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THE ENERGY CONTENT OF A WHITE DWARF AND ITS RATE OF COOLING

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Summary

It is known that in a zero-temperature 'black dwarf' star, the ions arrange themselves in a nearly rigid lattice structure, with the associated Coulomb energy reducing the total energy somewhat below that of the Chandrasekhar model. In a white dwarf of moderate internal temperature the lattice structure is preserved, and besides the zero-point oscillation energy of the lattice there is a thermal oscillation energy, partly kinetic and partly Coulomb. At higher temperatures the lattice begins to melt, but since the melting takes place at nearly constant volume, there should be no discontinuous change in the specific heat. Over the whole temperature range of interest for most white dwarfs it is probably correct to use the Dulong–Petit specific heat: only at temperatures high enough for vapourization will the ionic specific heat become that appropriate to a classical gas.

As the star cools and contracts slightly, gravitational energy is released comparable to the change in the thermal energy; however, almost all of it is absorbed by the increase in exclusion energy accompanying the density increase, so that it is semantically correct to speak of a white dwarf as radiating at the expense of its thermal energy. Because of the Coulomb contribution to the thermal energy, the computed lifetimes are twice earlier estimates.

1. *Energy change in a cooling white dwarf.* As a white dwarf cools towards its final zero-temperature state there is a slight decrease in its radius; in addition to its loss of thermal energy, there is a consequent decrease in gravitational potential energy and an increase in electron degeneracy energy as the electron Fermi level is raised. All three of these energy changes are generally of the same order of magnitude, although they are not usually considered together in estimates of the energy budget and cooling time for white dwarfs. We shall present some elementary considerations on the relations among these energy changes, and upon the net expendable energy, which bear on white dwarf evolution times. Only spherically-symmetric models are considered—centrifugal and magnetic perturbations are ignored.

We define in the usual way the total gravitational energy Ω of a white dwarf and similarly Ω_0 for a zero-temperature black dwarf. The total kinetic energy U is the sum of separate contributions from electrons and ions:

$$U = U(e) + U(i), \tag{1}$$

where the main contribution to $U(e)$ is the zero-point exclusion or degeneracy energy $U_d(e)$ of the electrons. There is also an important contribution from the Coulomb interaction energy V, the sum of electrostatic interactions among electrons and ions. This Coulomb energy V_0 for the black dwarf is the (negative) difference between the Coulomb energy of the ions in a lattice and that of a uniform charge distribution (Auluck & Mather 1959, Salpeter 1961), together with the average potential energy associated with the quantum zero-point motion of the ions in the lattice. Because of vibrations, V for the white dwarf depends on temperature as well as on density—the thermal energy includes a contribution from V as well as from U.* The ratios V/U and V/Ω are numerically of order $Z^{2/3}r_e/a_0$, where Z is the ion atomic number, r_e the inter-electron spacing and a_0 the Bohr radius. This number is always well below unity over the 'pressure-ionized' bulk of a white dwarf (Salpeter 1961); however, we shall see that the *change* in V between black and white dwarf states is comparable with the changes in Ω and U, and cannot be ignored.

Because of its finite thermal energy, a white dwarf has slightly lower densities than a black dwarf. For small changes we may write

$$\Delta U \equiv U - U_0 = \Delta^\rho U + \Delta^T U, \tag{2}$$

where Δ^T refers to the thermal change in U from the black dwarf value U_0 calculated with the density distribution fixed at that of the black dwarf, and Δ^ρ refers to the change in U contributed by the change in density alone evaluated at fixed temperature $T=0$. Similarly

$$\Delta V = \Delta^\rho V + \Delta^T V. \tag{3}$$

Because Ω depends explicitly only upon the density distribution

$$\Delta\Omega = \Delta^\rho \Omega. \tag{4}$$

The total change in energy is given by

$$\Delta E = \Delta^T(U+V) + \Delta^\rho(U+V+\Omega). \tag{5}$$

But in any small deviation from the black dwarf equilibrium state, the change in the total energy $(U+V+\Omega)$, computed at zero temperature, must vanish to the first order:

$$\Delta^\rho(U+V+\Omega) = 0. \tag{6}$$

This particular application of a general principle may be readily verified; if the radius r of the mass sphere m is altered by Δr, the first-order change in Ω for a star of mass M_s is

$$\begin{aligned}\Delta^\rho\Omega &= \int_0^{M_s} \frac{Gm}{r^2}\,\Delta r\,dm \\ &= -\int_0^{M_s} \frac{dp}{dm}\,(4\pi r^2 \Delta r)\,dm \\ &= \int_0^{M_s} p\,\frac{d}{dm}\,(4\pi r^2\Delta r)\,dm \\ &= \int_0^{M_s} p\Delta\left(\frac{1}{\rho}\right) dm \\ &= -\Delta^\rho(U+V),\end{aligned} \tag{7}$$

where use has been made of the thermodynamic relation for pressure p at zero temperature in terms of the kinetic energy density u and the Coulomb energy density v

$$p = -\frac{\partial\left(\dfrac{u+v}{\rho}\right)}{\partial\left(\dfrac{1}{\rho}\right)}. \tag{8}$$

* This point was made to one of us indirectly by Dr E. E. Salpeter.

