F(\langle x \rangle) + \frac{\sigma_x^2}{2} \partial_x^2 F(\langle x \rangle) + \ldots, \tag{6}$$

where σ_x^2 is the variance of x so that $\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle$. If the second and higher terms in the Taylor expansion are negligible, the equations for the centroid become

$$\langle \dot{x} \rangle = \frac{\langle p \rangle}{m} \quad , \tag{7}$$

and

$$\langle \dot{p} \rangle = F(\langle x \rangle, t) \quad . \tag{8}$$

And these equations for the centroid are identical to the equation of motion for the classical particle! If we somehow arrange to start the system with a sharply localized Wigner function, the motion of the centroid will start out by being classical, and Equation (6) indicates precisely how sharply peaked the Wigner function needs to be.

However, the Wigner function of an unobserved quantum particle rarely remains localized even if for some reason it starts off that way. In fact, when an otherwise noninteracting quantum particle is subject to a nonlinear force, that is, a force with a nonlinear dependence on x, the evolution usually causes the Wigner function to develop a complex structure and spread out over large areas of phase space. In the sequence of plots in Figure 2(a–d), the Wigner function is shown to spread out in phase space under the influence of a nonlinear force. Once the Wigner function has spread out in this way, the evolution of the centroid bears no resemblance to a classical trajectory.

So, the key issue in understanding the quantum-to-classical transition is the following: Why should the Wigner function localize and stay localized thereafter? As stated in the introduction, this is an outcome of continuous observation (measurement). We therefore now turn to the theory of continuous measurements.

Continuous Measurement

In simple terms, any process that yields a continuous stream of information may be termed continuous observation. Because in quantum mechanics measurement creates an irreducible disturbance on the observed system and we do not wish to disturb the system unduly, the desired measurement process must yield a limited amount of information in a finite time. Simple projective measurements, also known as von Neumann

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[2] According to Ehrenfest's theorem, a quantum-mechanical wave packet obeys the equation of motion of the corresponding classical particle when the position, momentum, and force acting on the particle are replaced by the expectation values of these quantities.
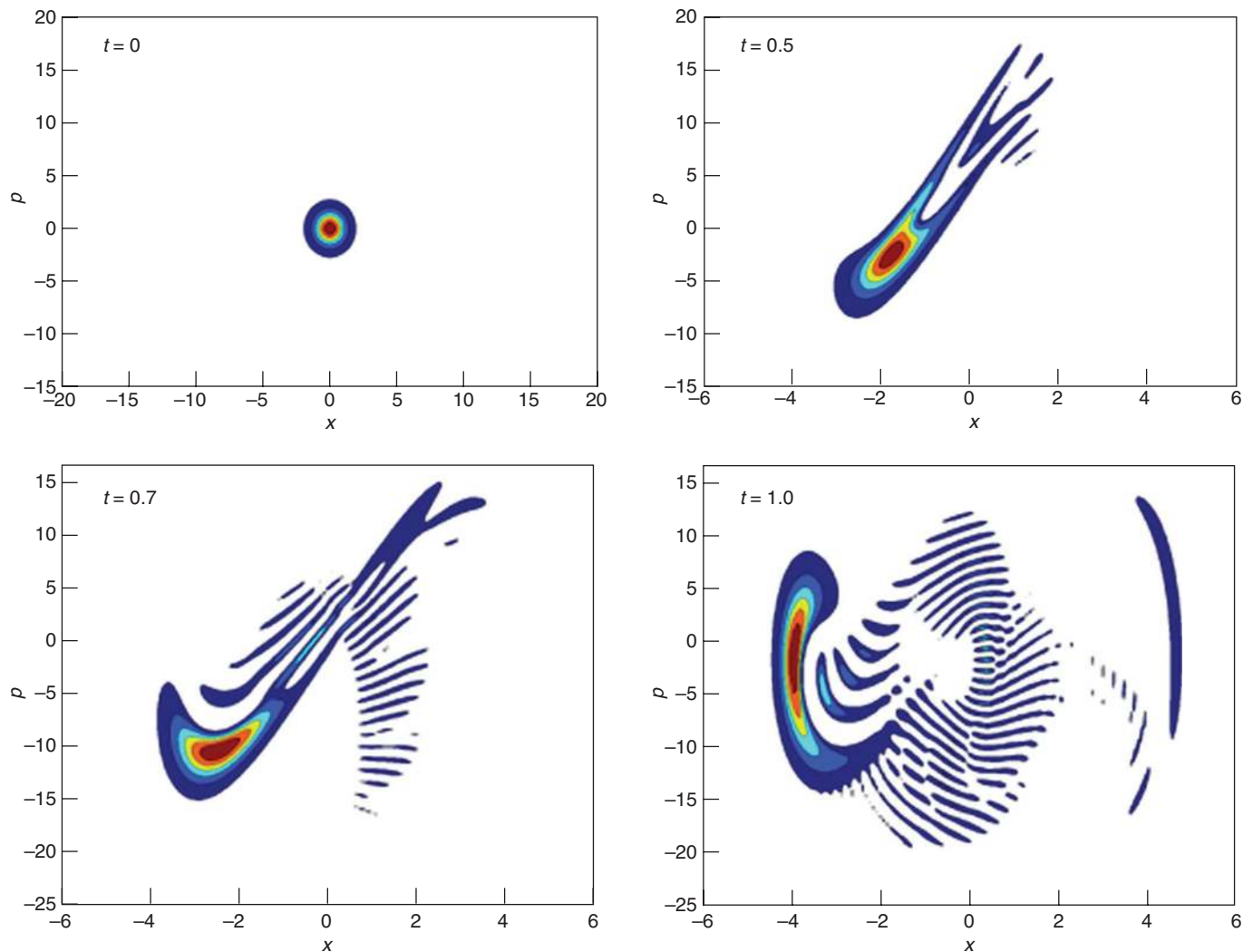
Figure 2. Evolution of the Wigner Function under a Nonlinear Force

