The energy–momentum complex in non-local gravity

Salvatore Capozziello *1,2,3, Maurizio Capriolo † 4,2, and Gaetano Lambiase ‡4,5

¹Dipartimento di Fisica "E. Pancini", Università di Napoli "Federico II", Compl.

Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy,

²INFN Sezione di Napoli, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy,

³Scuola Superiore Meridionale, Largo S. Marcellino 10, I-80138, Napoli, Italy,

⁴Dipartimento di Fisica Università di Salerno, via Giovanni Paolo II, 132, Fisciano, SA I-84084, Italy.

⁵INFN Sezione di Napoli, Gruppo Collegato di Salerno, via Giovanni Paolo II, 132, Fisciano, SA I-84084, Italy.

April 6, 2023

Abstract

In General Relativity, the issue of defining the gravitational energy contained in a given spatial region is still unresolved, except for particular cases of localized objects where the asymptotic flatness holds for a given spacetime. In principle, a theory of gravity is not selfconsistent, if the whole energy content is not uniquely defined in a specific volume. Here we generalize the Einstein gravitational energy-momentum pseudotensor to non-local theories of gravity where analytic functions of the non-local integral operator \Box^{-1} are taken into account. We apply the Noether theorem to a gravitational Lagrangian, supposed invariant under the one-parameter group of diffeomorphisms, that is, the infinitesimal rigid translations. The invariance of non-local gravitational action under global translations leads to a locally conserved Noether current, and thus, to the definition of a gravitational energy-momentum pseudotensor, which is an affine object transforming like a tensor under affine transformations. Furthermore, the energy-momentum complex remains locally conserved, thanks to the non-local contracted Bianchi identities. The continuity equations for the gravitational pseudotensor and the energymomentum complex, taking into account both gravitational and matter components, can be derived. Finally, the weak field limit of pseudotensor is performed to lowest order in metric perturbation in view of astrophysical applications.

capozziello@unina.it

[†]mcapriolo@unisa.it

[‡]glambiase@unisa.it

1 Introduction

Recently, non-local contributions to the gravitational action have been considered from various points of view as possible solutions of the problem of renormalization and regularization of gravitational field [1–3]. In this context, a non-local gravitational energy-momentum pseudo-tensor can be proposed as a manifestation of non-locality of gravity and, therefore, as a possible manifestation of quantum nature of gravity.

Theories of gravity can be endowed with non-local properties in three ways [4]. Firstly through integral operators acting on functions whose value at a given point depends on the values of fields at another point in spacetime [5–7]. Secondly, through gravitational Lagrangians involving an analytic non-polynomial function \mathcal{F} of the operator \Box , which can be expanded in convergent series with real coefficients as

$$\mathcal{F}(\Box) = \sum_{h=1}^{\infty} a_h \Box^h , \qquad (1.1)$$

known as Infinite Derivative Theories of Gravity (IDG) [8–14]. Thirdly, through a suitable constitutive law where, like in electrodynamics, temporal dispersion, anisotropy and non-homogeneity of medium, i.e. the spatial dispersion, are due to temporal and spatial non-locality, respectively [15–19].

At infra-red scales, non-local models of gravity can naturally explain late-time acceleration without introducing exotic material components such as dark matter and dark energy [5]. In addition, they can potentially fix some cosmological and astrophysical problems plaguing the Λ CDM model [20–24], black hole stability [25], or stability and traversability of wormhole solutions [26].

On the other hand, many authors such as Einstein, Tolman, Landau, Lifshitz, Papapetrou, Møller and Weinberg have proposed definitions for gravitational pseudotensor [27–35], to describe the energy and momentum of gravitational field in General Relativity. These prescriptions are based either on the introduction of a super-potential or on expanding the Ricci tensor in metric perturbation $h_{\mu\nu}$ or on manipulating the Einstein equations. Although these definitions are different, it has been shown they coincide for Kerr-Schild metric [36]. Many prescriptions for gravitational pseudotensor in higher-order curvature theories, in metric and Palatini approach, have been proposed [37–44]. Also for teleparallel gravity, it is possible to formulate self-consistent definition of gravitational pseudotensor [45, 47].

Here, we want to propose a generalization of Einstein gravitational pseudotensor to non-local gravity models involving $f(\Box^{-1})$ operators. It will be derived from a variational principle using the Noether theorem applied to a gravitational Lagrangian invariant under global translations [46]. This object remains an affine tensor, i.e. a pseudotensor, but it is a non-local quantity. Indeed, its non-local corrections involve non-local $\Box^{-1}R$ terms, which assume, at a point x, a value depending on the values assumed by the metric tensor $g_{\mu\nu}$ in all points of the integration domain. Then, we show that the covariant conservation of the energy-momentum, associated to the gravitational and matter fields, holds in non-local $f(\Box^{-1}R)$ gravity, thanks to the non-local pseudotensor, fundamental for astrophysical calculations such as the power carried by gravitational waves.

The paper is organized as follows. In the Sec. 2, we firstly define the non-local integral operator \Box^{-1} , then we prove both that it is the inverse operator of d'Alembertian \Box and a generalization of the Green second identity to the \Box -operator on the manifold. In addition, we perform the total variation of non-local gravitational action with respect to both the metric tensor and the coordinates. Then we derive the field equations from a variational principle. Sec. 3 is devoted to

the application of the Noether theorem to the non-local gravitational action for global translations. The procedure allows us to derive the related Noether current, i.e., the locally conserved energymomentum pseudotensor of the gravitational field in non-local gravity. Hence, in Sec. 4, we prove the non-local generalized contracted Bianchi identities and then analyze the energy-momentum complex for gravitational and matter fields, in particular its non-local nature and its conservation. In Sec. 5, we carry out the expansion to lower order in the metric perturbation $h_{\mu\nu}$ of non-local gravitational energy-momentum pseudotensor. Finally, we discuss results and draw conclusions in Sec. 6.

2 Variational principle and field equations for non-local gravity

Let the spacetime \mathcal{M} be a differentiable 4-manifold endowed with a Lorentzian metric g and Ω be a i four-dimensional region in \mathcal{M} . We can define the integral operator \Box^{-1} as follows

Definition 2.1. Let G(x, x') be the retarded Green function of the differential operator \Box , i.e., the solution of the partial differential equation

$$\sqrt{-g(x)} \square_x G(x, x') = \delta^4(x - x') ,.$$
 (2.1)

It is subject to retarded boundary condition, due to the causality principle. It is

$$G(x, x') = 0 \quad \forall t < t' , \qquad (2.2)$$

with the d'Alembert operator defined as

$$\Box = \nabla^{\mu} \nabla_{\mu} = \frac{1}{\sqrt{-g}} \partial_{\sigma} \left(\sqrt{-g} g^{\sigma \lambda} \partial_{\lambda} \right) \,. \tag{2.3}$$

If $p \in C_o^{\infty}(\mathbb{R}^4)$ is an element of the space of infinitely differentiable functions with compact support, then the operator

$$\Box^{-1}: C_o^{\infty}(\mathbb{R}^4) \to C_o^{\infty}(\mathbb{R}^4) , \qquad (2.4)$$

is given by

$$(\Box^{-1}p)(x) = \int_{\Omega} d^4x' \sqrt{-g(x')} G(x, x') p(x') , \qquad (2.5)$$

where $\Omega \subseteq \mathbb{R}^4$ and $supp(p) = \overline{\Omega}$.

