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The Energy-Levels and Transition Probabilities for a Bounded Linear Harmonic Oscillator

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The first five energy levels and the oscillator strengths for transitions involving the first three states have been numerically computed for a bounded linear harmonic oscillator for various values of the boundary parameter. It is found that for $l > 3l_0$, where l is the length of the box within which the oscillator is confined and l_0 is the classical amplitude of the oscillator when it has energy $h\nu$, the bounded oscillator behaves more or less like a free oscillator (in the first few energy levels), while for $l < l_0$ it has properties closely approaching those of a free particle enclosed in a box.

Introduction

The study of artificially bounded atomic systems on the basis of quantum mechanics was initiated by Sommerfeld and Welker[1] in 1938. They dealt with the problem of the hydrogen atom confined in a spherical enclosure of radius R. The wave function instead of vanishing at infinity, as usual, now vanishes on the surface of the sphere. They found that for $R > 1.835a$ where a is the radius of the first Bohr orbit, the electron is bound to the nucleus while for $R < 1.835\,a$, it is not (energy positive). The corresponding problem of the bounded linear harmonic oscillator has been attacked by Auluck and Kothari[2], and they have obtained expressions for the energy levels of the oscillator in two extreme cases, viz., $l \ll l_0$ and $l \gg l_0$, where l is the length of the box within which the oscillator is confined and l_0 is the classical amplitude of the oscillator when it has energy $h\nu$ (ν=classical frequency of the oscillator). In this paper we have numerically evaluated the energy levels of the bounded oscillator for intermediate values of l, and have also considered the problem of the transition probabilities of the oscillator. In the case of a free oscillator (wave function vanishing at ∞), transitions can occur only between adjacent levels. For a bounded oscillator, however, there is a non-zero probability of transition between any two states of different parities (we label the states as 1 (lowest energy), 2, 3,$\cdots$; the 'odd' states 1, 3, 5$\cdots$ and the 'even' states 2, 4, 6,$\cdots$, are referred to as states of different parities). We have numerically worked out the oscillator strengths for all transitions involving the first three states (Tables III, IV, and V). It is of interest to note that recently Corson & Kaplan[3] have referred to a possible role of the bounded oscillator in the theory of the specific heats of solids. It might also find an application in the study of the second order phase transitions[4]. We hope to discuss these questions in a subsequent paper.

§ 1. The wave equation

For a particle of mass m moving in a field of force of potential $(1/2)\ \alpha x^2$, the Schrödinger wave-equation is

$$d^2\psi/dx^2+2m/\hbar^2\ (E-1/2\cdot\alpha x^2)\psi=0. \tag{1}$$

Putting $$E=(n+1/2)\hbar\omega,$$

and $$x=(\hbar/2m\omega)^{1/2}\ \xi, \tag{2}$$

where $\omega=(\alpha/m)^{1/2}$ is the 'classical' angular frequency of the oscillator, we get

$$\frac{d^2\psi}{d\xi^2}+(n+1/2-1/4\cdot\xi^2)\psi=0. \tag{3}$$

This equation has solutions

$$\psi(\xi)=\xi^{-1/2}\ M_{n/2+1/4,\pm 1/4}\ (\xi^2/2),$$

where the M's are the confluent hypergeometric functions given by[5]

$$2^{1/4}\ \xi^{-1/2}\ M_{n/2+1/4,-1/4}\ (\xi^2/2)=e^{-\xi^2/4}\left[1-\frac{n}{2!}\ \xi^2+\frac{n(n-2)}{4\,!}\ \xi^4-\cdots\right], \tag{4}$$

$$2^{3/4}\ \xi^{-1/2}\ M_{n/2+1/4,+1/4}(\xi^2/2)=e^{-\xi^2/4}\left[\xi-\frac{(n-1)}{3!}\ \xi^3+\frac{(n-1)\,(n-3)}{5\,!}\ \xi^5-\cdots\right]. \tag{5}$$

In what follows we shall denote the two solutions (4) and (5) by $W_{n,\,-1/4}(\xi)$ and $W_{n,+1/4}\ (\xi)$ respectively.*

We assume the oscillator to be enclosed between infinitely high and steep potential walls at $x=\pm l/2$. The energy eigenvalues are then determined from the condition

$$\psi(x)\longrightarrow 0\ \ as\ \ x\longrightarrow \pm l/2,$$

or in terms of ξ

$$\psi(\xi)\longrightarrow 0\ \ as\ \ \xi\longrightarrow \pm\xi_0,$$

where $$\xi_0=l/2\cdot(2m\omega/\hbar)^{1/2}=l/l_0\,; \tag{6}$$

$l_0=(2\hbar/m\omega)^{1/2}$ is the 'classical amplitude' of the oscillator when it has energy $\hbar\omega$.