These four snapshots show the Wigner function at different times during the simulations of the Duffing oscillator. At $t = 0$, the Wigner function is localized around a single point. As time passes, however, the Wigner function becomes increasingly delocalized under the nonlinear potential of the Duffing oscillator.

measurements, introduced in undergraduate quantum mechanics courses, are not adequate for describing continuous measurements because they yield complete information instantaneously. The proper description of measurements that extract information continuously, however, results from a straightforward generalization of von Neumann measurements (Davies 1976, Kraus 1983, Carmichael 1993). All we need to do is let the system interact weakly with another one, such as a light beam, so that the state of the auxiliary system should gather very little information about the main one over short periods and thereby the system of interest should be perturbed only slightly. Only a very small part of the information gathered by a projective measurement of the auxiliary system then pertains to the system of interest, and a continuous limit of this measurement process can then be taken. By the mid 1990s, this generalization of the standard measurement theory was already being used to describe continuous position measurement by laser beams. In our analysis, we use the methods developed as part of this effort.

A simple, yet sufficiently realistic, analogy to measuring position by direct observation is measuring the position of a moving mirror by reflecting a laser beam off the mirror and continuously monitoring the phase of the reflected light. As the knowledge of the system is initially imprecise, there is a random component in the measurement record. Classically, our knowledge of the system state may be refined to an arbitrary accuracy over time, and the random component is thereby reduced. Quantum mechanically,

however, the measurement itself disturbs the system, and our knowledge cannot be improved arbitrarily. As a result, the measurement record continues to have a random component.