From now on, we shell identify the region Ω of the manifold \mathcal{M} with its image $\phi(\Omega)$ through the chart $\phi : \Omega \subseteq \mathcal{M} \to \phi(\Omega) \subseteq \mathbb{R}^4$. It always exists because the manifold is differentiable and therefore covered by an atlas. Likewise, we can identify the boundary of the region $\partial\Omega$ with the action of chart ϕ on it, i.e., $\phi(\partial\Omega)$. Therefore, let us consider functions, and more generally, vector and tensor fields on the manifold, as defined on the open set of \mathbb{R}^4 by means of the graph ϕ . Thus we have

Theorem 2.1. Let $\Omega \subseteq \mathbb{R}^4$ be an open set and $f, h \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be two twice continuously differentiable functions in the open and once in its closure. If the boundary $\partial\Omega$ is a closed, regular and orientable three-dimensional hypersurface, then

$$\int_{\Omega} d^4x \sqrt{-g} (f \Box h - h \Box f) = \int_{\partial \Omega} dS_{\mu} \sqrt{-g} (f \nabla^{\mu} h - h \nabla^{\mu} f) , \qquad (2.6)$$

where dS_{μ} is the infinitesimal hypersurface element. Under the further assumption that functions f and h vanish on boundary, i.e., f = h = 0 on $\partial\Omega$, we get

$$\int_{\Omega} d^4x \sqrt{-g} (f \Box h) = \int_{\Omega} d^4x \sqrt{-g} (h \Box f) .$$
(2.7)

Proof. By means of the Leibniz rule applied to the functions f and h, we find the differential identity

$$f\Box h = h\Box f + \nabla_{\mu}(f\nabla^{\mu}h - h\nabla^{\mu}f) .$$
(2.8)

So the integral (2.6) can be written as

$$\int_{\Omega} d^4x \sqrt{-g} (f \Box h) = \int_{\Omega} d^4x \sqrt{-g} (h \Box f) + \int_{\Omega} d^4x \sqrt{-g} \nabla_{\mu} (f \nabla^{\mu} h - h \nabla^{\mu} f) , \qquad (2.9)$$

that, thanks to the Gauss theorem, transforms the second volume integral of Eq (2.9) into a surface integral as

$$\int_{\Omega} d^4x \sqrt{-g} (f \Box h) = \int_{\Omega} d^4x \sqrt{-g} (h \Box f) + \int_{\partial \Omega} dS_{\mu} \sqrt{-g} (f \nabla^{\mu} h - h \nabla^{\mu} f) \ .$$

If f and h are zero on $\partial\Omega$, then the integral on the boundary $\partial\Omega$ vanishes and we get Eq (2.7).

Then, we show that \Box^{-1} operator (2.5) is the inverse operator of the d'Alembert operator \Box . We can enunciate the following proposition

Theorem 2.2. For all
$$p \in C_o^{\infty}(\mathbb{R}^4)$$
, \Box^{-1} is the inverse of \Box , i.e.,
 $(\Box\Box^{-1})p = (\Box^{-1}\Box)p = \mathbb{1}p = p$
(2.10)

Proof. From the definition of product between two operator, we have

$$(\Box\Box^{-1})p(x) \equiv \Box(\Box^{-1}p)(x) = \Box_x \int_{\Omega} d^4x' \sqrt{-g(x')}G(x,x')p(x')$$
$$= \int_{\Omega} d^4x' \sqrt{-g(x')}\Box_x G(x,x')p(x')$$
$$= \frac{1}{\sqrt{-g(x)}} \int_{\Omega} d^4x' \sqrt{-g(x')}\delta^4(x-x')p(x') = p(x) , \quad (2.11)$$

where we used definition (2.5) and the following identity involving the Dirac δ distribution function

$$f(x) = \int_{\Omega} d^4 x' \,\delta(x - x') f(x') , \qquad (2.12)$$

non-null in $x \in \Omega$ and zero elsewhere. We have to prove now the second identity in Eq. (2.10), by means of Theorem (2.1). Hence we have

$$(\Box^{-1}\Box)p(x) \equiv \Box^{-1}(\Box p)(x) = \int_{\Omega} d^{4}x' \sqrt{-g(x')}G(x,x')\Box_{x'}p(x')$$
$$= \int_{\Omega} d^{4}x' \sqrt{-g(x')}\Box_{x'}G(x,x')p(x')$$
$$= \int_{\Omega} d^{4}x' \sqrt{-g(x')}\frac{\delta^{4}(x'-x)}{\sqrt{-g(x')}}p(x') = p(x) , \quad (2.13)$$

Let us now consider the following gravitational Lagrangian

$$S_g = \frac{1}{2\chi} \int_{\Omega} d^4x \sqrt{-g} \left(R + Rf(\Box^{-1}R) \right) , \qquad (2.14)$$

where f is an analytic function of $\Box^{-1}R$ and $\chi = 8\pi G/c^4$ is a dimensional constant that measures the coupling between matter and geometry. The variation of gravitational action (2.14) with respect to both metric tensor and coordinates, denoted by $\tilde{\delta}$, reads as

$$\tilde{\delta}S_g = \frac{1}{2\chi} \int_{\Omega} d^4x \left[\delta(\sqrt{-gR}) + \delta(\sqrt{-gR}) f(\Box^{-1}R) + \sqrt{-gR}\delta\left(f(\Box^{-1}R)\right) + \partial_{\mu}(R + Rf(\Box^{-1}R)\delta x^{\mu}) \right], \quad (2.15)$$

where δ is the variation at fixed coordinates. Also, we have to introduce a further theorem useful for the variation of gravitational action (2.15), which allows us, under suitable assumptions, to move the \Box^{-1} operator from a factor to another of the product in the integral.

Theorem 2.3. Let $f, h \in C^{\infty}(\Omega)$ be two infinitely differentiable functions on $\Omega \subseteq \mathbb{R}^4$, that is, $f, h : \Omega \to \mathbb{C}$. If \Box^{-1} is the inverse integral operator of the d'Alembert operator \Box as defined in (2.5), then

$$\int_{\Omega} d^4x \sqrt{-g(x)} f(x) \left(\Box^{-1} h \right)(x) = \int_{\Omega} d^4x \sqrt{-g(x)} h(x) \left(\Box^{-1} f \right)(x) .$$
 (2.16)

Proof. Let us prove Theorem (2.3) considering the identity (2.12). It follows

$$\begin{split} \int_{\Omega} d^{4}x \sqrt{-g(x)} f(x) \left(\Box^{-1}h \right) (x) \\ &= \int_{\Omega} d^{4}x \sqrt{-g(x)} \int_{\Omega''} d^{4}x'' f(x'') \delta(x-x'') \int_{\Omega'} d^{4}x' \sqrt{-g(x')} G(x',x) h(x') \\ &= \int_{\Omega'} d^{4}x' \sqrt{-g(x')} h(x') \int_{\Omega''} d^{4}x'' \left(\int_{\Omega} d^{4}x \sqrt{-g(x)} G(x',x) \delta(x-x'') \right) f(x'') \\ &= \int_{\Omega'} d^{4}x' \sqrt{-g(x')} h(x') \int_{\Omega''} d^{4}x'' \sqrt{-g(x')} G(x',x'') f(x'') \\ &= \int_{\Omega'} d^{4}x' \sqrt{-g(x')} h(x') \int_{\Omega''} d^{4}x'' \sqrt{-g(x')} h(x') \left(\Box^{-1}f \right) (x') . \quad (2.17) \end{split}$$

Here Ω , Ω' and Ω'' are the same region covered by different charts.