The zeros of (4) and (5) can be evaluated numerically (this is done in Section III). Zeros of $W_{n,-1/4}\ (\xi)$ give the first (lowest), third, fifth$\cdots$ energy levels, while those of $W_{n,+1/4}(\xi)$ determine the even energy levels.

An approximate formula for the energy levels in terms of ξ_0 can be obtained by treating the term $(1/2)\ \alpha x^2$ in the Hamiltonian as a small perturbation. When this term is neglected, we have the Hamiltonian for a free particle, and the energy levels are given by

$$E_q=\pi^2q^2/4\xi_0^{\,2}\cdot\hbar\omega,\quad q=1,\,2,\,3,\cdots.$$

*) These W-function are not identical with those defined by Whittaker and Watson. Reader should be careful not to confuse them.

The first order correction term according to the ordinary perturbation theory is

$$(q|1/2\cdot\alpha x^2|q)=\frac{\xi_0^2}{12}\Big(1-\frac{6}{\pi^2q^2}\Big)\hbar\omega,$$

and the second order term is

$$\sum_{p\neq q}\frac{|(q|1/2\cdot\alpha x^2|p)|^2}{(E_q-E_p)}=\frac{256}{\pi^6}\xi_0^6\sum_{p\neq q}\frac{q^2p^2}{(q^2-p^2)^5}\hbar\omega,$$

where p takes on odd or even integral values according as q is odd or even. Thus we have for the energy of the q-th level,

$$E_q=(n_q+1/2)\hbar\omega$$
$$=\frac{\pi^2q^2}{4\xi_0^2}\Big[1+\frac{\xi_0^4}{3\pi^2q^2}\Big(1-\frac{6}{\pi^2q^2}\Big)+\frac{1024}{\pi^8}\xi_0^8\sum_{p\neq q}\frac{p^2}{(q^2-p^2)^5}\Big]\hbar\omega\,. \tag{7}$$

This formula gives correct values of energy, (correct to 1 in 10^5) for the first few levels up to $\xi_0\leqq 1.5$. For higher levels the approximation is still better.

We list below a few recurrence relations between the W's that are of help in evaluating the matrix elements (Sec. II). These formulae may be verified by direct substitution.

$$nW_{n-1,1/4}+\xi W_{n,-1/4}-(n+1)W_{n+1,1/4}=0\,, \tag{8a}$$

$$W_{n+1,-1/4}+\xi W_{n,1/4}-W_{n-1,-1/4}=0\,, \tag{8b}$$

$$W'_{n,1/4}-(1/2)\,\xi W_{n,1/4}-W_{n+1,-1/4}=0\,, \tag{8c}$$

$$W'_{n,-1/4}-(1/2)\,\xi\,W_{n,-1/4}+(n+1)W_{n+1,1/4}=0\,. \tag{8d}$$

('Dash' denotes differentiation with respect to ξ).

§ 2. The matrix elements

The dipole matrix element between two states q and p is

$$(q|x|p)=c_{n_q,\pm1/4}\cdot c_{n_p,\pm1/4}\int_{-l/2}^{l/2}\psi_q x\psi_p\,dx$$
$$=c_{n_q,\pm1/4}\cdot c_{n_q,\pm1/4}\Big(\frac{\hbar}{2m\omega}\Big)\int_{-\xi_0}^{\xi_0}W_{n_q,\pm1/4}\xi W_{n_p,\pm1/4}\,d\xi,\cdots \tag{9}$$

where the c's are the normalisation constants given by

$$c^2_{n_q,\pm1/4}=(2m\omega/\hbar)^{1/2}\Big/\int_{-\xi_0}^{\xi_0}W^2_{n_q,\pm1/4}\,d\xi. \tag{10}$$

