The Journal of Geometric Analysis
Volume 14, Number 1, 2004

The Entropy Formula for Linear Heat Equation

By *Lei Ni*

ABSTRACT. We derive the entropy formula for the linear heat equation on general Riemannian manifolds and prove that it is monotone non-increasing on manifolds with nonnegative Ricci curvature. As applications, we study the relation between the value of entropy and the volume of balls of various scales. The results are simpler version, without Ricci flow, of Perelman's recent results on volume non-collapsing for Ricci flow on compact manifolds. We also prove that if the entropy for the heat kernel achieves its maximum value zero at some positive time, on any complete Riamannian manifold with nonnegative Ricci curvature, if and only if the manifold is isometric to the Euclidean space.

1. Introduction

In a recent article of Perelman [20], an entropy formula for Ricci flow was derived. The formula turns out to be of fundamental importance in the study of Ricci flow (cf. [20, Sections 3, 4, 10]) as well as the Kähler–Ricci flow [21]. The derivation of the entropy formula in [20, Section 9] resembles the gradient estimate for the linear heat equation proved by Li–Yau in another fundamental article [18] on the linear parabolic equation. This suggests that there may exist a similar entropy formula for the linear heat equation. The purpose of this short note is to show such entropy formula and derive some applications of the new entropy formula.

Let M be a complete Riemannian manifold. We study the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = 0\,. \tag{1.1}$$

Following [20], we define

$$\mathcal{W}(f,\tau) = \int_M \left(\tau|\nabla f|^2 + f - n\right) \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\, dv \tag{1.2}$$

restricted to (f,τ) satisfying

$$\int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\, dv = 1 \tag{1.3}$$

Math Subject Classifications. Primary: 58G11.
Key Words and Phrases. Entropy formula, heat equation, logarithmic Sobolev.
Acknowledgements and Notes. Research partially supported by NSF grant DMS-0328624.

ISSN 1050-6926

with $\tau > 0$.

Theorem 1.1. *Let M be a closed Riemannian manifold. Assume that u is a positive solution to the heat Equation* (1.1) *with $\int_M u\, dv = 1$. Let f be defined as $u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$ and $\tau = \tau(t)$ with $\frac{d\tau}{dt} = 1$. Then*

$$\frac{d\mathcal{W}}{dt} = -\int_M 2\tau \left(\left| \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right|^2 + R_{ij} f_i f_j \right) u\, dv\,. \tag{1.4}$$

In particular, if M has nonnegative Ricci curvature, $\mathcal{W}(f, \tau)$ is monotone decreasing along the heat equation.

Notice that in the case that M is Ricci flat, the result above is indeed a special case of Perelman's result. We show that the monotonicity of the entropy holds for all complete manifolds with nonnegative Ricci curvature.

The result can be derived out of a point-wise differential inequality. The proof of Theorem 1.1 and the argument of [20] gives the following differential inequality for the fundamental solution to the heat equation.

Theorem 1.2. *Let M be a closed Riemannian manifolds with nonnegative Ricci curvature. Let H be the positive heat kernel. Then*

$$t\left(2\Delta f - |\nabla f|^2\right) + f - n \leq 0\,, \tag{1.5}$$

for $t > 0$ with $H = \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$.

Notice that Li–Yau's gradient estimate $\frac{u_t}{u} - \frac{|\nabla u|^2}{u^2} + \frac{n}{2t} \geq 0$ is equivalent to

$$t(2\Delta f) - n \leq 0\,. \tag{1.6}$$

The inequality (1.6) can be viewed as a generalized Laplacian comparison theorem. In deed, the Laplacian comparison theorem on M is a consequence of (1.6) by applying the inequality to the heat kernel and letting $t \to 0$. This suggests that one can view $\bar{L}(x, t) = 4tf$ as the square of a time-dependent 'distance function'; then (1.6), which says that $\Delta \bar{L} \leq 2n$, simply generalizes the standard Laplacian comparison $\Delta r^2 \leq 2n$ on any Ricci non-negative manifold to such generalized 'distance function.' From this point of view, one can think (1.5) as a Laplacian comparison theorem in the space-time since it says $\Delta \bar{L} + \bar{L}_t \leq 2n$. The similar inequality [20, (7.15)] was one of the important new discoveries of Perelman. Applying similar consideration, one can also view the entropy estimate in [20] as a generalization of the space-time Laplacian comparison theorem. This is also related to the reduced volume monotonicity of [20]. It was pointed to us later by Professor S.T. Yau that he and Hamilton also noticed (1.5) for the Ricci flow a few years ago. The relation between (1.4) and Li-Yau's estimate (1.6) will be shown in the addendum to this paper.

For closed manifolds, following [20], one can define

$$\mu(\tau) = \inf_{\int_M u\, dv = 1} \mathcal{W}(f, \tau)\,. \tag{1.7}$$

A direct consequence of Theorem 1.1 is the following.

Corollary 1.3. *On manifolds with nonnegative Ricci curvature, $\mu(\tau)$ is a monotone decreasing function of τ. Moreover, $\mu(\tau) \le 0$ with $\lim_{\tau\to 0}\mu(\tau) = 0$.*

Whenever it makes sense (for example, when M is simply connected, negatively curved with lower bound on its curvature), as in [20], one can also define $\nu = \inf_\tau \mu(\tau)$ and it can be thought as some sort of isoperimetric constant. When $M = \mathbb{R}^n$, $\nu = 0$. Thanks to the gradient estimates of Li–Yau [18], the above results still hold on complete noncompact manifolds with nonnegative Ricci curvature. As an application of the entropy formula obtained in Theorem 1.1 we prove the following result.