An equivalent way of understanding this random component is to note that the measurement process may be characterized by the rate at which information is obtained. A more powerful measurement is one in which information is obtained at a faster rate. Because of the Heisenberg uncertainty relation, if we obtain information about position, we lose information about momentum. But uncertainty in momentum turns into uncertainty in position at the very next instant. This random feedback guarantees that a continuous measurement will cause the system to be driven by noise: The higher the rate at which information is obtained, the more the noise. For a position measurement, the rate of information extraction is usually characterized by a constant, k, that measures how fast the precision in our knowledge of position, $1/\sigma_x^2$, would increase per unit time in the absence of the accompanying disturbance. In the laser measurement of position, k is determined by the power of the laser. The more powerful the laser, the stronger the measurement, and the more noise introduced by the photon collisions.

Now we are in a position to see how and under what circumstances continuous measurement transforms quantum into classical dynamics, resulting in the quantum-to-classical transition. We can include the effects of the observation on the motion of the particle by writing down a stochastic Liouville equation, that is, a Liouville equation with a random component. This equation is given in the box "Conditions for Approximate Classical Motion under Continuous Measurement" on the next page. The resulting equations of motion for the centroid of the Wigner function are

$$\langle \dot{x} \rangle = \frac{\langle p \rangle}{m} + \sqrt{8k}\sigma_x^2 \xi(t) , \qquad (9)$$

and

$$\langle \dot{p} \rangle = F(\langle x \rangle, t) + \sqrt{8k}C_{xp}\xi(t) , \qquad (10)$$

where $C_{xp} = (1/2)(\langle xp \rangle + \langle px \rangle - 2\langle x \rangle \langle p \rangle)$ is the covariance of x and p, and $\xi(t)$ is a Gaussian white noise.[3]

We have now reached the crux of the quantum-to-classical transition. To keep the Wigner function well localized, a strong measurement, or a large k, is needed. But Equations (9) and (10) show that a strong measurement introduces a lot of noise. In classical mechanics, however, we deal with systems in which the amount of noise, if any, is imperceptible compared with the scale of the distances traveled by the particle. We must therefore determine the circumstances under which continuous measurement will maintain sufficient localization for the classical equations to be approximately valid without introducing a level of noise that would affect this scale of everyday physics.

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[3] White noise is random noise that has constant energy per unit bandwidth at every frequency. In reality, the actual recording of the measurement always occurs at a finite rate. So, effectively, the white noise gets filtered through a low-pass filter, which cuts out high frequencies.

Conditions for Approximate Classical Motion

The evolution of the Wigner function f_W for a single particle subjected to a continuous measurement of position is given by the stochastic Liouville equation:

$$f_W(x,p,t+dt) = \left[1 - dt\left[\left(\frac{p}{m}\right)\partial_x + F(x,t)\partial_p\right] + dt\sum_{\lambda=1}^{\infty}\frac{1}{(2\lambda+1)!}\left(\frac{\hbar}{2i}\right)^{2\lambda}\partial_x^{2\lambda+1}V(x,t)\partial_p^{2\lambda+1}\right]f_W(x,p)$$

$$+ \sqrt{8k}\,dt\,\xi\left(x-\langle x\rangle\right)f_W(x,p) \ , \tag{1}$$

where F is the force on the particle, ξ is a Gaussian white noise, and k is a constant characterizing the rate of information extraction. Making a Gaussian approximation for the Wigner function, which according to numerical studies is a good approximation when localization is maintained by the measurement, the equations of motion for the variances of x and p, σ_x^2 and σ_p^2, are

$$\dot{\sigma}_x^2 = \frac{2}{m}C_{xp} - 8k\sigma_x^4 \ , \ \text{ and}$$

$$\dot{\sigma}_p^2 = 2\hbar^2 k - 8kC_{xp}^2 + 2\partial_x F C_{xp} \ , \ \text{ where the covariance of } x \text{ and } p \text{ is } C_{xp} = \frac{1}{2}\left(\langle xp\rangle + \langle px\rangle - 2\langle x\rangle\langle p\rangle\right) \ .$$

$$\dot{C}_{xp} = \frac{1}{m}\sigma_p^2 - 8k\sigma_x^2 C_{xp} + \partial_x F\sigma_x^2 \ . \tag{2}$$

the noise has negligible effect in these equations when the Wigner function stays Gaussian.