We establish, furthermore, a new relation that connects the variation of \Box and that of \Box^{-1} . **Theorem 2.4.** Let \Box be the d'Alembert operator with its inverse operator \Box^{-1} satisfying the identity

$$\Box \left(\Box^{-1} \right) = \Box^{-1} (\Box) = \mathbb{1} .$$

$$(2.18)$$

For all $p \in C^{\infty}(\mathbb{R}^4)$, we get

$$\left(\delta \Box^{-1}\right)p = -\Box^{-1}\delta(\Box)\Box^{-1}p , \qquad (2.19)$$

where δ is the first variation of the operator part only.

Proof. Varying both sides of identity (2.18) and taking into account that variation of the Identity operator $\mathbb{1}$ is zero, we have

$$\delta(\Box\Box^{-1}) = \delta\mathbb{1} = 0 , \qquad (2.20)$$

and then, from Eq. (2.18), we get

$$(\delta\Box)\Box^{-1} + \Box\delta(\Box^{-1}) = 0.$$
(2.21)

By means of the action of \Box^{-1} operator on the left side of Eq. (2.21), we obtain

$$\Box^{-1}(\delta\Box)\Box^{-1} + \delta(\Box^{-1}) = 0 , \qquad (2.22)$$

from which follows the relation (2.19).

Thanks to the above theorems, we are ready to split Eq. (2.15) in three parts. The first part is the same as in General Relativity

$$\frac{1}{2\chi} \int_{\Omega} d^4 x \,\delta(\sqrt{-g}R) = \frac{1}{2\chi} \int_{\Omega} d^4 x \,\sqrt{-g} \,G_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} \,\nabla_{\sigma} \Big[g_{\mu\nu} \nabla^{\sigma} \delta g^{\mu\nu} - \nabla_{\lambda} \delta g^{\sigma\lambda} \Big] \,, \qquad (2.23)$$

while the second one is

$$\frac{1}{2\chi} \int_{\Omega} d^{4}x \left[\delta(\sqrt{-g}R) f(\Box^{-1}R) \right] = \frac{1}{2\chi} \int_{\Omega} d^{4}x \left\{ \sqrt{-g} f G_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} f \nabla_{\sigma} \left[g_{\mu\nu} \nabla^{\sigma} \delta g^{\mu\nu} - \nabla_{\lambda} \delta g^{\sigma\lambda} \right] \right\} \\
= \frac{1}{2\chi} \int_{\Omega} d^{4}x \left\{ \sqrt{-g} (G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu}) f \delta g^{\mu\nu} + \sqrt{-g} \nabla_{\sigma} \left[\left(g_{\mu\nu} \nabla^{\sigma} \delta g^{\mu\nu} - \nabla_{\lambda} \delta g^{\sigma\lambda} \right) f - \left(g^{\lambda\sigma} g_{\mu\nu} \delta g^{\mu\nu} - \delta g^{\lambda\sigma} \right) \nabla_{\lambda} f \right] \right\}, \quad (2.24)$$

where $G_{\mu\nu}$ is the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R . \qquad (2.25)$$

Finally, we have for the third part of Eq. (2.15), from Eqs. (2.16) and (2.19), the following form

$$\frac{1}{2\chi} \int_{\Omega} d^4x \sqrt{-g} R\delta\left(f(\Box^{-1}R)\right) = \frac{1}{2\chi} \int_{\Omega} d^4x \sqrt{-g} Rf'\delta\left(\Box^{-1}R\right) \\
= \frac{1}{2\chi} \int_{\Omega} d^4x \left[\sqrt{-g} Rf'\left(\delta(\Box^{-1})R + \Box^{-1}[\delta R]\right)\right] \\
= \frac{1}{2\chi} \int_{\Omega} d^4x \left[\sqrt{-g} Rf'\Box^{-1}[\delta R] - \sqrt{-g} Rf'\Box^{-1}\delta(\Box)\Box^{-1}R\right], \quad (2.26)$$

where $f' = \frac{\partial f(\Box^{-1}R)}{\partial(\Box^{-1}R)}$. The first piece of Eq. (2.26) in the last line, from the identity (2.16), gives

$$\frac{1}{2\chi} \int_{\Omega} d^4x \sqrt{-g} Rf' \Box^{-1}[\delta R] = \frac{1}{2\chi} \int_{\Omega} d^4x \sqrt{-g} \Box^{-1}[Rf'] \delta R$$

$$= \frac{1}{2\chi} \int_{\Omega} d^4x \left\{ \sqrt{-g} \Box^{-1}[Rf'] R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) \Box^{-1}[Rf'] \delta g^{\mu\nu} + \sqrt{-g} \nabla_{\sigma} \left[\left(g_{\mu\nu} \nabla^{\sigma} \delta g^{\mu\nu} - \nabla_{\lambda} \delta g^{\sigma\lambda} \right) \Box^{-1}[Rf'] - \left(g^{\lambda\sigma} g_{\mu\nu} \delta g^{\mu\nu} - \delta g^{\lambda\sigma} \right) \nabla_{\lambda} \Box^{-1}[Rf'] \right\}. \quad (2.27)$$

While the second piece of Eq. (2.26) in the last line, by means of the d'Alembert operator (2.3) and from Eq. (2.16), yields

$$\frac{1}{2\chi} \int_{\Omega} d^{4}x \left[-\sqrt{-g}Rf' \Box^{-1}\delta(\Box)\Box^{-1}R \right] \\
= \frac{1}{2\chi} \int_{\Omega} d^{4}x \left[-\sqrt{-g}\Box^{-1}[Rf']\delta(\Box)\Box^{-1}R \right] \\
= \frac{1}{2\chi} \int_{\Omega} d^{4}x \left[-\sqrt{-g}\Box^{-1}[Rf']\delta\left(\frac{1}{\sqrt{-g}}\right)\partial_{\sigma}\left(\sqrt{-g}g^{\sigma\lambda}\partial_{\lambda}\right)\Box^{-1}R \right] \\
- \sqrt{-g}\Box^{-1}[Rf']\frac{1}{\sqrt{-g}}\partial_{\sigma}\left(\delta\left(\sqrt{-g}g^{\sigma\lambda}\right)\partial_{\lambda}\right)\Box^{-1}R \right] \\
= \frac{1}{2\chi} \int_{\Omega} d^{4}x \left\{ \sqrt{-g} \left[-\frac{1}{2}g_{\mu\nu}R\Box^{-1}[Rf']\delta g^{\mu\nu} \right] + \partial_{\sigma}\left(\Box^{-1}[Rf']\right)\partial_{\lambda}\left(\Box^{-1}R\right)\delta\left(\sqrt{-g}g^{\sigma\lambda}\right) \\
- \partial_{\sigma} \left[\Box^{-1}[Rf']\partial_{\lambda}\left(\Box^{-1}R\right)\delta\left(\sqrt{-g}g^{\sigma\lambda}\right) \right] \right\}. \quad (2.28)$$