In the last line mass conservation has been applied in the form $1/\rho = 4\pi r^2\, \partial r/\partial m$. From equations (5) and (6) we have for small changes from the black dwarf state

$$\Delta E = \Delta^T(U+V). \tag{9}$$

Thus the first-order change in energy of the white dwarf is just the change with temperature in the Coulomb plus kinetic energies evaluated at fixed density, i.e.

$$\frac{dE}{dT} = \int dV C_V(T), \tag{10}$$

where $C_V(T)$ is the heat capacity per unit volume evaluated at fixed volume and the integral is over the entire volume of the star. It is thus correct to speak of a white dwarf without nuclear sources as radiating at the expense of the thermal energy (kinetic and potential); with a decline in temperature the star contracts slightly and some gravitational energy is released, but most of it is absorbed by the increase with density in the kinetic plus electrostatic potential energy (chiefly the electron exclusion energy, from (6) and (12) below).

At sufficiently high temperatures the ion lattice structure is dissolved completely, and any further change in density will not yield any corresponding changes in $\Delta^\rho V$ and $\Delta^\rho U(i)$: use of the black dwarf formulae is no longer valid. However, we shall see in (23–25) below that ΔE, $\Delta^\rho U(e)$ and $\Delta^\rho \Omega$ are usually comparable. Since numerically

$$\Delta^\rho U(i) \leqslant \Delta^\rho V \ll \Delta^\rho U(e) \tag{11}$$

for the white dwarf, there is a fractional error in ΔE only of order $Z^{2/3} r_e/a_0$ in assuming both

$$\Delta^\rho U(i) = 0,$$
$$\Delta^\rho V = 0 \tag{12}$$

at all temperatures, and we shall now make these approximations. Equation (9) is valid as long as $|\Delta\rho/\rho| \ll 1$ relative to the black dwarf state, and it may be applied to the difference in energy between any two states of a white dwarf which satisfy this criterion.

2. *Relations among changes in gravitational, Coulomb and kinetic energies.* We now give estimates of the relevant magnitude of $\Delta\Omega$, $\Delta^\rho U(e)$, and ΔE for a cooling white dwarf.

The dominant term in the pressure which balances the self-gravitation is that of a fully degenerate gas of free electrons (Chandrasekhar 1939)

$$p_d = Af(x) \equiv \frac{\pi m_e^4 c^5}{3h^3} [(x^2+1)^{1/2} x(2x^2-3) + 3 \sinh^{-1} x], \tag{13}$$

with x related to the mass density ρ by

$$\rho = Bx^3 \equiv \frac{8\mu_e M_H}{m_e c^2} Ax^3. \tag{14}$$

Here m_e is the electron mass, M_H the proton mass, c the velocity of light, h Planck's constant, and $\mu_e = A/Z$ is the ratio of the mean atomic weight to the mean ionic charge of the nuclear species present. The associated degeneracy energy density is

$$u_d = Ag(x) \equiv \frac{\pi m_e^4 c^5}{3h^3} \{8x^3[(x^2+1)^{1/2} - 1] - f(x)\}. \tag{15}$$

We write

$$\frac{p_d}{u_d}=\frac{f(x)}{g(x)}\equiv\frac{2}{3}\eta(x), \tag{16}$$

where $\eta(x)$ decreases monotonically from unity for $x\ll 1$ (non-relativistic energies) to $\frac{1}{2}$ at $x\gg 1$ (relativistic energies).