First, we solve these equations for the steady state and then impose on this solution the conditions required for classical dynamics to result. In order for the Wigner function to remain sufficiently localized, the measurement strength k must stop the spread of the wave function at the unstable points, $\partial_x F > 0$:[*]

$$8k \gg \left|\frac{\partial_x^2 F}{F}\right|\sqrt{\frac{|\partial_x F|}{2m}} \ . \tag{3}$$

If noise is to bring about only a negligible perturbation to the classical dynamics, it is sufficient that, at a typical point on the trajectory, the measurement satisfy

$$\frac{2|\partial_x F|}{s} \ll \hbar k \ll \frac{|\partial_x F|s}{4} \ , \tag{4}$$

where s is the typical value of the system's action[†] in units of $\hbar$. Obviously, as s becomes much larger than $2\sqrt{2}$ this relationship is satisfied for an ever-larger range of k. At the spot where this range is sufficiently large, we obtain the classical limit.

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[*] If the nonlinearity is large on the quantum scale, $\hbar\left|\left(\partial_x^2 F\right)/F\right| \geq 4\sqrt{m|\partial_x F|}$, then $8k$ needs to be much larger than $\left(\partial_x^2 F\right)^2 \hbar/4mF^2$ irrespective of the sign of $\partial_x F$. This observation does not change the argument in the body of the paper.

[†] We are assuming that both $[mF^2/(\partial_x F)^2]|F/p|$ and $E\,|p/4F|$ evaluated at a typical point of the trajectory are comparable to the action of the system, and we define that action to be $\hbar s$.

With analytical tools alone, this problem cannot be solved. However, one can take a semianalytical approach by accepting two important results that come from numerical simulations: (1) Any Wigner function localizes under a sufficiently strong measurement, and (2) under such a measurement, once the Wigner function becomes localized, it is approximately described by a narrow Gaussian at all later times. Therefore, we assume a Gaussian form for the Wigner function, write the equations determining how the variances and covariances change with time, and solve those equations to find their values in a steady state. Having all these ingredients, we can then find the conditions under which the noise terms are small and the system remains well localized (see the box on the opposite page). Our central conclusion is that a quantum system will behave almost classically for an ever-increasing range of measurement strengths when the action of the system is large compared with the reduced Planck constant $\hbar$.

This concept may be understood heuristically in the following way: Because of the uncertainty principle, the effective area where the localized Gaussian Wigner function is nonzero can never be less than $\hbar$. If this limiting area is so large compared with the scale of the problem that it cannot be considered localized, we certainly do not expect classical behavior. Conversely, as long as the measurement extracts information at a sufficiently low rate to avoid squeezing the Wigner function to a smaller scale than the limiting one, the quantum noise remains on the scale of the variances themselves. As a result, the system behaves almost classically.

There are systems, however, whose phase space is sufficiently small for quantum effects to be manifest or even dominant. This is true, for example, of isolated spin systems with small total angular momenta. Even when they are observed and interacting with the environment, these spin systems are expected to be indescribable by the classical laws of motion. A spin coupled to other degrees of freedom such as position is a more interesting case, especially when the position of the system would have followed a classical trajectory in the absence of that interaction. To what extent, if at all, that coupling stops position from following a classical trajectory is the subject of ongoing research (Ghose et al. 2002).

Chaos in a Quantum System under Continuous Observation

As an illustration of these general ideas, we consider the Duffing oscillator, a single particle sitting in a double-well potential and driven sinusoidally—see Figure 1(d). We chose this nonlinear system because it has been studied in depth and it allows us to choose parameters that produce chaotic behavior over most of the system's phase space. Our test will indicate whether chaotic classical motion is a good approximate description of this quantum system when it is under continuous observation. To diagnose the presence of chaos, we calculate the maximal Lyapunov exponent, the most rigorous measure of chaotic behavior,[4] and compare our calculated value for the quantum system with the classical value.