Since $W_{n_q,+\frac{1}{4}}$ is an odd function of ξ and $W_{n_q,-1/4}$ is an even function of ξ

$$\int_{-\xi_0}^{\xi_0}W_{n_q,1/4}\,\xi W_{n_q,1/4}\,d\xi=\int_{-\xi_0}^{\xi_0}W_{n_q,-1/4}\,\xi W_{n_p,-1/4}\,d\xi=0.$$

Hence transitions between states labelled by integers of the same parity are forbidden. Next we consider $\int_{-\xi_0}^{\xi_0}W_{n_q,1/4}\,\xi W_{n_q,-1/4}\,d\xi$. We have

$$\frac{d^2W_{n_q,-1/4}}{d\xi^2}+(n_q+1/2-1/4\cdot\xi^2)\,W_{n_q,1/4}=0\,, \tag{i}$$

$$\frac{d^2W_{n_p,-1/4}}{d\xi^2}+(n_p+1/2-1/4\cdot\xi^2)\,W_{n_p,-1/4}=0\,. \tag{ii}$$

From (i) and (ii) we easily obtain

$$(n_p-n_q)\int_{-\xi_0}^{\xi_0} W_{n_q,1/4}\,\xi W_{n_p,-1/4}\,d\xi=2\int_{-\xi_0}^{\xi_0} W_{n_q,1/4}\,W'_{n_p,-1/4}\,d\xi,$$

or using recurrence relation (8d),

$$(n_p-n_q-1)\int_{-\xi_0}^{\xi_0} W_{n_q,1/4}\,\xi W_{n_p,-1/4}\,d\xi=-2\,(n_p+1)\int_{-\xi_0}^{\xi_0} W_{n_q,1/4}\,W_{n_p+1,1/4}\,d\xi.$$

Again

$$(n_p-n_q+1)\int_{-\xi_0}^{\xi_0} W_{n_q,1/4}\,W_{n_p+1,1/4}d\xi=2W_{n_p+1,1/4}(\xi_0)\,W_{n_q+1,-1/4}(\xi_0),$$

hence

$$\int_{-\xi_0}^{\xi_0} W_{n_q,1/4}\,\xi W_{n_p,-1/4}\,d\xi=-\frac{4\,(n_p+1)}{(n_p-n_q)^2-1}\,W_{n_p+1,1/4}(\xi_0)\,W_{n_q+1,-1/4}(\xi_0)\,. \tag{11}$$

The normalisation constants can be evaluated as follows : it may be shown from the wave equation[2] that

$$\int_{-\xi_0}^{\xi_0} u^2(n_q,\ \xi)\,e^{-\xi^2/2}\,d\xi=2e^{-\xi_0^2/2}(\partial u/\partial n)_{n_q,\xi_0}(\partial u/\partial\xi)_{n_q,\xi_0}$$

where

$$u(n,\ \xi)=e^{\xi^2/4}\ W_{n,\pm 1/4}\,(\xi).$$

From $u(n_q,\ \xi_0)=0$, we have

$$dn_q/d\xi_0=(-\partial u/\partial\xi)_{n_q,\xi_0}/\,(\partial u/\partial n)_{n_q,\xi_0}, \tag{13}$$

so that

$$\int_{-\xi_0}^{\xi_0} u^2(n_q,\ \xi)\,e^{-\xi^2/2}\,d\xi=-2e^{-\xi_0^2/2}\left[(\partial u/\partial\xi)_{n_q,\xi_0}\right]^2/\,(dn_q/d\xi_0).$$

From this equation and the recurrence formulae (8c) and (8d), it is readily shown that

$$c_{n_q,1/4}=(m\omega/2\hbar)^{1/4}\ \frac{(-dn_q/d\xi_0)}{W_{n_q+1,-1/4}(\xi_0)}\ , \tag{14}$$

$$c_{n_p,-1/4}=(m\omega/2\hbar)^{1/4}\ \frac{(-dn_p/d\xi_0)}{(n_p+1)\,W_{n_p+1,1/4}(\xi_0)}. \tag{15}$$