Theorem 1.4. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Then $\mathcal{W}(f, t) \ge 0$, with $u = \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$ being the heat kernel, for some $t > 0$ if and only if M is isometric to $\mathbb{R}^n$.*

In [11] (see also [23, 26]), a sharp logarithmic Sobolev inequality (in different disguises in [23, 26]) was proved on $\mathbb{R}^n$. When $M = \mathbb{R}^n$, the inequality is equivalent to

$$\int_M \left(\frac{1}{2}|\nabla f|^2 + f - n\right)\frac{e^{-f}}{(2\pi)^{\frac{n}{2}}}\, dv \ge 0 \tag{1.8}$$

for all f with $\int_M \frac{e^{-f}}{(2\pi)^{\frac{n}{2}}}\, dv = 1$.

Since (1.8) is equivalent to $\mu(\frac{1}{2}) \ge 0$, a simple corollary of Theorem 1.4 is the following result on the relation between the logarithmic Sobolev inequality and the geometry of the manifolds, which is originally due to Bakry, Concordet and Ledoux [1].

Corollary 1.5. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Then (1.8) holds on M if and only if M is isometric to $\mathbb{R}^n$.*

It can be shown that the (1.8) holds on any manifold with sharp isoperimetric inequality, or equivalently the sharp L^1-Sobolev inequality. Under the request of some readers of the preliminary version, we include a proof of this fact, which is communicated to the author by Perelman in 2002 (cf. Proposition 4.1). The proof only uses spherical symmetrization to compare with the Euclidean case. It does not give a proof to (1.8) in the Euclidean case. One can find the simple elegant proof of (1.8) by Beckner and Pearson in [3], which makes use of the fact that the product of Euclidean spaces is still Euclidean together with the sharp L^2-Sobolev inequality.

It turns out that $\mathcal{W}(f, \tau)$ being finite, where $u = \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$ is the heat kernel, also has strong geometric and topological consequences. For example, in the case M has nonnegative Ricci curvature, it implies that M has finite fundamental group. In fact we can show that

M is of maximum volume growth if and only if the entropy $\mathcal{W}(f, t)$ is uniformly bounded for all $t > 0$, where $u = \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$ is the heat kernel.

The analogy above was discovered originally in [20] for the ancient solution to Ricci flow with bounded nonnegative curvature operator, where he claimed that an ancient solution to the Ricci flow with nonnegative curvature operator is κ-non-collapsed if and only if the entropy is uniformly bounded for any fundamental solution to the conjugate heat equation.

With some care in the estimates, we can show that the asymptotic value of the entropy $\mathcal{W}(x, t)$, as $t \to \infty$, is given by $\log\theta_\infty$. Here θ_∞ is the asymptotic volume ratio of the Riemannian manifold with nonnegative Ricci curvature. This certainly sharps the above statement on M being maximum volume growth and gives a quantified version of Theorem 1.4.

Without assuming the nonnegativity of the Ricci curvature, the bound on $\mu(\tau)$ also implies the uniform lower bound on the volume of balls of certain scales. Namely, it implies the volume noncollapsing, as in the κ-noncollapsing theorem of Perelman [20, Theorem 4.1], therefore an uniform upper bound of the diameter, if the manifold has finite volume. In some sense, $\mu(\tau)$ reflects the isoperimetric property of M for the scale parametrized by τ.

2. Proof of Theorems 1.1 and 1.2

We start with the following two lemmas.

Lemma 2.1. *Let M be a complete Riemannian manifold. Let u be a positive solution to* (1.1). *Then*

$$\left(\frac{\partial}{\partial t}-\Delta\right) w=-2\left(|\nabla_i\nabla_j f|^2+R_{ij}f_i f_j\right)-2<\nabla w,\nabla f> \tag{2.1}$$

where $w=2\Delta f-|\nabla f|^2$ and $f=-\log u$.

Proof. Direct calculation shows that

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right) w &= -2|\nabla_i\nabla_j f|^2-2R_{ij}f_i f_j-2<\nabla(\Delta f),\nabla f>+2<\nabla(f_t),\nabla f> \\ &\quad -2\Delta(f_t)+2f_{tt} \\ &= -2|\nabla_i\nabla_j f|^2-2R_{ij}f_i f_j-2<\nabla\left(|\nabla f|^2\right),\nabla f>-2\left(|\nabla f|^2\right)_t \\ &= -2|\nabla_i\nabla_j f|^2-2R_{ij}f_i f_j-2<\nabla\left(|\nabla f|^2+2f_t\right),\nabla f> \\ &= -2|\nabla_i\nabla_j f|^2-2R_{ij}f_i f_j-2<\nabla w,\nabla f>\ .\end{aligned}$$

Here we have used $w=2f_t+|\nabla f|^2$ and $\left(\frac{\partial}{\partial t}-\Delta\right)f=-|\nabla f|^2$. □

Lemma 2.2. *Let M and u be as in Lemma 2.1. Let $W=\tau(2\Delta f-|\nabla f|^2)+f-n$, where we write $u=\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$. Here $\tau=\tau(t)$ with $\frac{d\tau}{dt}=1$. Then*

$$\left(\frac{\partial}{\partial t}-\Delta\right) W=-2\tau\left(\left|\nabla_i\nabla_j f-\frac{1}{2t}g_{ij}\right|^2+R_{ij}f_i f_j\right)-2<\nabla W,\nabla f>\ . \tag{2.2}$$

Proof. One can proceed directly. Here we use Lemma 2.1 to simplify the calculation a little. Let $\bar{f}=-\log u$. We then have that

$$W=\tau w+\bar{f}-\frac{n}{2}\log(4\pi\tau)-n\ .$$

Keep in mind that $\left(\frac{\partial}{\partial t} - \Delta\right) \bar{f} = -|\nabla \bar{f}|^2$. The direct calculation shows that