The Hamiltonian for the particle in the double-well potential is

$$H = \frac{p^2}{2m} + Bx^4 - Ax^2 + \Lambda x \cos(\omega t) \,, \tag{11}$$

where m, A, B, Λ, and ω are parameters that determine the size of the particle and the

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[4] The maximal Lyapunov exponent is one of a number of coefficients that describe the rates at which nearby trajectories in phase space converge or diverge.

spatial extent of the phase space. The action should be large enough so that the particle can behave almost classically, yet small enough to illustrate how tiny it needs to be before quantum effects on the particle become dominant. Bearing this requirement in mind, we choose a mass $m = 1$ picogram, a spring constant $A = 0.99$ piconewton per meter, a nonlinearity $A/B = 0.02$ square micrometer, a peak driving force of $\lambda = 0.03$ attonewton, and a driving frequency $\omega = 60$ rad per second. Because of the weakness of the nonlinearity, the distance between the two minima of the double well is only about 206 nanometers, and the height of the potential is only 33 nano-electron-volts. The frequency of the driving force is 10 hertz. For these values, a measurement strength k of 93 per square picometer per second, which corresponds to a laser power of about 0.24 microwatt, is adequate to keep the motion classical, or the Wigner function well localized.

To study the system numerically, we allow the particle's Wigner function to evolve according to the stochastic Liouville equation for approximately 50 periods of the driving force and then check that it remains well localized in the potential. We find, indeed, that the width of the Wigner function in position (given by the square root of the position variance σ_x^2) is always less than 2 nanometers. Thus the position of the particle is always well resolved by the measurement as the system evolves. In addition, an inspection of the centroid's trajectory shows that the noise is negligible. In order to verify that the motion is, in fact, that of a classical Duffing oscillator, we perform two tests. The first is to plot a stroboscopic map showing the particle's motion in phase space and then compare that map with the corresponding one of the classical Duffing oscillator driven by a small amount of noise. The observed quantum map and the classical map are displayed in Figure 3.

The two stroboscopic maps are very similar and show qualitatively that the quantum dynamics under continuous measurement exhibits chaotic behavior analogous to classical chaos. To verify this finding quantitatively, we conduct a second test and calculate the Lyapunov exponent for both systems. As we already mentioned, trajectories that are initially separated by a very small phase-space distance, $d(0)$, diverge exponentially as a function of time in chaotic systems. The Lyapunov exponent λ, which determines the rate of this exponential increase, is defined to be

$$\lambda = \lim_{t \to \infty} \lim_{d(0) \to 0} \frac{\ln d(t)}{t} \ . \tag{12}$$