Substituting from (11), (14) and (15) in (9), we obtain

$$(q|x|p)=(\hbar/2m\omega)^{1/2}\frac{2}{(n_q-n_p)^2-1}(dn_q/d\xi_0)^{1/2}(dn_p/d\xi_0)^{1/2}, \tag{16}$$

q and p being of different parities. The oscillator strengths f_{qp} are given by

$$\begin{aligned} f_{qp}&=\frac{2m}{\hbar^2}(E_p-E_q)\,|(q|x|p)|^2 \\ &=\frac{4(n_p-n_q)}{[(n_p-n_q)^2-1]^2}(dn_q/d\xi_0)(dn_p/d\xi_0). \end{aligned} \tag{17}$$

The f's satisfy the Thomas-Kuhn rule

$$\sum_p f_{qp}=1. \tag{18}$$

§ 3. Numerical calculations

In order to calculate the oscillator strengths f_{qp}, we need the values of n_q's and $(dn_q/d\xi_0)$'s. These values have been computed in this paper for $\xi_0 \leqq 3$. For $q>5$, the values of n_q obtained from formula (7) are correct to 1 in 10^5 for $\xi_0 \leqq 3$. For the lower levels (7) gives correct n_q's for $\xi_0 \leqq 1.5$ only, and for larger values of ξ_0 the n_q's have to be evaluated directly from the equations $W_{n,\pm 1/4}(\xi_0)=0$. Equation (7) is still useful in as much as it provides us with the rough values of the zeros of $W_{n,\pm 1/4}(\xi_0)$. The correct roots are computed by applying Newton's rule which states that if n' is an approximate root of $f(n)=0$, a better value for the root is

$$n''=n'-\frac{f(n')}{(df/dn)_{n'}}. \tag{19}$$

In our case the functions $W_{n,\pm 1/4}(\xi_0)$ correspond to $f(n)$. This procedure was adopted to calculate the first three roots of $W_{n,-1/4}(\xi_0)$ and the first two roots of $W_{n,+1/4}(\xi_0)$ for $1.5<\xi_0 \leqq 3$. Usually one Newton approximation was sufficient to give the roots correct to 1 in 10^5. The derivatives $dn_q/d\xi_0$ were obtained from (7) for $q>5$. For the lower levels, they were calculated from the relations (for $\xi_0>1.5$)

$$\left.\begin{aligned} dn_q/d\xi_0&=\frac{(n_q+1)\,W_{n_q+1,1/4}(\xi_0)}{(\partial W_{n,-1/4}/\partial n)_{n_q,\xi_0}}, \quad q=1,\,3,\,5\cdots, \\ &=\frac{W_{n_q-1,\,-1/4}(\xi_0)}{(\partial W_{n,1/4}/\partial n)_{n_q,\xi_0}}, \quad q=2,\,4,\,6\cdots, \end{aligned}\right\} \tag{20}$$

which follows from (13) with the help of recurrence formulae. (For $\xi_0<1.5$, (7) was again used). The calculation of the derivatives does not involve much additional labour since $\partial W_{n\pm 1/4}/\partial n$ become available from the calculation of the n_q's. The overall accuracy of the calculations is 1 in 10,000.

Table 1.

$n_q=(E_q/\hbar\omega-1/2)$ as function of $\xi_0=l/l_0$ for $q=1$, 2, 3, 4 and 5.

q \ ξ_0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
1	61.186	14.926	6.3657	3.3762	2.0000	1.2603	0.82236	0.54624
2	246.24	61.196	26.941	14.966	9.4402	6.4555	4.6735	3.5350
3	554.66	138.30	61.213	34.247	21.784	15.0332	10.982	8.3736
4	986.46	246.25	109.191	61.236	39.059	27.0311	19.7993	15.1268
5	1541.6	385.04	170.877	95.935	61.266	42.454	31.1314	23.8041

q \ ξ_0	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
1	0.36496	0.24300	0.16002	0.10355	0.065476	0.040247	0.023946	
2	2.7728	2.24566	1.87342	1.60747	1.41680	1.28064	1.18450	
3	6.6061	5.36323	4.4654	3.80463	3.3125	2.94404	2.66842	
4	11.9451	9.69148	8.0468	6.81909	5.88730	5.17207	4.61953	
5	18.8026	15.2475	12.6402	10.6806	9.17941	8.01278	7.0966	

Table 2.