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) W &= \tau \left(\frac{\partial}{\partial t} - \Delta\right) w + w - \left|\nabla \bar{f}\right|^2 - \frac{n}{2\tau} \\ &= -2\tau|\nabla_i\nabla_j f|^2 - 2\tau R_{ij} f_i f_j - 2\tau < \nabla w, \nabla f > + \left|\nabla \bar{f}\right|^2 + 2\bar{f}_t - \left|\nabla \bar{f}\right|^2 - \frac{n}{2\tau} \\ &= -2\tau|\nabla_i\nabla_j f|^2 - 2 < \nabla W, \nabla f > + 2\Delta \bar{f} - \frac{n}{2\tau} - 2\tau R_{ij} f_i f_j \\ &= -2\tau \left|\nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}\right|^2 - 2 < \nabla W, \nabla f > - 2\tau R_{ij} f_i f_j \ .\end{aligned}$$

Here we have used $\nabla f = \nabla \bar{f}$.

Remarks. 1. Lemma 2.1 has its corresponding version for the Ricci flow, namely, if g_{ij} satisfies the backward Ricci flow equation $\frac{\partial}{\partial t} g_{ij} = 2R_{ij}$ and u is a solution to $(\frac{\partial}{\partial t} - \Delta + R)u = 0$. Define $w = (2\Delta f - |\nabla f|^2 + R)$. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right) w = -2|R_{ij} + f_{ij}|^2 - 2 < \nabla w, \nabla f > \ . \tag{2.3}$$

Here $u = e^{-f}$. One can easily see that (2.3) implies the formula (2.2) of [20]. This also gives another derivation of the first monotonicity formula in [20].

2. The above approach of the proof to Lemma 2.2 was motivated by the statistical analogy in [20, Section 6]. One can also use the similar approach to derive Proposition 9.1 of [20] from (2.3) above. This would simplify the calculation a little and reflect the relation between the energy and the entropy quantity. □

Proof of Theorem 1.1. The proof of Theorem 1.1 follows from the simple observation $u\nabla f = -\nabla u$, therefore

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(Wu) &= -2\tau \left(\left|\nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}\right|^2 + R_{ij} f_i f_j\right) u \\ &\quad - 2 < \nabla W, \nabla f > u - 2 < \nabla W, \nabla u > \\ &= -2\tau \left(\left|\nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}\right|^2 + R_{ij} f_i f_j\right) u \ ,\end{aligned}$$

and integration by parts. □

Proof of Theorem 1.2. We can apply Perelman's argument in the proof of Corollary 9.3 of [20]. For any $t_0 > 0$, let h be any positive function. We solve the backward heat equation starting from t_0 with initial data h. We then have that

$$\begin{aligned}\frac{d}{dt}\int_M hWu\, dv &= \int_M (h_t)(Wu) + h(Wu)_t\, dv \\ &= \int_M (h_t + \Delta h)\, Wu + h((Wu)_t - \Delta(Wu))\, dv \\ &\le 0 \ .\end{aligned}$$

Using the fact that $(\int_M hWu\, dv)|_{t=0} = 0$, when u is the fundamental solution, we have that

$$\int_M h(Wu)\, dv \le 0$$

for any $t_0 > 0$ and any positive function h. This implies that $Wu \le 0$. Therefore $W \le 0$. □

3. Extensions and the value of $\mu(0)$

The first extension is to complete noncompact manifolds. From the proof, it is easy to see that Theorems 1.1 and 1.2 hold as long as the integration by parts can be justified. We focus on the case M having nonnegative Ricci curvature. Since we have the gradient estimate of Li–Yau for the positive solutions, one can make the integration by part rigorous, keeping in mind that u is assumed integrable in our consideration of the entropy. One of the references where one can find the estimates on derivatives of u is [4, Section 3].

Another extension of Theorems 1.1 and Theorem 1.2 is for manifolds with boundary. In this case, it is not hard to show that the theorem holds on manifolds with convex boundary. In fact, in this case

$$\begin{aligned}\frac{\partial W}{\partial \nu} &= \left(2\tau f_\tau + \tau|\nabla f|^2 + f - n\right)_\nu\\ &= 2\tau\sum_i^{n-1} f_i f_{i\nu}\\ &= -2\tau h_{ij} f_i f_j\\ &\le 0\,.\end{aligned}$$

Here h_{ij} denotes the second fundamental form of ∂M. Therefore, Theorems 1.1 and 1.2 hold for positive solution u with the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$.

Corollary 3.1. *Let M be a compact manifold with boundary. Let u be a positive solution to* (1.1) *with the Neumann boundary condition. Let f and τ be as in Theorem 1.1. Then*

$$\frac{d}{dt}\mathcal{W} = -\int_M 2\tau\left(\left|\nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}\right|^2 + R_{ij} f_i f_j\right)u\,dv - 2\int_{\partial M}\tau II\left((\nabla f)^T, (\nabla f)^T\right)dA. \quad (3.1)$$

Here $II(\cdot,\cdot)$ is the second fundamental form of ∂M and $(\nabla f)^T$ is the tangential projection of ∇f on ∂M. In particular, in the case M has nonnegative Ricci and ∂M is convex, $\mathcal{W}$ is monotone decreasing. Moreover, if u is the fundamental solution,

$$t\left(2\Delta f - |\nabla f|^2\right) + f - n \le 0\,, \quad (3.2)$$

for $t > 0$.