To calculate this exponent, we first choose a single fiducial trajectory in which the centroid of the Wigner function starts at the phase-space point given by $\langle x \rangle = -98$ nanometers and $\langle p \rangle = 2.6$ picograms micrometers per second (pg μm/s). At 17 intervals along this trajectory, each separated by approximately 20 periods of the driving force, we obtain neighboring trajectories by varying the noise realization. We calculate how these trajectories diverge from the initial trajectory and average the difference over the 17 sample trajectories. We then carry out this procedure for 10 fiducial trajectories, all starting at the same initial point but differing because of different noise realizations. Plotting the logarithm of this average divergence as a function of time results in a line whose slope is the Lyapunov exponent. In Figure 4, we plot the logarithm of the average divergence for both the observed quantum system and the classical system driven with a small amount of noise. The slope of the lines drawn through the curves gives the Lyapunov exponent, which in both cases is 5.7(2) per second. To show that the noise has a negligible effect on the dynamics, we also calculate the Lyapunov exponent for the classical system with no noise, using trajectories starting in a small region around the point given by $x = -98$ nanometers
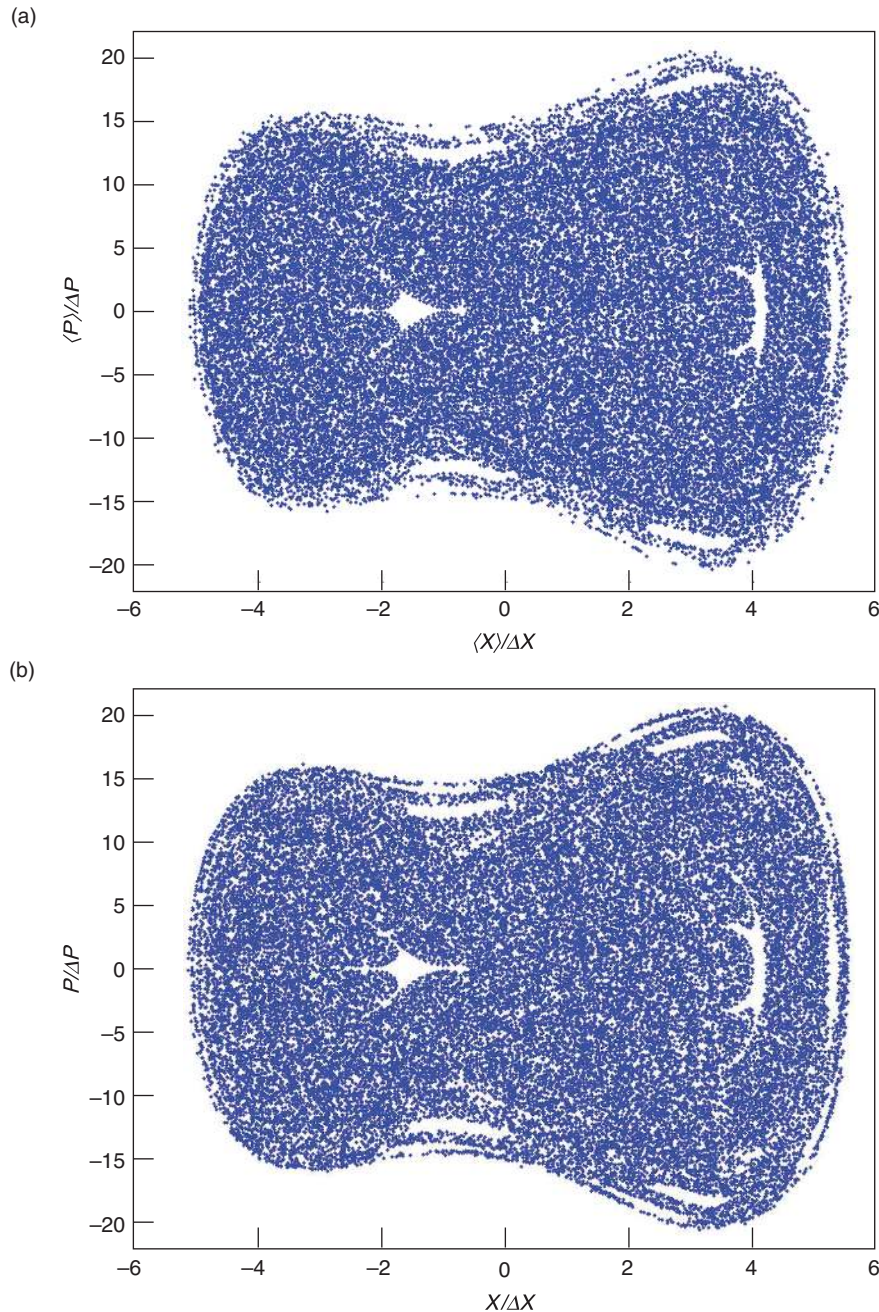
(a)

(b)

**Figure 3. Stroboscopic Maps for the Quantum and Classical Duffing Oscillators**
The results of the Duffing oscillator simulations are plotted as stroboscopic maps. (a) The map for the continuously observed quantum Duffing oscillator displays the centroids of the Wigner function at time intervals separated by the period of the driving force. This map is a pastiche from several different runs with different initial conditions, for a total duration of 39,000 periods of the temporal drive. (b) The map for the classical Duffing oscillator driven with a small amount of noise displays the calculated locations of particles in phase space at time intervals separated by the period of the driving force. The two maps are very similar. The quantum system under continuous measurement exhibits qualitatively the same chaotic behavior as the classical system driven with a small amount of noise. In these figures, ΔX = 33 nm, and ΔP = 324 pg nm/s.

and p = 2.6 pg μm/s. Those trajectories give a Lyapunov exponent of 5.6(1) per second, which is in agreement with the previous value.

