THE ENDOMORPHISM SEMIGROUP OF A SPECIAL SEMIGROUP

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ABSTRACT
The endomorphism semigroup for a class of commutative semigroups, called special semigroups, will be studied their structures will be determined in some important cases.

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1. INTRODUCTION

Majumdar and Mallick[3] introduced the concept of special abelian groups. They studied these groups and classified them in most cases. Generalising this idea to semigroups, Hossain and Majumdar[2] defined special semigroups and proved some of their properties including the fact that many classes of special semigroups are closed under direct sums. It has been observed that there are three semigroups:

(i) Z^+, the additive semigroup of all positive integers,

(ii) $Q+$, the additive semigroup of all positive rational numbers, and

(iii) $N(2)$, the additive semigroup of all positive rational numbers which have denominators of the form 2^r, r being a non-negative integer, form the building blocks for many classes of special abelian semigroups including free commutative semigroups and divisible commutative semigroups, where the direct sum is used to connect the building blocks together.

In this paper, we shall consider the endomorphism semigroups. Here, all special semigroups will be written additively, and by the endomorphism semigroup of a special semigroup S we will mean the additive commutative semigroup *Hom (S,S)*. We use the terminology and notations of [1,3,4].

2. We now recall some definitions and results for convenience. An additive commutative semigroup S is said to be *special* if S has a subset X such that non-identity (i.e., non-zero) element s of S has unique expression $s = \sum_{x_i \in X} n_i x_i$, where n_i = 0 or 1, only finitely many n_i, s being equal to 1; X is called a set of special generators.
A semigroup S is called an inverse semigroup[1,4] if for each element $a \in S$, there exists an element $b \in S$ such that $aba = a$, $bab = b$. b is uniquely determined by a. b is usually written a^{-1} and called the inverse of a. It is clear that if a belongs to a set X of special generators, then $a^{-1} \notin X$.
It was shown in[2] that Z^+, Q^+, $N(2)$ are special semigroups. If $\{A_a\}$ is a family of subsemigroups of an additive semigroup S, the sum $\sum_a A_a$ is the subsemigroup of S consisting of all fintite sums $a_{a_i} + ... + a_{a_r}$, where $a_{a_i} \in A_{a_i}$ and are all distinct. Let $\hat{A}_\beta = \sum_{a \neq \beta} A_a$. If none of the Aa' s is a monoid and $A_a \cap \hat{A}_\beta = \Phi$, for each $\beta(\alpha \neq \beta)$,

then $\sum_a A_a$ is called the direct sum of the A_a's. The direct sum is then written as $\sum_a \oplus A_a$. Now $\sum_a \oplus A_a$ is called a free commutative semigroup if each A_a is isomorphic to Z^+. The following theorem was established in[2]:

Theorem 1: $\sum_a \oplus A_a$ *is special if and only if each Aa is special.*

We use this result to prove the following:

Theorem 2 : *A free semigroup S is special.*
Proof : It is obvious from the definition of direct sum that $S = \sum_{x \in X} \oplus S_x$ where S_x is the infinite cyclic semigroup generated by x, *i.e.*, $S_x = \{x, 2x, 3x, \ldots\ldots, nx, \ldots\}$. Clearly, each $S_x \cong Z^+$, the additive semigroup of all positive integers. It thus follows from the theorem 1 that S is special.
An additive commutative semigroup S will be called *divisible* if, for each $s \in S$ and for each positive integer n, there exists an element $s' \in S$ such that $s = ns'$. Obviously Q^+ is a divisible semigroup whereas Z^+ and $N(2)$ are not divisible.
Also, it is clear that if $\{S_a\}$ is a family of divisible semigroups, then $\sum_a \oplus S_a$ is divisible.