Since we know that $\mu(\tau)$ is monotone, it is nice to know the value of $\mu(\tau)$ as $\tau \to 0$. We can adapt the argument of [20] to prove that $\lim_{\tau\to 0}\mu(\tau) = 0$.

Proposition 3.2. *Let M be a closed manifold.*

$$\lim_{\tau\to 0}\mu(\tau) = 0\,. \quad (3.3)$$

Proof. It is easy to see that $\mu(\tau) \le 0$ by Theorem 1.2. Assume that there exists $\tau_k \to 0$ such that $\mu(\tau_k) \le c < 0$ for all k. We show that this will contradict the logarithmic Sobolev inequality. We are going to blow up the metric by $\frac{1}{2}\tau^{-1}$. First we can decompose M into open subsets $U_1, U_2, \cdots, U_N$ such that each U_j is contained inside some normal coordinates and each U_j also contains $B(o_j, \delta)$, a ball of radius δ, for some small $\delta > 0$. Now let $g^\tau = \frac{1}{2}\tau^{-1}g_{ij}$ and $g_k = g^{\tau_k}$. It is clear that (U_j, g_k, o_j) converges to $(\mathbb{R}^n, g_0, 0)$ in C^∞ norm. We will also identify the compact subset of $\mathbb{R}^n$ with the compact subset of U_j.

It is easy to see that

$$\mathcal{W}_g(f, \tau) = \int_M \left(\frac{1}{2}|\nabla f|_\tau^2 + f - n\right) \frac{e^{-f}}{(2\pi)^{\frac{n}{2}}}\, dv_\tau$$

with restriction $\int_M \frac{e^{-f}}{(2\pi)^{\frac{n}{2}}}\, dv_\tau = 1$, where $|\cdot|_\tau$ is the norm with respect to $g^\tau = \frac{1}{2}\tau^{-1}g$ and dv_τ is the corresponding volume form. It is also convenient to write in more standard form.

$$\mathcal{W}(\psi, \tau) = \int_M \left(2|\nabla\psi|_\tau^2 - \left(\log\psi^2\right)\psi^2 - \left(\frac{n}{2}\log(2\pi) + n\right)\psi^2\right) dv_\tau \tag{3.4}$$

restricted to $\int_M \psi^2\, dv_\tau = 1$. Let φ_k be the minimizer realizing $\mu(\tau_k)$. Then we have that

$$-2\Delta_k\varphi_k - 2\varphi_k\log\varphi_k = \left(\mu(\tau_k) + n + \frac{n}{2}\log(2\pi)\right)\varphi_k \tag{3.5}$$

and

$$\int_M \varphi_k^2\, dv_k = 1\,. \tag{3.6}$$

Here Δ_k denote the Laplacian of g_{τ_k} and $dv_k = dv_{\tau_k}$. Due to the monotonicity, we can also assume that $\mu(\tau_k) \geq -A$ for some $A > 0$ independent of k.

Now we write $F_k(\psi) = 2|\nabla\psi|_{\tau_k}^2 - (\log\psi^2)\psi^2 - (\frac{n}{2}\log(2\pi) + n)\psi^2$. It is a easy matter to check that

$$\frac{\int F(\lambda\psi)\, dv_\tau}{\int(\lambda\psi)^2\, dv_\tau} = \frac{\int F(\psi)\, dv_\tau}{\int \psi^2\, dv_\tau} - \log\lambda^2\,. \tag{3.7}$$

By the assumption that $\mu(\tau_k) \leq c < 0$ we know that

$$\int_M F(\varphi_k)\, dv_k \leq c < 0\,.$$

By passing to subsequence we can assume that

$$\int_{U_1} F(\varphi_k)\, dv_k \leq \frac{c}{N} < 0\,.$$

It is easy to see that $\int_{U_1} \varphi_k^2\, dv_k \leq 1$. Combining the above with (3.5) and the fact that g_k converges to g_0 on every fixed compact subset of $\mathbb{R}^n$, the elliptic PDE theory implies that there exists a subsequence of φ_k (still denote by φ_k) such that it converges uniformly on every compact subset of $\mathbb{R}^n$. If the limit function φ_∞ exists and $\int_{\mathbb{R}^n} \varphi_\infty^2\, dv_0 > 0$ we claim that we will get contradiction to the logarithmic Sobolev inequality (1.8). In fact in this case we just denote $\epsilon^2 = \int_{\mathbb{R}^n} \varphi_\infty^2$. Clearly $0 < \epsilon \leq 1$ by the assumption. Since[1]

$$\int_{\mathbb{R}^n} F(\varphi_\infty)\, dv_0 \leq \frac{c}{N} < 0 \tag{3.8}$$

[1] This needs justification. One can consult [24] for a proof of Proposition 3.2 with Ricci flow, via the maximum principle.

then by (3.7) we have that

$$\int_{\mathbb{R}^n} F\left(\frac{1}{\epsilon}\varphi_\infty\right) dv_0 \le \frac{c}{N} + 2\log\epsilon < \frac{c}{N}\,.$$

Let $\frac{e^{-f_\infty}}{(2\pi)^{\frac{n}{2}}} = \left(\frac{1}{\epsilon}\varphi_\infty\right)^2$ we have that $\int_{\mathbb{R}^n} \frac{e^{-f_\infty}}{(2\pi)^{\frac{n}{2}}}\, dv_0 = 1$ and

$$\int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla f_\infty|_0^2 + f_\infty - n\right) \frac{e^{-f_\infty}}{(2\pi)^{\frac{n}{2}}}\, dv_0 \le \frac{c}{N} < 0\,.$$