According to Eqs. (2.23), (2.24), (2.27), (2.28) and the following relation

$$\delta\left(\sqrt{-g}g^{\sigma\lambda}\right) = \sqrt{-g}\left(\delta^{(\sigma}_{\mu}\delta^{\lambda)}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\sigma\lambda}\right) , \qquad (2.29)$$

the variation of the gravitational action (2.14) can be written as follows

$$\begin{split} \tilde{\delta}S_{g} &= \frac{1}{2\chi} \int_{\Omega} d^{4}x \sqrt{-g} \Biggl\{ \Biggl\{ G_{\mu\nu} + \left(G_{\mu\nu} + g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu} \right) \Biggl[f + \Box^{-1} [Rf'] \Biggr] \\ &+ \Biggl[\delta_{\mu}^{(\sigma} \delta_{\nu}^{\lambda)} - \frac{1}{2} g_{\mu\nu} g^{\sigma\lambda} \Biggr] \partial_{\sigma} \left(\Box^{-1} [Rf'] \right) \partial_{\lambda} \left(\Box^{-1} R \right) \Biggr\} \delta g^{\mu\nu} \\ &+ \sqrt{-g} \nabla_{\sigma} \Biggl[\left(g_{\mu\nu} \nabla^{\sigma} \delta g^{\mu\nu} - \nabla_{\lambda} \delta g^{\sigma\lambda} \right) + \left(\delta g^{\lambda\sigma} - g^{\lambda\sigma} g_{\mu\nu} \delta g^{\mu\nu} \right) \nabla_{\lambda} \left(f + \Box^{-1} [Rf'] \right) \\ &+ \left(g_{\mu\nu} \nabla^{\sigma} \delta g^{\mu\nu} - \nabla_{\lambda} \delta g^{\sigma\lambda} \right) \left(f + \Box^{-1} [Rf'] \right) \\ &- \left(\delta_{\mu}^{(\sigma} \delta_{\nu}^{\lambda)} - \frac{1}{2} g_{\mu\nu} g^{\sigma\lambda} \right) \nabla_{\lambda} \left(\Box^{-1} R \right) \Box^{-1} [Rf'] \delta g^{\mu\nu} + \left(R + Rf \right) \delta x^{\sigma} \Biggr] \Biggr\} . \quad (2.30)$$

From the least action principle $\delta S_g = 0$, if field variations and its derivatives vanish on boundary, the field equations in vacuum are obtained, i.e.,

$$G_{\mu\nu} + \Delta G_{\mu\nu} = 0 , \qquad (2.31)$$

with

$$\Delta G_{\mu\nu} = \left(G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu}\right) \left[f + \Box^{-1}[Rf']\right] + \left[\delta^{(\sigma}_{\mu}\delta^{\lambda)}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\sigma\lambda}\right]\partial_{\sigma}\left(\Box^{-1}[Rf']\right)\partial_{\lambda}\left(\Box^{-1}R\right) , \quad (2.32)$$

or if we define

$$\mathcal{G}[P](x) = \left(\Box^{-1}P\right)(x) , \qquad (2.33)$$

Eq. (2.31) can be rewritten as

$$G_{\mu\nu} + \left(G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu}\right) \left[f + \mathcal{G}[Rf']\right] + \left[\delta^{(\sigma}_{\mu}\delta^{\lambda)}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\sigma\lambda}\right]\partial_{\sigma}\left(\mathcal{G}[Rf']\right)\partial_{\lambda}\left(\mathcal{G}[R]\right) = 0. \quad (2.34)$$

We can find the field equations in presence of matter using the following action

$$S_m = \frac{1}{2\chi} \int_{\Omega} d^4 x \sqrt{-g} \mathcal{L}_m , \qquad (2.35)$$

and imposing the stationarity of total action, i.e.,

$$\delta(S_g + S_m) = 0 , \qquad (2.36)$$

with the matter energy-momentum tensor defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}} .$$
(2.37)

Hence, the field equations in presence of matter are [7]

$$G_{\mu\nu} + \Delta G_{\mu\nu} = \chi T_{\mu\nu} , \qquad (2.38)$$

or

$$G_{\mu\nu} + \left(G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu}\right) \left[f + \mathcal{G}[Rf']\right] + \left[\delta^{(\sigma}_{\mu}\delta^{\lambda)}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\sigma\lambda}\right] \partial_{\sigma}\left(\mathcal{G}[Rf']\right) \partial_{\lambda}\left(\mathcal{G}[R]\right) = \chi T_{\mu\nu} . \quad (2.39)$$

We shall use these considerations to derive the gravitational energy-momentum pseudotensor.

3 Gravitational energy-momentum pseudotensor in non-local gravity

Let us now use the Noether theorem to derive the non-local gravitational energy-momentum pseudotensor. If the infinitesimal coordinate transformations

$$x^{\prime\mu} = x^{\mu} + \delta x^{\mu} , \qquad (3.1)$$

leave the gravitational action (2.14) unchanged, $\tilde{\delta}S_g = 0$, and the domain of integration Ω can be chosen arbitrarily, by means of the variation (2.30) and the assumption that the metric tensor $g_{\mu\nu}$ is solution of the field equations in vacuum (2.34), we find a conserved current J^{σ} , i.e., the Noether current [46], which reads as

$$2\chi J^{\sigma} = R\delta x^{\sigma} - \left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right) \nabla_{\lambda}\delta g_{\mu\nu} + \left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right) \nabla_{\lambda} \left(f + \Box^{-1}[Rf']\right) \delta g_{\mu\nu} - \left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right) \left(f + \Box^{-1}[Rf']\right) \nabla_{\lambda}\delta g_{\mu\nu} - \left(\frac{1}{2}g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right) \nabla_{\lambda} \left(\Box^{-1}R\right) \Box^{-1}[Rf']\delta g_{\mu\nu} + Rf\delta x^{\sigma} ,$$

$$(3.2)$$

that obeys the following local continuity equation

$$\partial_{\sigma} \left(\sqrt{-g} J^{\sigma} \right) = 0 . \tag{3.3}$$

Integrating the continuity equation (3.3) over a three-dimensional volume V at a given time x^0 , from the Gauss theorem, we obtain

$$\frac{d}{dx^0} \int_V d^3x \sqrt{-g} J^0 = -\int_{\partial V} dS_i \sqrt{-g} J^i .$$
(3.4)

If the fields with their derivatives vanish on the boundary ∂V , the surface integral on the right of Eq. (3.4) vanishes, i.e., there is no current crossing the boundary, and we can derive the conserved Noether charge in the volume V, associated to symmetries (3.1)

$$Q = \int_{V} d^{3}x \sqrt{-g} J^{0} .$$
 (3.5)

So, if we consider the one-parameter group of diffeomorphisms for the global infinitesimal translations

$$x^{\prime\mu} = x^{\mu} + \epsilon^{\mu} , \qquad (3.6)$$

the local variation δ of tensor metric $g_{\mu\nu}$ becomes

$$\delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -g_{\mu\nu,\alpha}\epsilon^{\alpha}$$
 (3.7)

Hence, the conserved Noether current, related to the translational symmetry (3.6), becomes the energy-momentum density of the gravitational field, while, for isolated systems, where the spacetime is asymptotically flat at spatial infinity, the conserved Noether charge becomes the energy and momentum of the gravitational field. Therefore, the translation invariance of gravitational action, from Eq. (3.2), gives

$$\tau^{\sigma}_{\ \alpha} = \tau^{\sigma\,(GR)}_{\ \alpha} + \Delta \tau^{\sigma}_{\ \alpha} \ , \tag{3.8}$$

where $\tau^{\sigma\,(GR)}_{\alpha}$ is the Einstein pseudotensor

$$2\chi \tau^{\sigma}{}^{(GR)}_{\alpha} = R\delta^{\sigma}_{\alpha} + \left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right)\left(g_{\mu\nu,\alpha\lambda} - \Gamma^{\beta}{}_{\lambda\mu}g_{\beta\nu,\alpha}\right), \qquad (3.9)$$

while the correction $\Delta \tau^{\sigma}_{\ \alpha}$, is the gravitational energy-momentum pseudotensor of non-local part, i.e.,

$$2\chi\Delta\tau^{\sigma}_{\ \alpha} = Rf\delta^{\sigma}_{\alpha} + \left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right)\left(g_{\mu\nu,\alpha\lambda} - \Gamma^{\beta}_{\ \lambda\mu}g_{\beta\nu,\alpha}\right)\left(f + \Box^{-1}[Rf']\right) \\ - \left\{\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right)\nabla_{\lambda}\left(f + \Box^{-1}[Rf']\right)\right\} \\ - \left(\frac{1}{2}g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right)\nabla_{\lambda}\left(\Box^{-1}R\right)\Box^{-1}[Rf']\right\}g_{\mu\nu,\alpha}$$
(3.10)