It therefore follows that:

Theorem 3 : A commutative semigroup S is divisible and special if $\overline{S} = \sum_a \oplus \overline{A}_a$, where $\overline{S} = s - \{0\}$ *and* $\overline{A}_a \cong Q^+$, *for each a.*
Theorem 4 : *If S be an additive commutative semigroup and* $\overline{S} = S - (0)$, *then S is special if* $\overline{S} = \sum_a \oplus A_a \oplus \sum_\beta \oplus B_\beta \oplus \sum_\gamma \oplus C_\gamma$, *where each* $A_a \cong Z^+$, *each* $B_\beta \cong Q^+$, *and each* $C_\gamma \cong N(2)$.
Proof: This is a consequence of theorem 1, theorem 2, and theorem 6[2]

3. Endomorphisms of Special Semigroups

As a consequence of Theorem 4 we see that for determination of the structure of the endomorphism semigroup *End S = Hom* (*S*,*S*) of a special semigroup $S = \sum_\oplus S_a$ we need to
(i) determine *End* S_α, where S_α is any of Z^+, Q^+ and $N(2)$,
(ii) express *End* $\left(\sum_a \oplus S_a\right)$ in terms of End S_α

with an effective method of gluing them together so that the required structure can be read off from the structures of End S_a'*S.*
We first solve the problem (i) by proving the following theorem:

Theorem 5
$(i)\, End\ Z^+ \cong Z^+$
$(ii)\, End\ Q^+ \cong Q^+$

$(iii)\, End\,(N(2)) \cong N(2)$.

Proof: (i) If φ is an endomorphism of Z^+, let $\varphi(1) = n$. Then $\varphi(r) = rn$, for all $r \in Z^+$. Thus, $\varphi \xrightarrow{f} \varphi(1)$ is an isomorphism of $End\ Z^+$ onto Z^+.

(ii) As in (i) it can be proved that every endomorphism as well as every automorphism of Q^+ too is given by the map $\hat{r} : x \to rx$, for a fixed $r \in Q^+$. $r \to \hat{r}$ gives an isomorphism $Q^+ \to End\ Q^+$. Thus, $End\ Q^+ \cong Q^+$.

(iii) Since each element of $N(2)$ is a rational number of the form $\frac{m}{2'}(r \geq 0)$, it follows from (iii) that a map $\varphi : N(2) \to N(2)$ is an endomorphism if and only if $\varphi(x) = x.\ \varphi(1)$, for each $x \in N(2)$. Also $\varphi(1)$ may be any element of $N(2)$. Hence $\varphi \xrightarrow{f} \varphi(1)$ gives an isomorphism of $End\ N(2)$ onto $N(2)$.

We shall now try to solve the problem (ii) with the restriction that the set of α's is finite i.e., S a finite direct sum $(S_{\alpha_1} \oplus \ldots\ldots \oplus S_{\alpha_n})$.
We also have the following theorem:
Theorem 6: $End(S) \cong End(S_{\alpha_1} \oplus \ldots\ldots \oplus S_{\alpha_n}) \cong \sum_{i,j} \oplus Hom(S_{\alpha_i}, S_{\alpha_j})$.
(Here Hom (S_{a_i}, S_{a_j}), if non-empty, denotes the commutative semigroup of all homomorphisms of $f : S_{a_i} \to S_{a_j}$ when such homomorphisms exist, and the sum is over all such semigroups.)

Proof: The theorem follows by induction from the isomorphism

$\phi : Hom(\overline{A} \oplus \overline{B}, \overline{C}) \to Hom(\overline{A}, \overline{C}) \oplus Hom(\overline{B}, \overline{C})$ and

$\varphi : Hom(\overline{A}, \overline{B} \oplus \overline{C}) \to Hom(\overline{A}, \overline{B}) \oplus Hom(\overline{A}, \overline{C})$ given by $\phi(f) = (f\, t_A, f\, t_B)$ and $\varphi(g) = (\pi_B g, \pi_c g)$, where $\overline{A}, \overline{B}, \overline{C}$ are the obvious monoids into which A, B, C maybe imbedded, and t_A, t_B, π_B, π_C are the injections and the projections. Clearly, ϕ is well defined and φ yields isomorphisms

$\overline{\phi} : Hom(A \oplus B, C) \to Hom(A, C) \oplus Hom(B, C)$ and

$\phi : Hom(A, B \oplus C) \to Hom(A, B) \oplus Hom(A, C)$.

Hence, by induction, $(A_1 \oplus \ldots\ldots \oplus A_m, B_1 \oplus \ldots\ldots \oplus B_n) \cong \sum_{\beta=1}^{n} \sum_{\alpha=1}^{m} \oplus Hom(A_a, B_\beta)$. The problem (ii) has thus been solved.