This is a contradiction to the sharp logarithmic Sobolev inequality (1.8). On the other hand, if $\epsilon = 0$ it would imply $\varphi_\infty = 0$, contradicting (3.8). We therefore complete the proof of the proposition. □

4. Bounded entropy and volume growth

The main purpose of this section is to prove Theorem 1.4 and show that for the manifold with nonnegative Ricci curvature the finiteness of the entropy for the heat kernel is equivalent to the manifold haing maximum volume growth. We first include a short discussion on the logarithmic Sobolev inequality. We say M has logarithmic Sobolev inequality if

$$\int_M \left(\frac{1}{2}|\nabla f|^2 + f\right) \frac{e^{-f}}{(2\pi)^{\frac{n}{2}}}\, dv \ge -C_1 \tag{4.1}$$

for all f with restriction $\int_M \frac{e^{-f}}{(2\pi)^{\frac{n}{2}}}\, dv = 1$. This is equivalent to the finiteness of $\mu(\frac{1}{2})$. It is an easy matter to see that the regular Sobolev inequality implies (4.1). In particular, it holds on minimal submanifolds in $\mathbb{R}^n$, which is a special case of the general result in [10]. Since one has the L^2-Sobolev inequality on a closed manifold, any closed manifold satisfies (4.1). In this case the dependence of the constant C_1 can be explicitly traced, applying Lemma 2 of [16]. This would in turn give the explicit dependence of the κ constant on the geometry of the initial metric in the κ noncollapsing theorem of [20, Section 4].

The inequality (4.1) is also equivalent to the ultracontractivity as pointed out in [9], which then follows from conditions such as the lower bound on the Ricci curvature (i.e., $\mathrm{Ric} \ge -K$ for some $K \ge 0$) and $\inf V_x(1) \ge \delta > 0$. Therefore, $\mu(\frac{1}{2})$ is finite for a large class of manifolds.

It is interesting to find out on which manifolds the logarithmic Sobolev inequality holds with $C_1 = n$: namely, (1.8) holds. It was pointed out in [20] that the sharp isoperemetric inequality also implies the sharp logarithmic Sobolev inequality. The following was the proof suggested by the communication with Perelman.

Proposition 4.1 (Perelman). *Let M be a complete manifold such that*

$$A(\partial\Omega) \ge c_n V(\Omega)^{\frac{n-1}{n}}\,,$$

for any compact domain Ω with the Euclidean constant c_n. Here $A(\partial\Omega)$ is the area of the boundary $\partial\Omega$. Then (1.8) holds on M.

Proof. As we know in the proof of Proposition 3.2, (1.8) is equivalent to

$$\int_M 2|\nabla\phi|^2 - \left(\log\phi^2\right)\phi^2 - \left(\frac{n}{2}\log(2\pi) + n\right)\phi^2\, dv \ge 0\,. \tag{4.2}$$

It suffices to prove the result for compact supported nonnegative function ϕ. Let $M' = \{x|\phi(x) > 0\}$. Let $\bar{B}(R)$ be a ball of radius R in $\mathbb{R}^n$ such that $\mathrm{Vol}(\bar{B}(R)) = \mathrm{Vol}(M')$. We define that $F(t) = \mathrm{Vol}(\{x \in M'|\phi(x) \geq t\})$. We also denote $M_t = \{x \in M'|\phi(x) \geq t\}$. $\Gamma_t = \partial M_t$. Let $g(|y|)$ be a function on $\bar{B}(R)$ such that $\mathrm{Vol}(\{y|g(y) \geq t\}) = F(t)$ and $g(R) = 0$. We can define $\bar{M}_t$ and $\bar{\Gamma}_t$ similarly. Clearly $\mathrm{Vol}(M_t) = \mathrm{Vol}(\bar{M}_t)$ and $A(\Gamma_t) \geq A(\bar{\Gamma}_t)$ by the isoperimetric inequality. The simple integration by parts shows that

$$\int_0^\infty \lambda'(s)F(s)\,ds = \int_{M'} \lambda(f)\,dv \tag{4.3}$$

for any Lipschitz function $\lambda(t)$ with $\lambda(0) = 0$. This implies that

$$\int_{M'} \left(\log \phi^2\right)\phi^2 + \left(\frac{n}{2}\log(2\pi) + n\right)\phi^2\,dv = \int_{\bar{B}(R)} \left(\log g^2\right) g^2 + \left(\frac{n}{2}\log(2\pi) + n\right) g^2\,d\bar{v}\,.$$

On the other hand the isoperimetric inequality implies

$$\int_{M'} |\nabla\phi|^2\,dv \geq \int_{\bar{B}(R)} \left|\bar{\nabla} g\right|^2\,d\bar{v}\,. \tag{4.4}$$

In fact, the co-area formula shows that

$$\int_{M'} |\nabla\phi|^2\,dv = \int_0^\infty \int_{\Gamma_t} |\nabla\phi|\,dA\,dt$$

and

$$F(t) = \int_t^\infty \int_{\phi=s} \frac{1}{|\nabla\phi|}\,dA\,ds\,.$$

Combining with the fact $F(t) = \mathrm{Vol}(\bar{M}_t)$ we have that

$$\int_{\Gamma_t} \frac{1}{|\nabla\phi|}\,dA = \int_{\bar{\Gamma}_t} \frac{1}{\left|\bar{\nabla} g\right|}\,d\bar{A}\,.$$