The pseudotensor (3.10) has been obtained taking into account that the covariant derivative of variation for the metric tensor is

$$\nabla_{\lambda}\delta g_{\mu\nu} = \partial_{\lambda}\delta g_{\mu\nu} - \Gamma^{\alpha}{}_{\lambda\mu}\delta g_{\alpha\nu} - \Gamma^{\alpha}{}_{\lambda\nu}\delta g_{\alpha\mu} .$$
(3.11)

The symmetry of Levi Civita connection leads to

$$\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right)\Gamma^{\beta}_{\ \lambda\nu} = 0 , \qquad (3.12)$$

and the local conservation of pseudotensor can be read as

$$\partial_{\alpha} \left(\sqrt{-g} \, \tau^{\sigma}_{\ \alpha} \right) = 0 \,, \tag{3.13}$$

being

$$J^{\alpha} = \tau^{\sigma}_{\ \alpha} \epsilon^{\alpha} \ . \tag{3.14}$$

In terms of Eq. (2.33), in more compact form, one gets

$$2\chi\Delta\tau^{\sigma}_{\alpha} = Rf\delta^{\sigma}_{\alpha} + \left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right)\left(g_{\mu\nu,\alpha\lambda} - \Gamma^{\beta}_{\lambda\mu}g_{\beta\nu,\alpha}\right)\left(f + \mathcal{G}[Rf']\right) \\ - \left\{\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right)\partial_{\lambda}\left(f + \mathcal{G}[Rf']\right)\right\} \\ - \left(\frac{1}{2}g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right)\partial_{\lambda}\left(\mathcal{G}[R]\right)\mathcal{G}[Rf']\right\}g_{\mu\nu,\alpha}$$
(3.15)

It has to be emphasized that, from Eqs. (3.8), (3.9) and (3.10), it is clear that the geometric object τ^{σ}_{α} is a pseudotensor not a tensor. In other words, it transforms like a tensor under affine transformations but not under generic transformations. So τ^{σ}_{α} is at least an affine tensor. In an asymptotically flat spacetime the tensoriality is recovered and the integral (3.5) returns to being a four-vector for asymptotic linear coordinates, that is,

$$P^{\alpha} = \int_{V} d^{3}x \sqrt{-g} \tau^{\alpha}_{\ 0} , \qquad (3.16)$$

represents the energy and momentum in V of the gravitational field. Moreover the pseudotensor τ^{σ}_{α} is a non-local object because it involves non-local terms, such as $\Box^{-1}R$ or $\Box^{-1}[Rf']$, whose value depends on the values assumed by the metric in the integration domain.

4 The energy-momentum complex

The stationarity of gravitational action, $\tilde{\delta}S_g = 0$, with respect to the variation $\tilde{\delta}$, from Eqs. (2.30), (2.31), (2.32), (3.8) and (3.10), gives

$$\frac{1}{2\chi}\sqrt{-g}\left(G_{\mu\nu} + \Delta G_{\mu\nu}\right)\delta g^{\mu\nu} + \partial_{\sigma}\left(\sqrt{-g}\tau^{\sigma}_{\ \alpha}\epsilon^{\alpha}\right) = 0.$$
(4.1)

Hence, inserting the field equations in presence of matter (2.38) into Eq. (4.1), we get

$$-\frac{1}{2}\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} + \partial_{\sigma}\left(\sqrt{-g}\tau^{\sigma}_{\ \alpha}\epsilon^{\alpha}\right) = 0.$$
(4.2)

From rigid translations and coordinates independence from ϵ^{α} , it yields

$$\frac{1}{2}\sqrt{-g}T^{\mu\nu}g_{\mu\nu,\alpha} + \partial_{\sigma}\left(\sqrt{-g}\tau^{\sigma}_{\alpha}\right) = -\sqrt{-g}\nabla_{\sigma}T^{\sigma}_{\ \alpha} + \partial_{\sigma}\left(\sqrt{-g}T^{\sigma}_{\ \alpha}\right) + \partial_{\sigma}\left(\sqrt{-g}\tau^{\sigma}_{\ \alpha}\right) , \qquad (4.3)$$

where the identity

$$\sqrt{-g}\nabla_{\sigma}T^{\sigma}_{\ \alpha} = \partial_{\sigma}\left(\sqrt{-g}T^{\sigma}_{\ \alpha}\right) - \frac{1}{2}\sqrt{-g}\,g_{\mu\nu,\alpha}T^{\mu\nu} \ , \tag{4.4}$$

has been taken into account. From Eq. (4.3), we obtain

$$\partial_{\sigma} \left[\sqrt{-g} \left(T^{\sigma}_{\ \alpha} + \tau^{\sigma}_{\ \alpha} \right) \right] = \sqrt{-g} \nabla_{\sigma} T^{\sigma}_{\ \alpha} .$$

$$\tag{4.5}$$

According to the previous considerations, it is possible to prove generalized contracted Bianchi identities for non-local gravity [43, 48, 49]. They guarantee the conservation of energy–momentum complex of gravitational and matter components. Let us first demonstrate a lemma useful for our purpose.

Lemma 4.1. Let $f \in C^2(\Omega)$ be a twice continuously differentiable function on an open set Ω of \mathbb{R}^4 , ∇ be the covariant derivative, \Box be the d'Alembert operator and [,] be the commutator, we have

$$[\nabla_{\nu}, \Box]f = -R_{\mu\nu}\nabla^{\mu}f . \qquad (4.6)$$

Proof. From the commutator of two covariant derivatives ∇_{μ} and ∇_{ν} , which acts on the contravariant vector field A^{γ} , we get

$$[\nabla_{\mu}, \nabla_{\nu}]A^{\gamma} = R^{\gamma}{}_{\lambda\mu\nu}A^{\lambda} .$$

$$(4.7)$$

If we set $A^{\gamma} = \nabla^{\gamma} f$ and $\gamma = \nu$ in Eq. (4.7), we obtain

$$\begin{split} [\nabla_{\mu}, \Box] f &= \nabla_{\mu} \nabla_{\nu} \nabla^{\nu} f - \nabla_{\nu} \nabla^{\nu} \nabla_{\mu} f \\ &= \nabla^{\nu} [\nabla_{\mu}, \nabla_{\nu}] f - [\nabla_{\mu}, \nabla_{\nu}] \nabla^{\nu} f = \nabla^{\nu} [\nabla_{\mu}, \nabla_{\nu}] f - R_{\mu\nu} \nabla^{\nu} f . \end{split}$$
(4.8)

Thus, the commutativity of covariant derivatives of a function, that is,

$$[\nabla_{\mu}, \nabla_{\nu}]f = 0 , \qquad (4.9)$$

inserted into Eq. (4.8), gives us the result (4.6).