In the Chandrasekhar approximation to a *black* dwarf, the degenerate electron pressure is the only force opposing gravity. The virial thorem therefore yields for the gravitational energy

$$\begin{aligned}\Omega &= -3\int p_d dV = -2\int \eta u_d dV \\ &\equiv -2\bar{\eta}\int u_d dV = -2\bar{\mu}U_d,\end{aligned} \tag{17}$$

where the integrals are over the volume of the star; U_d is the total ' exclusion ' energy, and the average $\bar{\eta}$ is defined by equations (17). The total energy E_0 is therefore

$$E_0 = U_d+\Omega = -U_d(2\bar{\eta}-1) = \Omega(1-1/2\bar{\eta}). \tag{18}$$

In black dwarfs of low mass the electrons are everywhere non-relativistically degenerate, so that $\bar{\eta}\simeq 1$, and the relations (18) reduce to $E=-U_d=\Omega/2$, as for a non-degenerate star of monatomic gas. As the mass approaches the Chandrasekhar limit M_c, $\bar{\eta}\rightarrow\frac{1}{2}$, while simultaneously the radius goes to zero; although U_d and Ω each becomes infinite, the total energy (18) decreases to a finite limit

$$-GM_c^2/2\cdot 018 l_1$$

(Savedoff 1963). Here l_1 is the natural length unit $(2A/\pi G)^{1/2}/B$, and the factor $2\cdot 018$ comes from the $n=3$ polytrope, which is the limit of the density distribution as $M\rightarrow M_c$. For a normal hydrogen-free star, $\mu_e=2$, $M_c=1\cdot 44 M_\odot$ and the limiting energy is -7×10^{50} erg.

In the next approximation (still at zero temperature) we must include the effect of the electrostatic forces due to the ionic lattice structure and the associated energy V. Salpeter (1961) treats these forces as contributing a term $p_{e.s.}$ towards the total pressure, as in (8). Since the Coulomb energy per electron is $\propto\rho^{1/3}$, $p_{e.s.}=\frac{1}{3}v$, and the modified virial theorem, given from (17) by replacing p_d by $(p_d+p_{e.s.})$, becomes

$$\Omega+V+2\bar{\eta}U_d=0. \tag{19}$$

Thus the electrostatic term appears as a modification to the gravitational—in (17) and (18), we need only replace Ω by $(\Omega+V)$. This result is in fact quite general, because both Ω and V are potential energies of an inverse square law force, and so it can be applied to white dwarfs also. The other corrections to the Chandrasekhar models (Salpeter 1961) are smaller and can be ignored.

As the limiting mass is approached, the density becomes so high that the zero-point kinetic energy of the lattice ions $U_0(i)$ exceeds their mutual Coulomb energy, and the lattice structure dissolves. Savedoff's limiting energy would have as a small correction the zero-point energy of an extreme density ionic fluid, discussed recently by Abrikosov *et al.* (1963) and Ninham (1963). However, zero-temperature ' pycnonuclear ' reactions and inverse β-decays in fact preclude the existence of black or white dwarfs with these densities (Schatzman 1958, Hamada & Salpeter 1961).

We now turn to the corresponding relations for white dwarfs. An estimate of $\Delta\Omega$ and $\Delta U(e)$ may be found from a perturbation analysis of the equation of hydrostatic support. However the virial theorem supplies sufficient information:

$$2U(i)+2\bar{\eta}U(e)+\Omega+V=0, \tag{20}$$

where the factor 2 before $U(i)$ expresses the fact that at the temperatures of interest, the ionic energies are non-relativistic. We shall exploit (20) in the approximate form

$$2\Delta U(i)+2\bar{\eta}\Delta U(e)+\Delta\Omega+\Delta V\simeq 0, \tag{21}$$

where $(\Delta\bar{\eta})U(e)$ has been neglected next to $\bar{\eta}\Delta U(e)$. (The error vanishes in the non-relativistic and in the extreme relativistic limits; it is a maximum of 17% for electrons with energy $\sim 2 \cdot 4 m_e c^2$.) As long as the stellar interior remains very degenerate, with $\kappa T \ll E_F$ (the electron Fermi energy), the heat capacity of the electrons is negligible next to that of the ions and we may take

$$\Delta^T U(e)\simeq 0, \tag{22a}$$

so that

$$\Delta U(e)\simeq \Delta^\rho U(e). \tag{22b}$$

Then the combination of the approximate equations (22), (21) and (12) with equation (6) and equation (9) yields

$$-\Delta^\rho\Omega\simeq\Delta^\rho U(e)\simeq\frac{2\Delta^T U(i)+\Delta^T V}{1-2\bar{\eta}} \tag{23a}$$

$$\simeq\frac{\Delta^T E+\Delta^T U(i)}{1-2\bar{\eta}}. \tag{23b}$$

In the next section we distinguish two temperature regimes for white dwarf matter:

$$T\gg T_g \text{ for which } \Delta^T U(i)\gg\Delta^T V, \text{ and}$$
$$T\ll T_g \text{ for which } \Delta^T U(i)=\Delta^T V.$$