$(dn_q/d\xi_0)$ as function of $\xi_0=l/l_0$ for $q=1$, 2, 3, 4 and 5.

q \ ξ_0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
1	616.84	77.080	22.807	9.5861	4.8699	2.7785	1.7093	1.1048
2	2467.4	308.37	91.300	38.440	19.598	11.254	6.9979	4.5974
3	5551.6	693.89	205.52	86.620	44.258	25.515	15.967	10.594
4	9869.6	1233.6	365.44	154.08	78.796	45.500	28.549	19.019
5	15421.3	1927.6	571.06	240.83	123.207	71.200	44.732	29.858

q \ ξ_0	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
1	0.73657	0.49945	0.34041	0.23089	0.15444	0.10107	0.064267	
2	3.1381	2.1979	1.5639	1.1208	0.80288	0.57058	0.39945	
3	7.3345	5.2395	3.8265	2.8371	2.1210	1.5889	1.1853	
4	13.248	9.5454	7.0570	5.3186	4.0639	3.1326	2.4230	
5	20.859	15.093	11.2235	8.5265	6.5853	5.1497	4.0604	

Table III.
The oscillator strengths f_{1p} ($p=2, 4, 6\cdots$) as functions of $\xi_0=l/l_0$.
Initial level 1 (LOWEST)

p \ ξ_0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
2	0.9607	0.9607	0.9608	0.9610	0.9613	0.9620	0.9632	0.9649
4	0.0307	0.0307	0.0307	0.0305	0.0302	0.0296	0.0287	0.0274
6	0.0054	0.0054	0.0054	0.0054	0.0054	0.0053	0.0051	0.0049
8	0.0017	0.0017	0.0016	0.0016	0.0016	0.0016	0.0016	0.0015
10	0.0007	0.0007	0.0007	0.0007	0.0007	0.0006	0.0006	0.0006
12	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
14 (to ∞ Total)	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0004
Asymptotic Formula	$6.5\times p^{-4}$	$6.5\times p^{-4}$	$6.5\times p^{-4}$	$6.4\times p^{-4}$	$6.4\times p^{-4}$	$6.3\times p^{-4}$	$6.2\times p^{-4}$	$5.9\times p^{-4}$
Sum	1.0000	1.0000	1.0000	1.0000	1.00000	0.9999	1.0000	1.0000

p \ ξ_0	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
2	0.9673	0.9702	0.9738	0.9778	0.9823	0.9864	0.9904	
4	0.0255	0.0231	0.0202	0.0170	0.0135	0.0101	0.0071	
6	0.0046	0.0042	0.0038	0.0032	0.0027	0.0021	0.0015	
8	0.0014	0.0013	0.0012	0.0010	0.0008	0.0007	0.0005	
10	0.0006	0.0005	0.0005	0.0004	0.0003	0.0003	0.0002	
12	0.0003	0.0002	0.0002	0.0002	0.0002	0.0001	0.0001	
14 (to ∞ Total)	0.0004	0.0004	0.0004	0.0003	0.0003	0.0002	0.0002	
Asymptotic Formula	$5.6\times p^{-4}$	$5.2\times p^{-4}$	$4.8\times p^{-4}$	$4.2\times p^{-4}$	$3.6\times p^{-4}$	$2.9\times p^{-4}$	$2.3\times p^{-4}$	
Sum	1.0001	0.9999	1.0001	0.9999	1.0001	0.9999	1.0000	

Table IV.
The oscillator strengths f_{2p} ($p=1, 3, 5\cdots\cdots$) as functions of $\xi_0=l/l_0$.
Initial level=2.

p \ ξ_0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
1	−0.9607	−0.9607	−0.9608	−0.9610	−0.9613	−0.9620	−0.9632	−0.9649
3	1.8677	1.8677	1.8678	1.8681	1.8690	1.8704	1.8730	1.8766
5	0.0700	0.0700	0.0699	0.0698	0.0694	0.0688	0.0678	0.0663
7	0.0139	0.0139	0.0139	0.0139	0.0138	0.0137	0.0135	0.0132
9	0.0046	0.0046	0.0046	0.0046	0.0046	0.0045	0.0045	0.0044
11	0.0020	0.0020	0.0020	0.0020	0.0019	0.0019	0.0019	0.0019
13 (to ∞)	0.0026	0.0026	0.0026	0.0026	0.0025	0.0025	0.0025	0.0025
Asymptotic Formula	$25.9\times p^{-4}$	$25.9\times p^{-4}$	$25.9\times p^{-4}$	$25.9\times p^{-4}$	$25.7\times p^{-4}$	$25.5\times p^{-4}$	$25.2\times p^{-4}$	$24.7\times p^{-4}$
Sum	1.0001	1.0001	1.0000	1.0000	0.9999	0.9998	1.0000	1.0000