Theorem 4.1 (Non-local generalized contracted Bianchi identities). Let $G_{\mu\nu}$ be the Einstein tensor and $\Delta G_{\mu\nu}$ be the corrections to the field equations due to non-local terms as in Eq. (2.38), then the covariant 4-divergence of their sum vanishes, i.e.,

$$\nabla^{\mu} (G_{\mu\nu} + \Delta G_{\mu\nu}) = 0 . \qquad (4.10)$$

Proof. We carry out the 4-divergence of Eq. (2.32) and we have

$$\nabla^{\mu}\Delta G_{\mu\nu} = \left(\nabla^{\mu}G_{\mu\nu} + \nabla_{\nu}\Box - \Box\nabla_{\nu}\right)\left(f + \Box^{-1}[Rf']\right) + G_{\mu\nu}\nabla^{\mu}(f + \Box^{-1}[Rf']) \\ + \frac{1}{2}\left(\delta^{\lambda}_{\nu}\nabla^{\sigma} + \delta^{\sigma}_{\nu}\nabla^{\lambda} - g^{\sigma\lambda}\nabla_{\nu}\right)\nabla_{\sigma}\Box^{-1}[Rf']\nabla_{\lambda}\Box^{-1}R .$$
(4.11)

So, from the contracted Bianchi identities

$$\nabla^{\mu}G_{\mu\nu} = 0 , \qquad (4.12)$$

and performing some calculations, Eq. (4.11) can be rewritten as follows

$$\nabla^{\mu}\Delta G_{\mu\nu} = [\nabla_{\nu}, \Box] \left(f + \Box^{-1}[Rf'] \right) + G_{\mu\nu} \nabla^{\mu} (f + \Box^{-1}[Rf']) + \frac{1}{2} \left(\Box \Box^{-1}[Rf'] \nabla_{\nu} \Box^{-1}R + \nabla_{\sigma} \Box^{-1}[Rf'] \nabla^{\sigma} \nabla_{\nu} \Box^{-1}R + \nabla^{\sigma} \nabla_{\nu} \Box^{-1}[Rf'] \nabla_{\sigma} \Box^{-1}R + \nabla_{\nu} \Box^{-1}[Rf'] \Box \Box^{-1}R - \nabla_{\nu} \nabla^{\sigma}[Rf'] \nabla_{\sigma} \Box^{-1}R - \nabla_{\sigma} \Box^{-1}[Rf'] \nabla_{\nu} \nabla^{\sigma} \Box^{-1}R \right).$$
(4.13)

Now, the relation (4.13) and the lemma (4.1) lead to Eq. (4.10), that is, we find

$$\nabla^{\mu}\Delta G_{\mu\nu} = -R_{\mu\nu} \left(f + \Box^{-1}[Rf'] \right) + G_{\mu\nu} \nabla^{\mu} (f + \Box^{-1}[Rf']) + \frac{1}{2} Rf' \nabla_{\nu} \Box^{-1} R + \frac{1}{2} R \nabla_{\nu} \Box^{-1}[Rf'] = -R_{\mu\nu} f' \nabla^{\mu} \Box^{-1} R - R_{\mu\nu} \nabla^{\mu} \Box^{-1}[Rf'] + G_{\mu\nu} \nabla^{\mu} (f + \Box^{-1}[Rf']) + \frac{1}{2} g_{\mu\nu} Rf' \nabla^{\mu} \Box^{-1} R + \frac{1}{2} g_{\mu\nu} R \nabla^{\mu} \Box^{-1}[Rf'] = -G_{\mu\nu} \nabla^{\mu} \left(f + \Box^{-1}[Rf'] \right) + G_{\mu\nu} \nabla^{\mu} (f + \Box^{-1}[Rf']) = 0 . \quad (4.14)$$

According to the field equation in presence of matter (2.38), Eq. (4.10) leads to the standard covariant conservation of matter energy-momentum tensor, that is,

$$\nabla_{\mu}T^{\mu\nu} = 0 . \qquad (4.15)$$

It implicitly defines the trajectories of particles, that is, the time-like metric geodesics on the spacetime manifold. Finally, Eq. (4.5) gives the local conservation of energy-momentum complex $\mathcal{T}^{\sigma}_{\alpha}$ in non-local gravity, that is, the *continuity equation for energy-momentum complex in non-local gravity*

$$\partial_{\sigma} \left[\sqrt{-g} \left(T^{\sigma}_{\ \alpha} + \tau^{\sigma}_{\ \alpha} \right) \right] = 0 \quad .$$

$$(4.16)$$

We can define

$$\mathcal{T}^{\sigma}_{\ \alpha} = T^{\sigma}_{\ \alpha} + \tau^{\sigma}_{\ \alpha} \ , \tag{4.17}$$

involving all gravitational and matter contributions.

5 Weak field limit of non-local gravitaty energy-momentum pseudotensor

Let us now develop the low energy limit perturbing the metric tensor $g_{\mu\nu}$ around the Minkowskian metric $\eta_{\mu\nu}$. It is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} ,$$
 (5.1)

and then, we can calculate the pseudotensor (3.10) or (3.15) to lowest order in the perturbation $h_{\mu\nu}$, that is, up to second ordear in $h_{\mu\nu}$. Therefore we get, at the order h^2 ,

$$\left(\tau^{\sigma}_{\alpha}\right)^{(2)} = \left(\tau^{\sigma}_{\alpha}^{(GR)}\right)^{(2)} + \left(\Delta\tau^{\sigma}_{\alpha}\right)^{(2)} , \qquad (5.2)$$

where the Einstein pseudo-tensor is

$$2\chi \left(\tau^{\sigma \, (GR)}_{\alpha}\right)^{(2)} = R^{(2)}\delta^{\sigma}_{\alpha} + \left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\sigma\nu}\right)^{(1)}g^{(1)}_{\mu\nu,\alpha\lambda} , \qquad (5.3)$$

and, from Eq. (3.15), the non-local perturbation of pseudotensor takes the form

$$2\chi \left(\Delta \tau^{\sigma}_{\ \alpha}\right)^{(2)} = R^{(1)} f^{(1)} \delta^{\sigma}_{\alpha} + \left(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\sigma\nu}\right)^{(0)} \left(f^{(1)} + \mathcal{G}^{(1)}[Rf']\right)_{,\lambda} g^{(1)}_{\mu\nu,\alpha} - \left(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\sigma\nu}\right)^{(0)} \left(f^{(1)} + \mathcal{G}^{(1)}[Rf']\right) g^{(1)}_{\mu\nu,\alpha\lambda} .$$
(5.4)

Then, we expand f as

$$f(\mathcal{G}[R])(x) = f(0) + f'(0)\mathcal{G}[R](x) + \dots , \qquad (5.5)$$

and imposing the case f(0) = 0, the relation (5.5) to the first order takes the form

$$f^{(1)}(\mathcal{G}[R])(x) = f'(0)\mathcal{G}^{(1)}[R](x) .$$
(5.6)

Taking into account the following first order perturbations in a generic coordinate system, the Ricci scalar becomes

$$R^{(1)} = \left(h^{\beta\gamma}_{,\beta\gamma} - \Box^{(0)}h\right) , \qquad (5.7)$$

where

$$\Box^{(0)} = \eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta} , \qquad (5.8)$$

and the non-local operator \Box^{-1} at first order reads as

$$\mathcal{G}^{(1)}[R](x) = \left(\Box^{-1}R\right)^{(1)}(x) = -h(x) + \widetilde{\mathcal{G}}\left[h^{\beta\gamma}_{,\beta\gamma}\right](x) , \qquad (5.9)$$

where

$$\widetilde{\mathcal{G}}\left[h^{\beta\gamma}{}_{,\beta\gamma}\right](x) = \int_{\Omega} d^4x' G(x,x') h^{\beta\gamma}{}_{,\beta\gamma}(x') .$$
(5.10)