Using the fact that $|\bar{\nabla} g|$ is constant on $\bar{\Gamma}_t$, by Hölder inequality, we have that

$$\begin{aligned}
\left(\int_{\bar{\Gamma}_t} \left|\bar{\nabla} g\right|\,d\bar{A}\right)\left(\int_{\bar{\Gamma}_t} \frac{1}{\left|\bar{\nabla} g\right|}\,d\bar{A}\right) &= A^2\left(\bar{\Gamma}_t\right) \\
&\leq A^2(\Gamma_t) \\
&\leq \left(\int_{\Gamma_t} |\nabla\phi|\,dA\right)\left(\int_{\Gamma_t} \frac{1}{|\nabla\phi|}\,dA\right)
\end{aligned}$$

which then implies that

$$\int_{\Gamma_t} |\nabla\phi|\,dA \geq \int_{\bar{\Gamma}_t} \left|\bar{\nabla} g\right|\,d\bar{A}\,. \tag{4.5}$$

Since (4.4) implies (4.5) we complete the proof. □

We should point out that in [2], Beckner provides a proof of (1.8) from the isoperimetric inequality using the product structure of the Euclidean spaces. The above argument just reduces the (1.8) for any manifolds, with the sharp isoperimetric inequality, to the Euclidean space (with the same dimension) case. It does not prove the Euclidean case itself. Now we prove Theorem 1.4.

Proof of Theorem 1.4. By the assumption, one can find τ_0 such that $\mathcal{W}(f,\tau_0)\ge 0$. But on the other hand, Theorem 1.2 implies that $\mathcal{W}(f,t)\le 0$ for $H=\frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$ being the heat kernel. This implies $\mathcal{W}(f,t)=0$ for $0<t<\tau_0$. Applying the equality case in Theorem 1.1 we have that $f_{ij}-\frac{1}{2t}g_{ij}=0$, which implies that

$$2t\Delta f = n\ . \tag{4.6}$$

On the other hand, by [6, 25] we know that $\lim_{t\to 0} -4t\log H = r^2(x,y)$. In particular,

$$\lim_{t\to 0} 4tf = r^2(x,y)\ .$$

Then (4.6) implies that

$$\Delta r^2(x,y) = 2n\ . \tag{4.7}$$

Combining with the assumption that Ricci is nonnegative this implies that M is isometric to $\mathbb{R}^n$. In fact, from (4.7) one can easily obtain that

$$\frac{A_x(r)}{V_x(r)} = n$$

where $A(r)$ and $V(r)$ denote the area of $\partial B_x(r)$ and the volume of $B_x(r)$, which then imply that $V_x(r)$ is same as the volume function of Euclidean balls. The equality case of the volume comparison theorem implies $M=\mathbb{R}^n$ □

It is clear that Theorem 1.4 implies Corollary 1.5. The proof of [1] to Corollary 1.5 relies on a deep result of Peter Li [17] on the large time behavior of the heat kernel. Since we only uses the behavior of the heat kernel near $t=0$, our proof is dual to theirs in some sense. In [15], the author proved that the sharp Sobolev inequalities on a Ricci nonnegative manifold also implies the manifold is isometric to $\mathbb{R}^n$. The case of L^1-Sobolev, which is equivalent to the isoperimetric inequality in Proposition 4.1, is relatively simple. The other cases are more involved. Please see [15] for details. It was also asked in [15] if the sharp Nash inequality implies the same conclusion or not. That still remains open.

Proposition 4.2. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Assume that M has maximum volume growth, namely $\frac{V_o(r)}{r^n}\ge\theta$ for some $\theta>0$. Then there exists $A=A(\theta,n)>0$ such that*

$$\mathcal{W}(f,t)\ge -A \tag{4.8}$$

for $u=\frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$ being the heat kernel. On the other hand, (4.8) *implies that M has maximum volume growth. Namely $\frac{V_o(r)}{r^n}\ge\theta$ holds for some $\theta=\theta(n,A)$.*

Proof. Let $v=\sqrt{u}$. One can rewrite $\mathcal{W}(f,t)$ as

$$\mathcal{W} = 4t\int_M |\nabla v|^2\,dv - \int_M \log\left(v^2\right)v^2\,dv - \left(n+\frac{n}{2}\log(4\pi t)\right)\ . \tag{4.9}$$

On the other hand, by Li–Yau's heat kernel estimate

$$\begin{aligned} v^2 = H(x,y,t) &\le \frac{C(n)}{V_x(\sqrt{t})}\\ &\le \frac{C(n)\theta}{t^{\frac{n}{2}}}\ . \end{aligned}$$

Hence

$$\mathcal{W} \geq -\log(C(n)\theta) - n - \frac{n}{2}\log(4\pi)\,.$$

Here we have used the fact $\int_M v^2\,dv = 1$.

To prove the second half of the claim we need to use the lower bound estimate of Li–Yau as well as the gradient estimate for the heat kernel. We first estimate the first term in (4.9) using inequality (1.6), the Li–Yau's gradient estimate:

$$\begin{aligned} 4t\int_M |\nabla v|^2\,dv &= t\int_M \frac{|\nabla H|^2}{H}\,dv \\ &\leq t\int_M \left(H_t + \frac{n}{2t}H\right)\,dv \\ &= \frac{n}{2}\,. \end{aligned} \tag{4.10}$$

The second term can be estimated as

$$\begin{aligned} -\int_M \log(H)H\,dv &\leq -\int_M \log\left(\frac{C_5(n)}{V_x\left(\sqrt{t}\right)}\exp\left(-\frac{r^2(x,y)}{3t}\right)\right)H\,dv_y \\ &\leq C_6(n) + \log\left(V_x\left(\sqrt{t}\right)\right) + \frac{1}{3t}\int_M r^2(x,y)H(x,y,t)\,dv_y \\ &\leq C_7(n) + \log\left(V_x\left(\sqrt{t}\right)\right)\,. \end{aligned} \tag{4.11}$$

Here C_i are positive constants only depending on n. We also have used Theorem 3.1 of [19] to estimate the last term of the second line above. Putting the assumption $\mathcal{W} \geq -A$ and (4.9)–(4.11) together we have the lower bound (4.1) for the volume. □

The similar result as above was claimed in [20, Section 11] for the Ricci flow ancient solutions. The proof here is easier than the nonlinear case considered in [20]. In fact, Proposition 4.2 here can be used in the proof of Theorem 10.1 of [20].