Now we elaborate on the problem hinted at in the introduction. If observation realizes the classical world, do trees in remote forests fall quantum mechanically? Of course, the tongue-in-cheek answer is, "who knows?" At a deeper level, however, we note that even in a remote forest, trees continue to interact with the environment, and through this interaction, the components of the environment (reflected light, air molecules, and so on) acquire information about the system. According to unitarity, an important property of quantum mechanics, information can never be destroyed. The information that flowed into the environment must either return to its origin or stay somewhere in the environment—the decaying sound of the falling tree must yet record

Figure 4. Lyapunov Exponents for the Quantum and Classical Duffing Oscillators
In order to calculate the Lyapunov exponents, λ, for (a) a continuously observed quantum Duffing oscillator and (b) a classical Duffing oscillator driven with a small amount of noise, we plot against time the logarithm of the average separation of trajectories that begin very close together. The parameters defining the oscillator—the continuous-measurement strength in the quantum system and the noise in the classical system—are detailed on **pages 119–120** of this article. The slope of the line drawn through the curves gives the Lyapunov exponent, which in both cases is $\lambda = 0.57(2)$. Also in both cases, $\Delta_0 = 33$ nm.
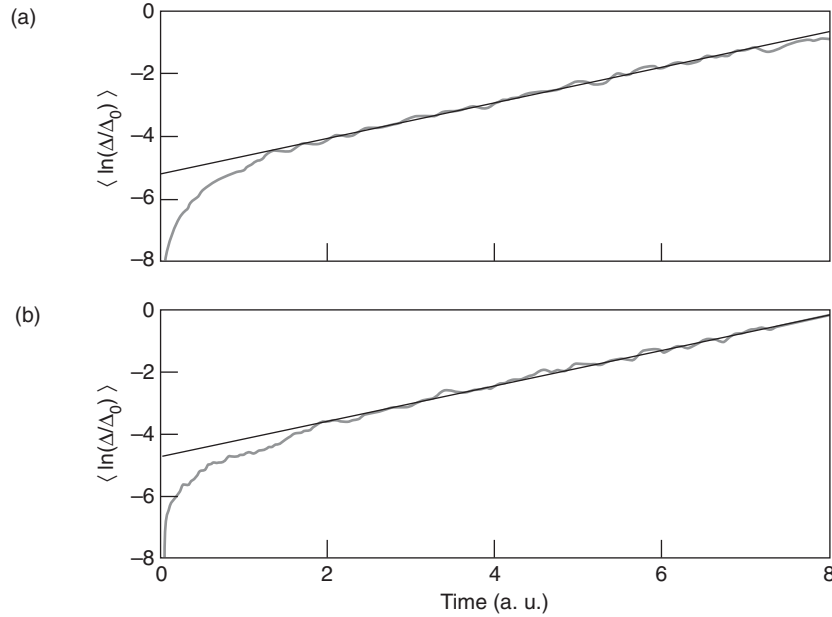
its presence faithfully, albeit perhaps only in a shaken leaf. And herein lies the key to understanding the unobserved: If a sufficiently motivated observer were to coax the information out of the environment, that action would become an act of continuous measurement of the current happenings even though actually performed in the future. But since the current state of affairs can't be influenced by what anyone does in the future, the behavior of the system at present cannot contradict anything that such a classical record could possibly postdict.