We have to prove the identity (5.9). Using Eqs. (2.1), (2.5), (5.7) and the theorem (2.1), it is

$$\left(\Box^{-1} R \right)^{(1)} (x) = \int_{\Omega'} d^4 x' \sqrt{-g(x')}^{(0)} G(x, x') R^{(1)}(x')$$

$$= \int_{\Omega} d^4 x' \sqrt{-g(x')}^{(0)} G(x, x') \left(h^{\beta\gamma}{}_{,\beta\gamma}(x') - \Box_{x'} h(x') \right)$$

$$= -\int_{\Omega} d^4 x' \Box_{x'} G(x, x') h(x') + \int_{\Omega} d^4 x' G(x, x') h^{\beta\gamma}{}_{,\beta\gamma}(x')$$

$$= -\int_{\Omega} d^4 x' \delta(x - x') h(x') + \widetilde{\mathcal{G}} \left[h^{\beta\gamma}{}_{,\beta\gamma} \right] (x) = -h(x) + \widetilde{\mathcal{G}} \left[h^{\beta\gamma}{}_{,\beta\gamma} \right] (x) .$$
 (5.11)

Furthermore, we perform the first-order perturbation of $\mathcal{G}[Rf']$, namely

$$\mathcal{G}^{(1)}[Rf'](x) = \int_{\Omega} d^4 \sqrt{-g(x')}^{(0)} G(x, x') R^{(1)}(x') f'^{(0)}[\mathcal{G}](x')$$
$$= f'(0) \int_{\Omega} d^4 \sqrt{-g(x')}^{(0)} G(x, x') R^{(1)}(x') = f'(0) \mathcal{G}^{(1)}[R](x) . \quad (5.12)$$

Finally substituting the Eqs. (5.7), (5.9) and (5.12) in the non-local perturbed gravitational energymomentum pseudotensor (5.4), we derive the non-local corrections of the gravitational pseudo-tensor τ^{σ}_{α} to the second order in $h_{\mu\nu}$, that is,

$$2\chi \left(\Delta \tau^{\sigma}{}_{\alpha}\right)^{(2)} = \left\{ \left(h^{\beta\gamma}{}_{,\beta\gamma} - \Box h\right) \left(-h + \widetilde{\mathcal{G}} \left[h^{\beta\gamma}{}_{,\beta\gamma}\right]\right) \delta^{\sigma}_{\alpha} + 2 \left(\eta^{\mu\nu} \eta^{\lambda\sigma} - \eta^{\mu\lambda} \eta^{\nu\sigma}\right) \left(-h + \widetilde{\mathcal{G}} \left[h^{\beta\gamma}{}_{,\beta\gamma}\right]\right)_{,\lambda} h_{\mu\nu,\alpha} - 2 \left(\eta^{\mu\nu} \eta^{\lambda\sigma} - \eta^{\mu\lambda} \eta^{\nu\sigma}\right) \left(-h + \widetilde{\mathcal{G}} \left[h^{\beta\gamma}{}_{,\beta\gamma}\right]\right) h_{\mu\nu,\alpha\lambda} \right\} f'(0) \right\}$$
(5.13)

The non-local contribution in Eq. (5.13) is evident and, as discussed in Refs. [50–52], it can contribute to gravitational radiation representing a signature for non-local gravity.

6 Discussion and Conclusions

In this paper, we investigated how non-locality gravity induces correction terms $\Delta \tau^{\sigma}_{\alpha}$ into the Einstein gravitational pseudotensor. Considering the Noether theorem and imposing the invariance of gravitational action under rigid translations, we found the associated conserved Noether current and charge. They can be interpreted as the gravitational density of the energy-momentum and the energy and momentum of gravitational field present in a spatial volume enclosing localized massive objects. The density and flux density of the gravitational energy and momentum expressed in Eq. (3.10) are not described by a covariant tensor, which means that, under general coordinate transformations, it does not transform like a tensor. The geometrical object (3.10) is an affine tensor or pseudotensor because it transforms like a tensor only under affine transformations. The non-tensorial character of Eq. (3.10) is closely linked to the non-localization of gravitational energy which holds also in non-local gravity. The non-locality of the gravitational pseudotensor intervenes through integral operators, like \Box^{-1} , where its value, at a given point x, takes into account the value assumed by the fields in other points x' of the spacetime. Then, by generalizing the contracted Bianchi identities to the non-local gravity, we have obtained an equation of continuity for the energy-momentum complex that ensures its local conservation. Finally, we studied the behavior at low energies of the non-local corrections of the gravitational pseudotensor (5.13), expanding it up to the second order in $h_{\mu\nu}$. The non-local gravitational energy-momentum pseudotensor is a crucial physical quantity because, thanks to the gravitational waves obtained and analyzed in the papers [50-52], it is possible to calculate the power emitted by a radiative system and transported by the waves with all its polarizations and multipole terms. The presence, in the gravitational radiation, of a scalar component with lower multipoles, in addition to the standard quadrupole tensor component, can be investigated thanks to the gravitational pseudotensor. In this perspective, it can give a relevant signature for the non-local gravity. In a forthcoming paper, we will investigate possible observational constraints on these features.

Acknowledgements

This paper is based upon work from COST Action CA21136 Addressing observational tensions in cosmology with systematics and fundamental physics (CosmoVerse) supported by COST (European

Cooperation in Science and Technology). Authors acknowledge the Istituto Nazionale di Fisica Nucleare (INFN) Sez. di Napoli, Iniziative Specifiche QGSKY and MOONLIGHT, and the Istituto Nazionale di Alta Matematica (INdAM), gruppo GNFM.

References

- [1] L. Modesto, Super-renormalizable quantum gravity Phys. Rev. D 86, 044005 (2012).
- [2] L. Modesto, L. Rachwał, Nonlocal quantum gravity: A review, Int. J. Mod. Phys. D 26, 1730020 (2017).
- [3] L. Modesto, L. Rachwał, I. L. Shapiro, Renormalization group in super-renormalizable quantum gravity, Eur. Phys. J. C 78, 555 (2018).
- [4] S. Capozziello and F. Bajardi, Nonlocal gravity cosmology: An overview, Int. J. Mod. Phys. D 31, 2230009 (2022).
- [5] S. Deser and R. P. Woodard, Nonlocal Cosmology, Phys. Rev. Lett. 99, 111301 (2007).
- [6] S. Deser and R. P. Woodard, Nonlocal cosmology II. Cosmic acceleration without fine tuning or dark energy, JCAP 06, 034 (2019).
- S. Deser and R. P. Woodard, Observational Viability and Stability of Nonlocal Cosmology, J. Cosmol. Astropart. Phys. 1311, 036 (2013).
- [8] L. Buoninfante, G. Lambiase, and L. Petruzziello, Quantum interference in external gravitational fields beyond General Relativity, Eur. Phys. J. C 81, 928 (2021).
- [9] L. Buoninfante, G. Lambiase, Y. Miyashita, W. Takebe, and M. Yamaguchi, Generalized ghostfree propagators in nonlocal field theories, Phys. Rev. D 101, 084019 (2020).
- [10] L. Buoninfante, G. Lambiase, and M. Yamaguchi, Nonlocal generalization of Galilean theories and gravity, Phys. Rev. D 100, 026019 (2019).
- [11] L. Buoninfante, A. Ghoshal, G. Lambiase, and A. Mazumdar, Transmutation of nonlocal scale in infinite derivative field theories, Phys. Rev. D 99, 044032 (2019).
- [12] L. Buoninfante, A. S. Cornell, G. Harmsen, A S. Koshelev, G. Lambiase, and A. Mazumadra, *Towards nonsingular rotating compact object in ghost-free infinite derivative gravity*, Phys. Rev. D 98, 084041 (2018).
- [13] L. Buoninfante, G. Lambiase, and A. Mazumdar, *Ghost-free infinite derivative quantum field theory*, Nucl. Phys. B 944, 114646 (2019).
- [14] L. Buoninfante, A. S. Koshelev, G. Lambiase, and A. Mazumdar, Classical properties of nonlocal, ghost- and singularity-free gravity, JCAP 09, 034 (2018).
- [15] B. Mashhoon, Nonlocal Gravity, International Series of Monographs on Physics 167 (Oxford University Press, 2017).
- [16] B. Mashhoon, Toward a non-local theory of gravitation, Ann. Phys. 519, 57 (2007).