p \ ξ_0	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
1	−0.9673	−0.9702	−0.9738	−0.9778	−0.9822	−0.9864	−0.9904	
3	1.8818	1.8888	1.8973	1.9076	1.9195	1.9322	1.9451	
5	0.0641	0.0611	0.0572	0.0524	0.0467	0.0403	0.0333	
7	0.0128	0.0122	0.0115	0.0106	0.0095	0.0083	0.0070	
9	0.0042	0.0041	0.0038	0.0035	0.0032	0.0028	0.0024	
11	0.0018	0.0017	0.0016	0.0015	0.0014	0.0012	0.0010	
13	0.0024	0.0023	0.0021	0.0020	0.0018	0.0016	0.0014	
(to ∞) Asymptotic Formula	$24.0\times p^{-4}$	$23.1\times p^{-4}$	$21.9\times p^{-4}$	$20.3\times p^{-4}$	$18.5\times p^{-4}$	$16.5\times p^{-4}$	$14.2\times p^{-4}$	
Sum	0.9998	1.0000	0.9998	0.9998	0.9999	1.0000	0.9998	

Table V.

The oscillator strengths f_{3p} ($p=2, 4, 6\cdots\cdots$) as functions of $\xi_0=l/l_0$.

Initial level=3.

p \ ξ_0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
2	−1.8676	−1.8676	−1.8678	−1.8681	−1.8690	−1.8704	−1.8730	−1.8766
4	2.7226	2.7227	2.7228	2.7232	2.7244	2.7264	2.7301	2.7353
6	0.1067	0.1067	0.1067	0.1065	0.1063	0.1058	0.1050	0.1038
8	0.0224	0.0224	0.0224	0.0224	0.0224	0.0223	0.0221	0.0219
10	0.0077	0.0077	0.0077	0.0077	0.0077	0.0077	0.0076	0.0075
12	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0033
14	0.0048	0.0048	0.0048	0.0048	0.0048	0.0047	0.0047	0.0047
(to∞Total) Asymptotic Formula	$58.3p^{-4}\times[1+27p^{-2}]$	$58.3p^{-4}\times[1+27p^{-2}]$	$58.3p^{-4}\times[1+27p^{-2}]$	$58.3p^{-4}\times[1+27p^{-2}]$	$58.1p^{-4}\times[1+27p^{-2}]$	$57.9p^{-4}\times[1+27p^{-2}]$	$57.6p^{-4}\times[1+27p^{-2}]$	$57.0p^{-4}\times[1+27p^{-2}]$
Sum	1.0000	1.0001	1.0000	1.0001	1.0000	0.9999	0.9999	0.9999

p \ ξ_0	1.8	2.0	2.2	2.4	2.6	2.8	3.0	
2	−1.8818	−1.8888	−1.8973	−1.9076	−1.9195	−1.9322	−1.9451	
4	2.7430	2.7532	2.7660	2.7821	2.8012	2.8229	2.8462	
6	0.1021	0.0996	0.0964	0.0921	0.0867	0.0801	0.0722	
8	0.0215	0.0210	0.0203	0.0195	0.0184	0.0170	0.0155	
10	0 0074	0.0073	0.0070	0.0068	0.0064	0.0059	0.0054	
12	0.0033	0.0032	0.0031	0.0030	0.0028	0.0026	0.0024	
14	0.0046	0.0045	0.0044	0.0042	0.0040	0.0037	0.0034	
(to∞Total) Asymptotic Formula	$56.2p^{-4}\times[1+27p^{-2}]$	$55.1p^{-4}\times[1+26p^{-2}]$	$53.6p^{-4}\times[1+26p^{-2}]$	$51.5p^{-4}\times[1+26p^{-2}]$	$49.0p^{-4}\times[1+25p^{-2}]$	$45.8p^{-4}\times[1+24p^{-2}]$	$42.0p^{-4}\times[1+24p^{-2}]$	
Sum	1.0001	1.0000	0.9999	1.0001	1.0000	1.0000	1.0000	