Corollary 4.3. *Let $w_\infty = \lim_{t\to\infty}\mathcal{W}(f,t)$ and $\theta_\infty = \lim_{r\to\infty}\frac{V_x(r)}{\omega_n r^n}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Then*

$$w_\infty = \log\theta_\infty.$$

Proof. The proof needs Corollary 1.2 in the addendum to this paper. □

5. Manifolds with bounded $\mu(\tau)$

The following result gives the geometric implication of the non-sharp logarithmic Sobolev inequality (4.3), or bounded $\mu(\tau)$. The result can be thought as the Riemannian version of the κ non-collapsing result of [20]. Notice that we do not even require M has nonnegative Ricci curvature. The results in this section are in line with Perelman's work on Kähler–Ricci flow [21]. However, the arguments in the nonlinear case are technically more involved than the case treated here, especially on the diameter bound.

Proposition 5.1. *Let M be a complete manifold. Assume that $\mu(\tau) \geq -A$, for all $0 \leq \tau \leq T$, for some constant $A > 0$. Then there exists a positive constant $\kappa(A,n) > 0$ such that*

$$V_x(R) \geq \kappa R^n \tag{5.1}$$

for all $R^2 \le T$. In particular, if $\mu(\tau) \ge -A$ for all $\tau \ge 0$, M has at least the Euclidean volume growth.

Proof. The first observation is that

$$\mu(\tau) \le \int_M \tau 4|\nabla h|^2 - \left(\log h^2 + \frac{n}{2}\log(4\pi\tau)\right) h^2\, dv \tag{5.2}$$

for compact supported nonnegative function h. If we have that

$$V_x\left(\frac{R}{2}\right) \ge \eta V_x(R) \tag{5.3}$$

for $\eta = \left(\frac{1}{3}\right)^n$ we will have the estimate (5.1). The reasoning is exactly as in [20], by choosing $h^2 = \frac{e^{-B}}{(4\pi R^2)^{\frac{n}{2}}}\zeta^2(r_x(y)/R)$, where ζ is a nonnegative cut-off function such that $\zeta(t) = 1$ for all $t \le \frac{1}{2}$, and $\zeta(t) = 0$ for $t \ge 1$. B is so chosen such that $\int_M h^2\, dv = 1$. Under the assumption (5.3) we have that

$$\log \frac{V_x(R)}{R^n} + C_1(n) \le B \le \log \frac{V_x(R)}{R^n} + C_2(n)\,.$$

Therefore, estimation on the right-hand side of (5.2) gives

$$-A \le \mu(R^2) \le C_3(n) + B$$

which implies (5.1) for some κ. Now argue by contradiction that (5.1) must holds. If not, we know that (5.3) cannot be true. Namely,

$$V_x\left(\frac{R}{2}\right) < \eta V_x(R)\,. \tag{5.4}$$

We focus on the smaller ball $B_x(\frac{R}{2})$. By the above argument we would conclude that

$$V_x\left(\frac{R}{4}\right) < \eta V_x\left(\frac{R}{2}\right)\,.$$

Otherwise we would have $V_x(\frac{R}{2}) \ge \kappa\left(\frac{R}{2}\right)^n$, which would imply $V_x(\frac{R}{2}) \ge \eta V_x(R)$ by the assumption (5.1) does not hold. Therefore, iterating the argument we have that

$$V_x\left(\frac{R}{2^k}\right) \le \eta^k V_x(R) \tag{5.5}$$

for all natural numbers k. This leads to

$$V_x(r) \le C r^{n\log_2 3}$$

for small r, which is a contradiction. □

The following result on the diameter of a manifold with bounded $\mu(\tau)$ is an easy consequence of Proposition 5.1

Corollary 5.2. *Let M be a Riemanian manifold such that $\mu(\tau) \ge -A$ for $1 \ge \tau \ge 0$. Assume also that $V(M) \le V_0$. Then there exists a constant $D = D(A, V_0, n)$ such that*

$$\text{Diameter}(M) \le D\,. \tag{5.6}$$

In fact, $D \leq 2([\frac{V_0}{\kappa}] + 1)$*. In particular, it implies that* M *is compact if it is complete.*

Remark 5.3. The argument in the proof of Proposition 5.1 was first indicated by Perelman in his MIT lectures to prove the volume non-collapsing for the Ricci flow assuming only the bound on the scalar curvature (in stead of the sectional curvature). The details are presented in the notes [8].

Concluding remarks 1) It would be interesting to find out if there is an interpolation between the entropy formula of Perelman and (1.4). Namely, to find a family monotonicity formulae connecting both. For the differential Harnack, or Li–Yau–Hamilton inequality, there is such interpolation in dimension two as shown by Chow [7]. The straightforward formulation seems not to work. (One could have some differential inequalities connecting both cases. But the differential inequalities do not give monotonicity formulae unless on two end points.)