- [17] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, (Pergamon, Oxford, 1960).
- [18] J. D. Jackson, *Classical Electrodynamics*, (Wiley, Hoboken N. J., 1999).
- [19] G. Chirco, C. Rovelli, and P. Ruggiero, *Thermally correlated states in Loop Quantum Gravity*, Class. Quant. Grav. **32**, 035011 (2015).
- [20] A. Acunzo, F. Bajardi and S. Capozziello, Non-local curvature gravity cosmology via Noether symmetries, Phys. Lett. B 826 (2022), 136907.
- [21] I. Dimitrijevic, B. Dragovich, Z. Rakic and J. Stankovic, Nonlocal de Sitter gravity and its exact cosmological solutions, JHEP 12 (2022), 054.
- [22] I. Dimitrijevic, B. Dragovich, Z. Rakic and J. Stankovic, New Cosmological Solutions of a Nonlocal Gravity Model, Symmetry 14 (2022), 3
- [23] S. Nojiri, S. D. Odintsov, M. Sasaki and Y. I. Zhang, Screening of cosmological constant in non-local gravity, Phys. Lett. B 696 (2011), 278.
- [24] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, *Ghost-free non-local F(R) Gravity Cosmology*, Phys. Dark Univ. 28 (2020), 100541.
- [25] G. Calcagni, L. Modesto and Y. S. Myung, Black-hole stability in non-local gravity, Phys. Lett. B 783 (2018), 19.
- [26] S. Capozziello and N. Godani, Non-local gravity wormholes, Phys. Lett. B 835, 137572 (2022).
- [27] S. S. Xulu, The Energy-Momentum Problem in General Relativity, Ph. D. thesis, University of Zululand 2003. https://doi.org/10.48550/arXiv.hep-th/0308070
- [28] D. Hestenes, Energy-Momentum Complex in General Relativity and Gauge Theory, Adv. Appl. Clifford Algebra 31, 51 (2021).
- [29] J. N. Goldberg, Conservation Laws in General Relativity, Phys. Rev. 111, 315 (1958).
- [30] D. L. Lee, A. P. Lightman, and W. T. Ni, Conservation laws and variational principles in metric theories of gravity Phys. Rev. D 10, 1685 (1974).
- [31] N. Rosen, The Energy of the Universe, Gen Rel Grav 26, 319 (1994).
- [32] G. Lessner, Møller's energy-momentum complex Once again, Gen Relat Gravit 28, 527 (1996).
- [33] T. N. Palmer, Gravitational energy-momentum: The Einstein pseudotensor reexamined, Gen Relat Gravit 12, 149 (1980).
- [34] M. Ferraris and M. Francaviglia, Covariant first-order Lagrangians, energy-density and superpotentials in general relativity, Gen Relat Gravit 22, 965 (1990).
- [35] F. I. Mikhail, M. I. Wanas, A. Hindawi, and E. I. Lashin, Energy-Momentum Complex in Møller's Tetrad Theory Of Gravitation, Int. J. Theor. Phys. 32, 1627 (1993).

- [36] J. M. Aguirregabiria, A. Chamorro, and K. S. Virbhadra, Energy and angular momentum of charged rotating black holes, Gen Relat Gravit 28, 1393 (1996).
- [37] S. Capozziello, M. Capriolo, and M. Transirico, The gravitational energy-momentum pseudotensor of higher order theories of gravity, Ann. Phys. 525, 1600376 (2017).
- [38] S. Capozziello, M. Capriolo and G. Lambiase, Energy-Momentum Complex in Higher Order Curvature-Based Local Gravity, Particles 5(3), 298 (2022).
- [39] H. Abedi, S. Capozziello, M. Capriolo, and A. M. Abbassi, *Gravitational energy-momentum* pseudo-tensor in Palatini and metric f(R) gravity, Annals of Physics **439**, 168796 (2022).
- [40] P. Wang, G. M. Kremer, D. S. M. Alves, and X. H. Meng, A note on energy-momentum conservation in Palatini formulation of L(R) gravity, Gen Relativ Gravit 38, 517 (2006).
- [41] T. Multamaki, A. Putaja, I. Vilja, and E. C. Vagenas, *Energy-momentum complexes in* f(R) theories of gravity, Class. Quant. Grav. **25**, 075017(2008).
- [42] B. Dongsu, D. Cangemi, R. Jackiw, Energy-momentum conservation in gravity theories, Phys. Rev. D 49, 5173(1994).
- [43] T. Koivisto, A note on covariant conservation of energy-momentum in modified gravities, Class. Quant. Grav. 23, 4289 (2006).
- [44] D. E. Barraco, E. Domínguez, and R. Guibert, Conservation laws, symmetry properties, and the equivalence principle in a class of alternative theories of gravity, Phys. Rev. D 60, 044012 (1999).
- [45] S. Capozziello, M. Capriolo, and M. Transirico, *The gravitational energy-momentum pseudo*tensor: the cases of f(R) and f(T) gravity, Int. J. Geom. Meth. Mod. Phys. **15**(supp01), 1850164 (2018).
- [46] F. Bajardi and S. Capozziello, Noether Symmetries in Theories of Gravity, Cambridge University Press, (2022) Cambridge, ISBN 978-1-00-920872-7, 978-1-00-920874-1, doi:10.1017/9781009208727
- [47] J. W. Maluf, The gravitational energy-momentum tensor and the gravitational pressure, Ann. Phys. 517, 723 (2005).
- [48] T. S. Koivisto, Dynamics of Nonlocal Cosmology, Phys. Rev. D 77, 123513 (2008).
- [49] T. S. Koivisto, Newtonian limit of nonlocal cosmology, Phys. Rev. D 78, 123505 (2008).
- [50] M. Capriolo, Gravitational radiation in higher order non-local gravity, Int. J. Geom. Methods Mod. Phys. 19(10), 2250159 (2022).
- [51] S. Capozziello and M. Capriolo, *Gravitational waves in non-local gravity*, Class. Quantum Grav. 38, 175008 (2021).
- [52] S. Capozziello, M. Capriolo, and S. Nojiri, Considerations on gravitational waves in higherorder local and non-local gravity, Phys. Lett. B 810, 135821 (2020).