Correspondingly, for $T\gg T_g$

$$-\Delta^\rho\Omega\simeq\Delta^\rho U(e)\simeq\frac{\Delta E}{1-2\bar{\eta}}=\frac{\Delta^T(U(i)+V)}{(1-2\bar{\eta})}, \tag{24}$$

and for $T\ll T_g$

$$-\Delta^\rho\Omega\simeq\Delta^\rho U(e)\simeq\frac{3\Delta E}{2-4\bar{\eta}}=\frac{\frac{3}{2}\Delta^T(U(i)+V)}{(1-2\bar{\eta})}. \tag{25}$$

Thus except for stars very near the Chandrasekhar limit, $|\Delta^\rho\Omega|=o[\Delta^T(U(i)+V)]$. If $M\simeq M_c$, so that $\bar{\eta}\simeq\frac{1}{2}$, a slight cooling causes large absolute changes $|\Delta^\rho\Omega|$ and $\Delta U(e)$, because the radius $R\to 0$ as $M\to M_c$. However,

$$\Delta^\rho\Omega/\Omega=o\{[\Delta^T(U(i)+V)]/(\bar{\eta}-\tfrac{1}{2})|\Omega|\}$$
$$=o\left(\frac{\Delta^T(U(i)+V)}{|E_0|}\right),$$

where E_0 is the total black dwarf energy; and this is again small compared with unity. It will be noted that $|\Delta E|$ is the same as $|\Delta^\rho\Omega|$ and $|\Delta^\rho U(e)|$ only for $T\gg T_g$ and in the non-relativistic limit ($\bar{\eta}\to 1$).

3. *Heat capacity of white dwarf matter.* For densities near 10^6 g/cm^3 the Fermi energy of the degenerate electrons in a white dwarf is about 1/10 MeV. At temperatures around 10^7 °K (1 keV)—typical of white dwarf interiors—the heat capacity of the electrons is much less than that of a non-degenerate electron gas and so is negligible next to that of the ions. Just because of the enormously high density, the heat capacity of the ions can be more reliably calculated than is usual for normal solids. In this high density limit the degenerate electrons have such a high Fermi energy that they are only very slightly perturbed by the Coulomb interaction between electrons and between electrons and ions: the screening radius around an ion is large compared with the interionic spacing. At low temperatures such ions will form a body-centred cubic lattice (Wigner 1934, Fuchs 1935) whose excitations and heat capacity can be calculated.

A fixed ion of atomic number Z embedded in a degenerate electron gas is the source of a screened Coulomb potential (e.g. Pines 1963)

$$V(r)=\frac{Ze}{r}\exp(-r/r_{sc}) \tag{26}$$

(except for some small short-wavelength oscillations). The ratio of the screening radius r_{sc} to the interionic separation r_i when the electrons are non-relativistic is given by

$$\frac{r_{sc}}{r_i}=\frac{0\cdot 6}{Z^{1/3}}\left(\frac{a_0}{r_e}\right)^{1/2}. \tag{27}$$

The interelectron spacing r_e is defined in terms of the number of electrons per unit volume n by

$$\frac{4}{3}\pi r_e^3=n^{-1}, \tag{28}$$

and

$$r_i=Z^{1/3}r_e. \tag{29}$$

In the very high density limit where the electron Fermi energy becomes much larger than the electron rest mass

$$\frac{r_{sc}}{r_i}\sim\frac{6}{Z^{1/3}}. \tag{30}$$

Since the ratio a_0/r_e in a white dwarf is generally about one hundred, the ratio (30) is characteristic of white dwarf matter.

In first approximation we neglect r_i relative to r_{sc} so that all ions interact with unscreened Coulomb repulsions and form a bcc. lattice at zero temperature. The excitation modes in such a lattice (Clark 1958) consist of a longitudinal plasmon and a pair of transverse phonons. Exact details depend upon the direction of propagation in the lattice. The typical spectrum of ω versus wave number k is given in Fig. 1 (Pines 1963, p. 29).

The frequency ω is measured in units of the ion plasma frequency

$$\Omega_p\equiv\left(\frac{4\pi Z^2e^2n_z}{M_z}\right)^{1/2}\simeq 5\cdot 4\times 10^{-10}\left(\frac{\rho}{10^6}\right)^{1/2}\frac{\text{erg}}{\hbar}, \tag{31}$$

where n_z and M_z are the ion number density and mass (assumed to be $2Z$ times the proton mass) and k_D is the Debye cut-off defined by