Table I gives the first five energy levels as functions of ξ_0 from $\xi_0=0$ to $\xi_0=3$ at sub-intervals of 0.2. The higher levels for the same interval $(0<\xi_0\leq 3)$ can be computed from (7) and are consequently not tabulated. Figure 1 is a plot of the first three energy levels against ξ_0. Table II gives $dn_q/d\xi_0$ as a function of ξ_0 for $q=1$, 2, 3, 4, and 5. The oscillator strengths f_{1p}, f_{2p} and f_{3p} are tabulated (tables III, IV, and V). Asymptotic expressions for the f_{qp}, when p is large, are obtained from (17) by assuming $(n_p-n_q)^2\gg 1$ and substituting for $dn_p/d\xi_0$ the approximate value

$$(dn_p/d\xi_0)_{p\ \text{large}}=-\frac{\pi^2p^2}{2\xi_0^3}\left(1-\frac{\xi_0^4}{3\pi^2q^2}\right).$$

This gives

$$f_{qp}\,(p\ \text{large})=\frac{128\,\xi_0^3}{\pi^4}\,\frac{dn_q}{d\xi_0}\,\frac{1}{p^4}\left[1+\frac{12\xi_0^2}{\pi^2p^2}\left(n_q+\frac{1}{2}-\frac{\xi_0^2}{9}\right)\right], \tag{21}$$

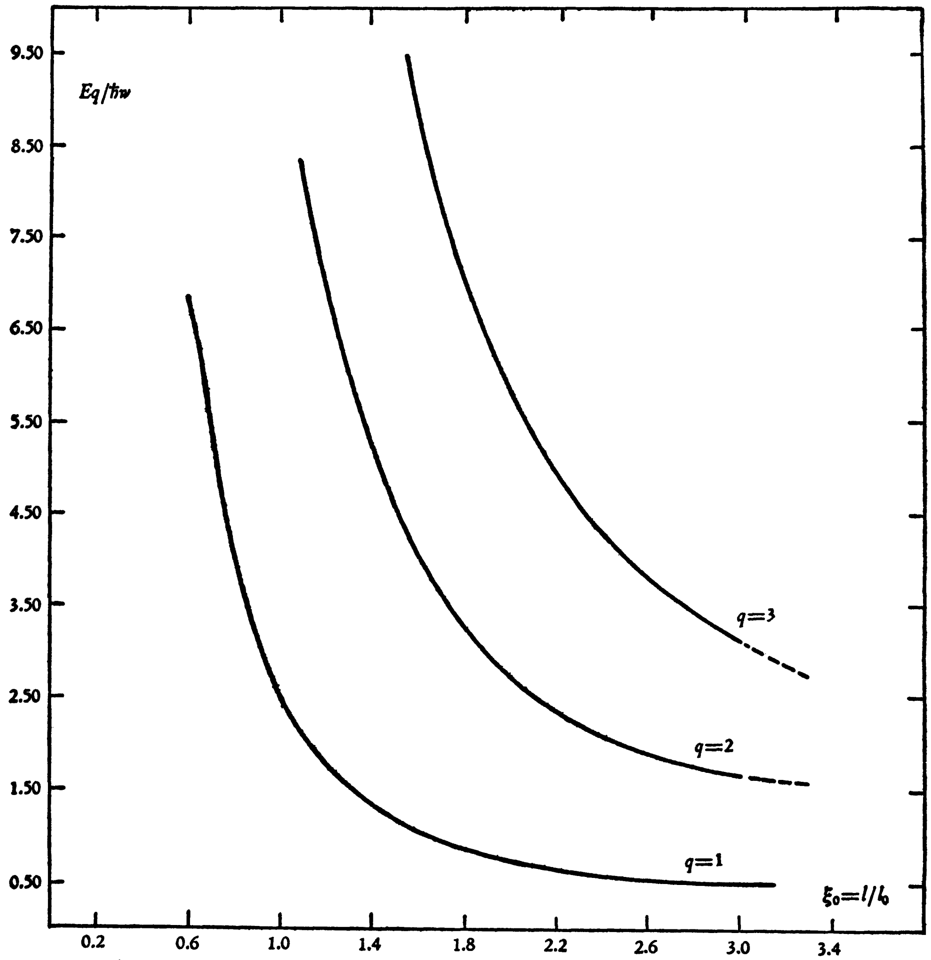

Fig. 1. The first three energy levels as functions of the boundary parameter $\xi_0=l/l_0$.