The determination of End (S) will therefore be completed from the following results.

Theorem 7: $Hom(Z^+, Q^+) \cong Q^+$.

Proof. Let t: $Z^+ \to Q^+$ be the inclusion map. $n \in Z^+, let \frac{t}{n}: Z^+ \to Q^+$ be the map $\frac{t}{n}(m) = \frac{m}{n}$. Then $\frac{t}{n}$ is a homomorphism. Also nt: $Z^+ \to Q^+$ given by $(nt)(m) = mn$ is a homomorphism. Thus, $\left\{\frac{m}{n}t : m,n \in Z^+\right\} \subseteq Hom(Z^+, Q^+)$.

Next, let $f \in Hom(Z^+, Q^+)$ and let $f(1) = x = \frac{m}{n}$, say, where $m, n \in Z^+$. Then,

$f = \frac{m}{n}t.$

Hence $Hom(Z^+, Q^+) = \left\{\frac{m}{n}t : m,n \in Z^+\right\} \cong Q^+$.

Theorem 8 : $Hom(Z^+, N(2)) \cong N(2)$.

Proof: Let $f(1) = x = \frac{a}{2^r}$, $a \in Z^+, r \geq 0$ so that $f(n) = \frac{an}{2^r}$. Thus $f \xrightarrow{\phi} f(1)$ is a homomorphism. Also, $g(1) = f(1) \Rightarrow g(n) = f(n), \forall n \Rightarrow g = f$ so that ϕ is 1-1.

Now, $\forall \frac{a}{2^r}, a \in Z^+, r \in Z, r \geq 0, \phi(f) = \frac{a}{2^r}$ where $f(n) = \frac{an}{2^r}$. Clearly, $f \in Hom(Z^+, N(2))$. Then $\phi(f) = f(1) = \frac{a}{2^r}$ i.e., ϕ is onto. Therefore ϕ is an isomorphism.

Theorem 9 : $Hom(Q^+, Z^+) = \Phi$.

Proof : If $Hom(Q^+, Z^+) \neq \Phi$, $let\ f \in Hom(Q^+, Z^+)$. $Let\ f(1) = m \in Z^+$. Then, for each $n \in Z^+, f\left(\frac{1}{n}\right) = \frac{m}{n}$, since $f(1) = n.\phi\left(\frac{1}{n}\right) = m$. Hence for each $n \in Z^+, \frac{m}{n} \in Z$ which is absurd. Hence the theorem.

Theorem 10: $Hom(N(2), Z^+) \cong \Phi$.

Proof: If $Hom(N(2), Z^+) \neq \Phi$, let $f \in Hom(N(2), Z^+)$ and let $f(1) = m$.
Then, $f\left(\frac{1}{2^r}\right) = \frac{m}{2^r}$, for each r ≥ 0. This means, $\frac{m}{2^r} \in Z^+$, for each r ≥ 0 which is absurd, Hence the result.

Theorem 11: $Hom(N(2), Q^+) \cong Q^+$.

Proof: Let $\varphi : Hom(N(2), Q^{+})$ and $\varphi(1) = \dfrac{m}{n}, m, n \in Z, n > 0.$ Let $x \in N(2)$, then

$x = \dfrac{a}{2^r}$, for some $a \in Z^{+}, r \in Z, r \geq 0.$ Since φ is a homomorphism,

$\varphi(x) = \dfrac{a}{2^r} . \dfrac{m}{n} = x . \dfrac{m}{n}$ by arguments used earlier. Clearly, $\varphi \to \varphi(1)$ is an isomorphism. The proof is thus complete.

Theorem 12: $Hom(Q^{+}, N(2)) = \Phi$

Proof: If $Hom(Q^{+}, N(N)) \neq \Phi$, let $f \in Hom(Q^{+}, N(2))$. Then $f(1) = \dfrac{a}{2^r}$ for some $a \in Z^{+}, r \in Z, \geq 0$ and so, $f\left(\dfrac{1}{2n+1}\right) = \dfrac{a}{(2n+1)2^r} \notin N(2)$. In particular, $f\left(\dfrac{1}{3}\right) = \dfrac{a}{3.2^r} \notin N(2)$. Hence $Hom(Q^{+}, N(2)) = \Phi$.

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