$$k_D^3=6\pi^2 n_z. \tag{32}$$

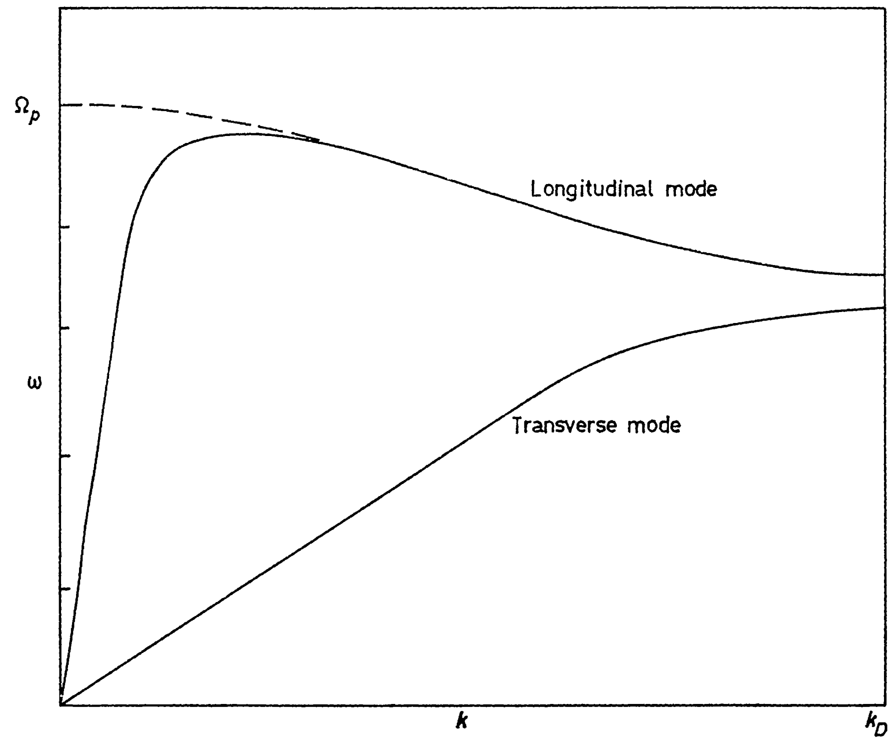

FIG. 1. *Spectrum of excitations for a body-centred cubic lattice with Coulomb interaction and long-range screening, based upon calculations of Clark* (1958) *and the survey of Pines* (1963). *The dashed line is the ideal case of no screening.*

The density ρ is measured in g/cm³.

When the finite screening radius is included the frequency of the longitudinal wave must approach zero linearly as k goes to zero, i.e. it must behave as a normal phonon instead of as an (ionic) plasmon. For small k (and non-relativistic electron Fermi energy E_F)

$$\omega^2\,(\text{longitudinal}) \to \left(\frac{2ZE_F}{3M_z}\right)\frac{k^2}{1+k^2/k_{sc}^2} \tag{33}$$

(Pines 1963, p. 243). Here k_{sc} is the wave number corresponding to (non-relativistic) r_{sc}, i.e.

$$k_{sc}^2 = \frac{6\pi e^2 n}{E_F}, \tag{34}$$

and equation (33) is valid for $k \ll k_{sc}$. As k goes to zero this is the dispersion relation for an acoustic phonon in which the inertia is that of the ions and the pressure that of the degenerate electrons.

The screening is not expected to affect significantly the spectrum of the transverse waves—their frequencies vanish proportionately with k even in the theory without screening. Graphs of $\omega(k)$ for the longitudinal and two transverse phonons are given by the solid lines in Fig. 1.

The energy content per unit volume of the ionic lattice at temperature T is given by

$$u(i)+v = \int_0^{k_D} \frac{k^2 dk}{2\pi^2} \sum_{\lambda=1}^{3} \hbar\omega_\lambda(k) \left\{\frac{1}{e^{\beta\hbar\omega_\lambda(k)}-1}+\frac{1}{2}\right\} \tag{35}$$

with $\beta \equiv (\kappa T)^{-1}$ and κ Boltzmann's constant. The summation is over the three possible phonon polarizations. The heat capacity per unit volume in the regime

$$T \gtrsim \frac{\hbar\Omega_p}{\kappa} \equiv \Theta = 4 \times 10^6 \sqrt{\frac{\rho}{10^6}}\ {}^\circ\mathrm{K} \tag{36}$$

is

$$C_V \sim 3n_z\kappa \left[1 - \frac{1}{36}\left(\frac{\hbar\Omega_p}{\kappa T}\right)^2 + \ \ldots\ \right]. \tag{37}$$