$dn_q/d\xi_0$ and n_q being obtained directly from the tables. The asymptotic formulae for f_{1p}, f_{2p}, and f_{3p} are listed in the corresponding tables. The Thomas-Kuhn rule, $\sum_p f_{qp}=1$, is seen to be satisfied in all the three cases and this serves as a check on the calculations. The oscillator strengths f_{12} and f_{14} are plotted against ξ_0 in Fig. 2. It will be noted that while the energy levels become increasingly sensitive to variations in ξ_0 as the latter decreases, the oscillator strengths become sensitive to changes in ξ_0 for relatively large values of ξ_0 ($\xi_0>1.5$). In fact, for $0<\xi_0<1$, the results are almost the same as those for a free particle enclosed in a box. The effects of the potential begin to show up as ξ_0 increases beyond 1, and for $\xi_0>3$, the bounded oscillator behaves more or less like a free oscillator (in the first few energy states.)

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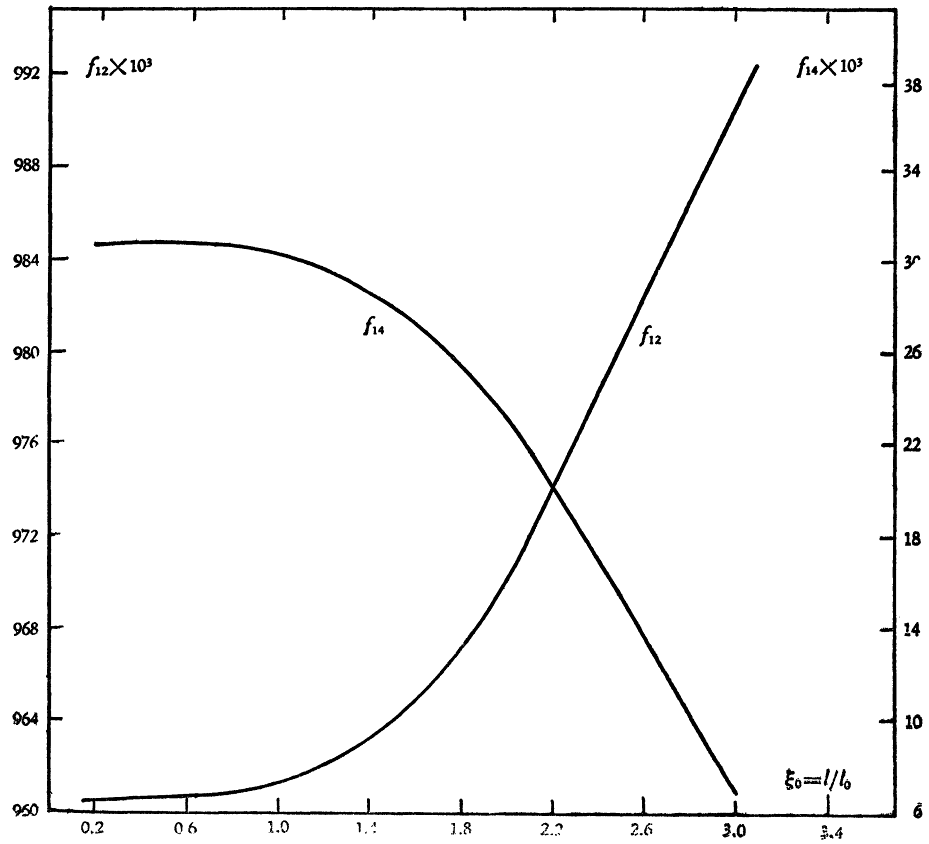

Fig. 2. f_{12} and f_{14} as functions of the boundary parameter $\xi_0=l/l_0$.

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