If the motion is not observed, no one knows which of the possible paths the object took, but the rest of the universe does record the path, which could, therefore, be considered as classical as any (Gell-Mann and Hartle 1993). All that happens when there is no observer is that our knowledge of the motion of the object is the result of averaging over all the possible trajectories. In that case, we are forced to describe the state of the system as being given by a probability distribution in phase space since we no longer know exactly where the system is as it evolves. This observation is, however, just as true for a (noisy) classical system as it is for a quantum system.

The Connection to the Theory of Decoherence

We can now explain how the analysis presented here relates to a standard approach to the quantum-to-classical transition often referred to as decoherence. The procedure employed in decoherence theory is to examine the behavior of the quantum system coupled to the environment by averaging over everything that happens to the environment. This procedure is equivalent to averaging over all the possible trajectories that the particle might have taken, as explained above. Thus decoherence gives the evolution of the probability density of the system when no one knows the actual trajectory. The relevant theoretical tools for understanding this process were first developed and applied in the 1950s and 1960s (Redfield 1957, Feynman and Vernon 1963), but more recent work (Hepp 1972, Zurek 1981, 1982, Caldeira and Leggett 1981, 1983a, 1983b,

Joos and Zeh 1985) was targeted at condensed-matter systems and a broader understanding of quantum measurement and quantum-classical correspondence. It was found that averaging over the environment or over the equivalent, unobserved, noisy classical system gives the same evolution (Habib et al. 1998). In this classical counterpart, different realizations of noise give rise to slightly different trajectories, and in a chaotic system, these trajectories diverge exponentially fast. As a result, probability distributions obtained by averaging over the noise tend to spread out very fast, and our knowledge of the system state is correspondingly reduced. In other words, discarding the information that is contained in the environment or, equivalently, the measurement record, as averaging over these data implies, leads to a rapid loss of information about the system. This increasing loss of information, characterized by a quantity called entropy, can then be used to study the phenomenon of chaos with varying degrees of rigor.

Averaging over the environment to produce classical probability distributions was, however, not completely satisfactory. Not only does this averaging procedure not allow us to calculate trajectory-based quantities, but it also restricts our predictions to those derivable by knowing only the probability densities at various times. But classical physics is much more powerful than that—it can predict the outcome of many "if ... then" scenarios. If I randomly throw a ball in some direction, the probability of it landing in any direction around me is the same, but if you see the ball north of me, you can predict with pretty good certainty that it won't land south of me. In the classical world, such correlations are numerous and varied, and the measurement approach we have taken here completes our understanding of the quantum-to-classical transition by treating all correlations on an equal footing. It is easy to see, however, that if the continuous measurement approach has to get all the correlations right, it must per force get the decoherence of probability densities right!

The realization that continuous measurement was the key to understanding the quantum-to-classical transition has emerged only in the last decade. First introduced in a paper by Spiller and Ralph (1994), this idea was then mentioned again by Martin Schlautmann and Robert Graham (1995). Subsequently, the idea was developed in a collection of papers (Schack et al. 1995, Brun et al. 1996, Percival and Strunz 1998, Strunz and Percival 1998). However, the scientific community was slow to pick up on this work, possibly because the authors used a stochastic model referred to as quantum state diffusion, which may have obscured somewhat the measurement interpretation. In 2000, we published the results presented in this article, namely, analytic inequalities that determine when classical motion will be achieved for a general single-particle system, and showed that the correct Lyapunov exponent emerges (Bhattacharya et al. 2000). For this purpose, we used continuous position measurement, which is ever present in the everyday world and therefore the most natural one to consider. This accumulation of work now provides strong evidence that continuous observation supplies a natural and satisfactory explanation for the emergence of classical motion, including classical chaos, from quantum mechanics. In addition, such an analysis also makes clear that the specific measurement model is not important. Any environmental interaction that provides sufficient information about the location of the system in phase space will induce the transition in macroscopic systems. Recently, Andrew Scott and Gerard Milburn (2001) have analyzed the case of continuous joint measurement of position and momentum and of momentum alone, and they verified that classical dynamics emerges in the same way as described in Bhattacharya et al. (2000). ■

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