In the evaluation of the r.h.s. of equation (37) we have used the Kohn sum rule (Pines 1963, p. 28) for $k \gtrsim k_{sc}$:

$$\sum_{\lambda=1}^{3} \omega_\lambda{}^2(k) = \Omega_p{}^2. \tag{38}$$

Thus the Dulong–Petit value for the heat capacity of the ionic lattice is a good approximation for white dwarf matter with $T > 10^7\,{}^\circ\mathrm{K}$ and $\rho \lesssim 10^6\ \mathrm{g/cm^3}$. When the temperature does not satisfy the criterion of equation (36) numerical integration is generally necessary. However in the special case

$$T \ll \Theta \tag{39}$$

we have

$$C_V \sim \frac{16\pi^4}{5}\left(\frac{T}{\Theta}\right)^3 \kappa n_z. \tag{40}$$

A more delicate issue is the determination of the regime in which equation (35) remains valid. At a sufficiently high temperature the lattice will melt. Finally at still higher temperatures the average Coulomb interaction energy will become much less than the ion kinetic energy and the heat capacity will approach the $(3/2)\kappa n_z$ of a perfect gas.

In the absence of a definitive theory of melting we apply Lindemann's rule (Lindemann 1910) to the ionic lattice to estimate its melting temperature: the lattice is expected to melt when the thermally-caused mean square fluctuation in position of an ion, $\langle \delta r_i{}^2 \rangle$, satisfies

$$\frac{\langle \delta r_i{}^2 \rangle}{r_i{}^2} \simeq \frac{1}{16}. \tag{41}$$

This rule is empirically satisfactory at normal densities. For normal modes of the lattice with

$$\kappa T \gtrsim \hbar\Omega_p \equiv \kappa\Theta,$$

$$\langle \delta r_i{}^2 \rangle \simeq \frac{\kappa T}{M_z n_z} \int_0^{k_D} \frac{k^2 dk}{2\pi^2} \sum_{\lambda=1}^{3} \frac{1}{\omega_\lambda{}^2(k)}. \quad (42)$$

We approximate (cf. Fig. 1) by taking

$$\omega_{1,\,2} \sim 0 \cdot 7 \frac{k}{k_D} \Omega_p, \quad (43$$

$$\omega_3 \sim 0 \cdot 7\ \Omega_p. \quad (44$$

The Kohn sum rule (38) is satisfied exactly for $k \sim 0 \cdot 7\ k_D$. Then

$$\langle \delta r_i{}^2 \rangle \sim \frac{14\kappa T}{M_z \Omega_p{}^2}. \quad (4$$

The predicted lattice melting temperature T_m is given by

$$T_m \sim 3\times 10^5 \left(\frac{\rho}{10^6}\right)^{1/3} Z^{5/3}\,^\circ\mathrm{K}, \tag{46}$$

in good agreement with indications of melting in computer models of a Coulomb gas (Brush, Sahlin & Teller 1965, Van Horn 1966). For $Z>2$, T_m generally exceeds Θ and the classical approximation of equation (42) is acceptable. (For the case of He they are about equal, and one would expect that if oscillation amplitude alone is the criterion that determines melting, then the zero-point fluctuations should be included in estimating $\langle \delta r_i^2\rangle$ for use in Lindemann's formula. However, for the transition between liquid and solid helium at normal density with the observed T_m much less than the Debye temperature, it is found that Lindemann's rule with $\langle \delta r_i^2\rangle$ given by equation (42) agrees much better with experiment, implying that the zero-point fluctuations make virtually no contribution to melting. A plausible qualitative explanation is that the zero-point oscillations are more concentrated into shorter wavelengths than the thermal oscillations (Domb & Dugdale 1957).)

Of course within the core of a white dwarf the lattice volume is pretty much fixed by the gravitationally-determined density of the degenerate electron environment. Since the melting takes place at essentially fixed volume (rather than pressure) the transition temperature is smeared; the temperature T_m defines a regime and there is no discontinuous phase change at a specific temperature.

There is no model for the liquid state that permits a confident estimate of the heat of melting associated with the above solid–liquid transition and the change in heat capacity. The significant difference between liquids and solids is the absence of shear forces over macroscopic distances in the liquid, so that transverse phonons of small k certainly are not present in the liquid state. However these contribute only a very small part of the heat capacity except in the regime $T\ll\Theta$. Over short distances (a few r_i), i.e. with $k>k_D/3$, the description of the liquid may well be almost the same as that of the solid: long-range order disappears but very short-range order remains. This is suggested by the observation that the two-particle correlation function for near neighbours is not much affected by melting. In this model the heat capacity in the solid and liquid would be similar. (Even if $\omega_\lambda(k)$ changed at large k the heat capacity would be unchanged as long as $\kappa T \gtrsim \hbar\omega(k)$.) This expected equality is compatible with the fact that at normal pressures there is, in general, less than 10% change in heat capacity between the liquid and solid forms of the elements which melt at temperatures greater than their Debye temperatures.