2) It seems that the entropy formula in [20] is essentially different from the known one of Hamilton [12] for the Ricci flow on Riemann surfaces since it can be used to derive the uniform scalar curvature bound and diameter bounds without appealing the Harnack inequality (in [21] Perelman proved these results for Kähler–Ricci flow with $c_1(M) > 0$), unlike the approach in [12], which used the Harnack inequality for the Ricci flow essentially (in [5], using the similar method of Hamilton on Riemann surfaces, the authors proved the scalar curvature and diameter bound for the case when the manifolds has positive bisectional curvature, which is a special, relatively easier, case of what treated in [21]). Is there any connection between Perelman's entropy formula and Hamilton's entropy formula at all?[2]

3) Is Theorem 1.4 still true in $n = 3$ by assuming instead the scalar curvature $\mathcal{R}(x)$ of M is nonnegative?

4) In [4], the authors proved the matrix Li–Yau–Hamilton inequality on Kähler manifolds with nonnegative bisectional curvature following an earlier work of Hamilton [13], which can be viewed as a generalized complex Hessian comparison theorem. The natural question is: does (1.5) have a matrix version? The same question applies to Perelman's entropy estimate Corollary 9.3.

Acknowledgments

The author would like to thank Professor Ben Chow for his encouragement to study Perelman's recent articles and discussions, and Professor Peter Li for helpful suggestions. Special thanks goes to Professor Perelman since most results in this article are the simpler versions of the corresponding results in [20] without Ricci flow.

References

[1] Bakry, D., Concordet, D., and Ledoux, M. Optimal heat kernel bounds under logarithmic Sobolev inequalities, *ESAIM Probab. Statist.,* **1**, 391–407, (1995/97).

[2] Beckner, W. Geometric asymptotics and the logarithmic Sobolev inequality, *Forum Math.,* **11**, 105–137, (1999).

[3] Beckner, W. and Pearson, M. On sharp Sobolev and the logarithmic Sobolev inequality, *Bull. London Math. Soc.,* **30**, 80–84, (1998).

[4] Cao, H.D. and Ni, L. Matrix Li–Yau–Hamilton estimates for the heat equation on Kaehler manifolds, submitted, arXiv: math.DG/0211283.

[2] In the addendum, we shall show a dual entropy formula, which relates Hamilton's and Perelman's entropies.

[5] Cao, H.D., Chen, B-L., and Zhu, X-P. Ricci flow on compact Kähler manifolds of positive bisectional curvature, preprint.

[6] Cheng, S.-Y., Li, P., and Yau, S.-T. On the upper estimate of heat kernel of a complete Riemannian manifold, *Am. J. Math.,* **103**, 1021–1063, (1981).

[7] Chow, B. Interpolating between Li–Yau's and Hamilton's Harnack inequalities on a surface, *J. Partial Differential Equations,* **11**(2), 137–140, (1998).

[8] Chow, B., Chu, S-C., Lu, P., and Ni, L. Notes on Perelman's papers on Ricci flow.

[9] Davies, E. Explicit constants for Gaussian upper bounds on heat kernels, *Am. J. Math.,* **109**, 319–334, (1987).

[10] Ecker. K. Logarithmic Sobolev inequalities on submanifolds of Euclidean spaces, *J. Reine Angew. Mat.,* **552**, 105–118, (2002).

[11] Gross, L. Logarithmic Sobolev inequalities, *Am. J. Math.,* **97**(4), 1061–1083, (1975).

[12] Hamilton, R. The Ricci flow on surfaces, Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, *Contemp. Math.,* **71**, Am. Math. Soc., Providence, RI, (1988).

[13] Hamilton, R. A matrix Harnack estimate for the heat equation, *Comm. Anal. Geom.,* **1**, 113–126, (1993).

[14] Kleiner, B. and Lott, J. Notes on Perelman's paper.

[15] Ledoux, M. On manifolds with non-negative Ricci curvature and Sobolev inequalities, *Comm. Anal. Geom.,* **7**, 347–353, (1999).

[16] Li, P. On the Sobolev constant and p-spectrum of a compact Riemannian manifold, *Ann. Scient. Éc. Norm. Sup.,* **13**, 451–469, (1980).

[17] Li, P. Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature, *Ann. of Math.,* **124**(1), 1–21, (1986).

[18] Li, P. and Yau, S.-T. On the parabolic kernel of the Schrödinger operator, *Acta Math.,* **156**, 139–168, (1986).

[19] Ni, L. Poisson equation and Hermitian–Einstein metrics on holomorphic vector bundles over complete noncompact Kahler manifolds, *Indiana Univ. Math. J.,* **51**(3), 679–704, (2002).

[20] Perelman, G. The entropy formula for the Ricci flow and its geometric applications, arXiv: math.DG/ 0211159.

[21] Perelman, G. Informal talks and discussions.

[22] Rothaus, O.S. Analytic inequalities, isoperimetric inequalities and logarithmic Sobolev inequalities, *J. Funct. Anal.,* **64**, 296–313, (1985).

[23] Stam, A.J. Some inequalities satisfied by the quantities of informations of Fisher and Shannon, *Inform. and Control,* **2**, 101–112, (1959).

[24] Sesum, N., Tian, G., and Wang, X. Notes on Perelman's paper on entropy formula for Ricci flow and its geometric applications.

[25] Varadhan, R.S. On the behavior of the fundamental solution of the heat equation with variable coefficients, *Comm. Pure Applied Math.,* **20**, 431–455, (1967).

[26] Weissler, F.B. Logarithmic Sobolev inequalities for the heat-diffusion semi-group, *Trans. AMS.,* **237**, 255–269, (1978).

Note added in proof:

An addendum with additional results will appear in issue Volume 14 Number 2.

Received August 15, 2003

Department of Mathematics, University of California, San Diego, La Jolla, CA 92093
e-mail: lni@math.ucsd.edu

Communicated by Peter Li