When simple monatomic solids melt at constant normal pressure the change in entropy is characteristically $\Delta S_m \sim 2$ cal/mol/ °K so that the heat of fusion (at constant pressure), is $L_m(p) \sim \kappa T_m$. But the major part of this can be attributed to work done against attractive forces when the solid expands as it makes a constant-pressure transition to the liquid. Work done in theoretically squeezing the liquid back to its solid volume is about the same as $L_m(p)$. Also the unchanged heat capacity implies no significant change in the number of degrees of freedom which must be 'filled' to κT so that almost all of $L_m(p)$ goes into stretching energy. Thus in the continuous transformation from solid to liquid *at essentially constant volume*, which is the case in the white dwarf interior, there should be only a relatively small entropy change beyond that associated with raising the temperature of the liquid–solid with the heat capacity of equation (37).* In the absence of a definitive description of melting and the liquid state we assume that equation (37) is appropriate in the solid–liquid transition region and also in the liquid state, until the temperature becomes so high that the ion kinetic energy exceeds its average vibrational potential energy and the lattice model can be replaced by that of a dense imperfect gas with

$$C_V \simeq \frac{3}{2}\kappa n_z. \tag{47}$$

This will certainly happen when the ratio $\langle \delta r_i^2 \rangle / r_i^2 \simeq 1$ which corresponds to a temperature T_g:

$$T_g \simeq 10^7 Z^{5/3}\left(\frac{\rho}{10^6}\right)^{1/3} {}^\circ\mathrm{K}. \tag{48}$$

To summarize: we adopt the Dulong–Petit value of the heat capacity in the regime $\Theta \ll T \ll T_g$, which includes almost all of the range of interest for white dwarfs with cores composed of carbon or heavier elements. Although there will be melting around the temperature T_m of equation (46) which may fall in this regime (for carbon at $\rho \sim 3 \times 10^6$ g/cm^3, $T_m \sim 10^7$ °K) it is expected to have an insignificant effect on the energy content.

4. *Cooling times.* Because of the high thermal conductivity of a degenerate electron gas, the bulk of a white dwarf has a nearly uniform temperature T. The energy leak L through the thin but opaque, non-degenerate envelope is given (Mestel 1952a, Schwarzschild 1958, Schatzman 1958) by

$$L = KMT^{7/2}, \tag{49}$$

where K depends primarily on the opacity of the envelope. It is known (Ledoux & Sauvenier-Goffin 1950, Mestel 1952b, 1965, Schatzman 1958) that the hypothesis of nuclear generation as the source of a white dwarf's luminosity leads to severe physical difficulties from secular and vibrational instabilities, as well as astronomical difficulties in explaining the origin of white dwarfs on plausible evolutionary grounds. No physical (and to date no overwhelming astronomical) difficulties follow from the cooling hypothesis, in which the luminosity is supplied by a decline in the total energy E. For most white dwarfs, equations (10), (37) and (49) yield as the equation of cooling

$$-\frac{d}{dt}\left(3\,\frac{\mathcal{R}MT}{A}\right) = KMT^{7/2}, \tag{50}$$

$\mathcal{R}$ being the gas constant κ/M_H and A the mean atomic number. Because of the strong L–T dependence, most of the cooling time is spent near the present temperature $\bar{T}$: (50) yields a white dwarf lifetime

$$\tau = 1 \cdot 2\,\frac{\mathcal{R}M\bar{T}}{A\bar{L}} \tag{51}$$

where $\bar{L}$ is the present luminosity $KM\bar{T}^{7/2}$. This differs by a factor 2 from the older formula (Kaplan 1950, Mestel 1952a) because of the inclusion of the thermal

* Brush, Sahlin and Teller (1965), on the contrary, do find evidence for a possible heat of melting of about κT_m. However, their computer model uses only thirty-five ions, and does not (for example) predict the correct zero-temperature lattice. The effect of a significant heat of melting in stabilizing the temperature of a cooling white dwarf has been studied by Van Horn (1966).

part of the lattice potential energy. With $\bar{T}$ computed using the improved opacity of Schwarzschild (1958), the age of Sirius B now becomes $3\times 10^9/A$ years. This agrees with the estimated age of 10^8 years for Sirius A provided $A\simeq 30$, implying that substantial nucleosynthesis beyond carbon and oxygen has occurred in the pre-white dwarf star. However, an error of 2—e.g. a doubled lifetime for Sirius A, or halved internal temperature for Sirius B—would reduce A to 15, in the C–O range.

Luyten and Zwicky have recently reported the discovery of pygmy stars (Zwicky 1966) whose density is greatly in excess of that normally found in white dwarfs. The white pygmy star LP 101–16 is thought to be a hot star of solar mass and density of order 10^9 g/cm³, about the maximum which could be stable against K-capture of electrons. Such a star would have $\Theta \sim 1\cdot 2\times 10^8$ °K and

$$T_g \sim Z^{5/3}\times 10^8\,°\mathrm{K}.$$

The zero-point ionic motion is so large that the stellar core may be a quantum fluid down to $T=0$ if $Z=2$. Application of equation (40) (which can be applied with confidence only if the ions do form a lattice) gives instead of equation (50)

$$-\frac{d}{dt}\left(\frac{4\pi^4}{5}\left(\frac{\bar{T}}{\Theta}\right)^3\frac{\mathscr{R}M\bar{T}}{A}\right)=KM\bar{T}^{7/2}. \tag{52}$$

The lifetime for cooling of such a star to a present internal temperature $\bar{T}$ and luminosity $\bar{L}$ is

$$\tau=\frac{32\pi^4}{5}\left(\frac{\bar{T}}{\Theta}\right)^3\left(\left(\frac{T_0}{\bar{T}}\right)^{1/2}-1\right)\frac{\mathscr{R}M\bar{T}}{A\bar{L}}, \tag{53}$$

where T_0 ($T_0\ll\Theta$) is the initial temperature from which the pygmy star has cooled. This is shorter than the 'classical' result (51) if $\bar{T}/\Theta<10^{-1}$ approximately.*

A possible neutrino luminosity would decrease the calculated cooling time, and could reduce still further the lifetimes of pygmy stars. However the present luminosity $\leqslant 3\times 10^{-3}L_\odot$ of normal white dwarfs exceeds the hypothetical but unconfirmed neutrino radiation (Inman & Ruderman 1964) by well over three orders of magnitude at temperatures $\bar{T}\leqslant 1\cdot 1\times 10^7$ °K, and so should not significantly alter the cooling time (51).

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* Greenstein (1963) has reported on two very faint white dwarfs in the Hyades cluster, with lifetimes as given by (51) longer than that of the cluster (as determined from the turn-off point from the main sequence). One possible explanation is that their densities are high enough for (53) to be the relevant formula.

References

Abrikosov, A., Gorkov, L. & Dzyaloshinski, I., 1963. *Quantum Field Theory in Statistical Physics*, p. 189. Prentice Hall.
Auluck, F. C. & Mathur, V. S., 1959. *Z. Astrophys.*, **48,** 28.
Brush, S. G., Sahlin, H. L. & Teller, E., 1965. *U.C.R.L. Report*, No. 14467.
Chandrasekhar, S., 1939. *Introduction to the Study of Stellar Structure*. Chicago University Press.
Clark, C. B., 1958. *Phys. Rev.*, **109,** 1133.
Domb, C. & Dugdale, J. S., 1957. Solid Helium, *Prog. low Temp. Phys.*, **2,** 338.
Fuchs, K., 1935. *Proc. R. Soc.*, **151,** 585.
Greenstein, J. L., 1963. Personal communication.
Hamada, T. & Salpeter, E. E., 1961. *Astrophys. J.*, **134,** 683.
Inman, C. L. & Ruderman, M. A., 1964. *Astrophys. J.*, **140,** 1025.
Kaplan, S. A., 1950. *Astr. Zh.*, **27,** 31.
Ledoux, P. J. & Sauvenier-Goffin, E., 1950. *Astrophys. J.*, **111,** 611.
Lindemann, F., 1910. *Phys. Z.*, **11,** 609.
Mestel, L., 1952a. *Mon. Not. R. astr. Soc.*, **112,** 583.
Mestel, L., 1952b. *Mon. Not. R. astr. Soc.*, **112,** 598.
Mestel, L., 1965. White dwafs, *Stars and Stellar Systems*, **8**, 297. Chicago University Press.
Ninham, B. W., 1963. *Phys. Lett.*, **4,** 178.
Pines, D., 1963. *Elementary Excitations in Solids*. Benjamin, New York.
Salpeter, E. E., 1961. *Astrophys. J.*, **134,** 669.
Savedoff, M. P., 1963. *Astrophys. J.*, **138,** 291.
Schatzman, E., 1958. *White Dwarfs*. Interscience, New York.
Schwarzschild, M., 1958. *Structure and Evolution of the Stars*. Princeton University Press.
Van Horn, H., 1966. Personal communication.
Wigner, E. P., 1934. *Phys. Rev.*, **46,** 1002.
Zwicky, F., 1966. *Science*, **153,** 53.