THE EQUATION $a^M = b^N c^P$ IN A FREE GROUP

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1. INTRODUCTION

The question of finding all solutions for the equation $a^M = b^N c^P$ in a free group is of interest only if none of the exponents is 0 or 1; we assume, then, that $M, N, P \geq 2$. The equation possesses obvious solutions for which a, b, and c are all powers of a common element; it will be shown that these are all solutions.

R. Vaught conjectured that $a^2 = b^2 c^2$ had only these obvious solutions, and R. C. Lyndon [3] verified this conjecture by a combinatorial argument. His result carries with it the case that all three exponents are even. That there are only the obvious solutions in the case where the three exponents have a common prime divisor was established independently by G. Baumslag [1], E. Schenkman [4], and J. Stallings [6], all of whom employed more characteristically group theoretic methods. The proof here, for general $M, N, P \geq 2$, is of a combinatorial nature.

In Section 2 we record some properties of the free monoid F of words representing elements in a free group G. In Section 3 we reduce the problem of finding all the solutions of the equation $a^M = b^N c^P$ in G to that of finding all solutions of each of two equations in F. In Sections 4 and 5 we show in turn that each of these equations has only the obvious solutions.

The greater part of the argument deals with the case that one of the exponents is 2 or 3. This suggests that arbitrary equations in powers of elements from a free group have only more or less obvious solutions when the exponents are sufficiently large. More generally, one may expect that in some sense more complicated equations have fewer solutions, with only rather special equations possessing genuinely nondegenerate solutions. Thus the equation $a^M = b^N c^P d^Q$, which possesses a wealth of nontrivial solutions when all four exponents are 2, appears to have only obvious solutions when all exponents are large.

2. COMBINATORIAL LEMMAS

Let G be a group freely generated by a set X of generators x. Let F be the monoid freely generated by the set $X \cup \bar{X}$, where $\bar{X}$ is a set, disjoint from X, of elements $\bar{x}$ in one-to-one correspondence with the elements x of X. The elements of F are *words*. A word a *represents* the group element ϕa, where ϕ is the epimorphism from F onto G carrying x into x and $\bar{x}$ into x^{-1}. The *length* $|a|$ of a word a is the number of factors in its expression as a product of the *letters* x and $\bar{x}$. The *formal inverse* $\bar{a}$ of a word a is its image under the involutory antiautomorphism of F that interchanges x and $\bar{x}$. Clearly $\phi\bar{a} = (\phi a)^{-1}$.

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A word is *reduced* if it contains no factor $a\bar{a}$ for $a \neq 1$. Each element of G is represented by a unique reduced word.

A word is *cyclically reduced* if it is reduced and is not of the form $ab\bar{a}$ for $a \neq 1$. Each reduced word is uniquely representable in the form $ab\bar{a}$ for b cylically reduced.

A word is *primitive* if it is not of the form a^m for any $m > 1$.

Two words are *cyclically conjugate* if they are of the forms ab and ba, respectively. If one is cyclically reduced or primitive, then so is the other.

The first of the following more or less obvious and familiar properties of F is due to F. W. Levi [2].

LEMMA 1. *If* $ab = cd$ *and* $|a| \leq |c|$, *then* $c = ae$ *and* $b = ed$ *for some* e.

The proof is immediate.

LEMMA 2. *If* $ab = bc$ *and* $a \neq 1$, *then* $a = uv$, $b = (uv)^k u$, *and* $c = vu$ *for some* $u, v \in F$ *and* $k \geq 0$.

Proof. If $|b| \leq |a|$, then by Lemma 1, $a = bv$ and $c = vb$ for some v, and the conclusion holds with $u = b$ and $k = 0$. If $|a| < |b|$, then again by Lemma 1, $b = ab'$ for some b', whence $a^2b' = ab'c$. Thus $ab' = b'c$. Since $a \neq 1$, $|b'| < |b|$, and the desired conclusion follows by induction on $|b|$, the initial case being trivial.

LEMMA 3. *If* $ab = ba$, *then* a *and* b *are powers of a common element.*

Proof. If $a = 1$, the conclusion is immediate. Otherwise, applying Lemma 2, from the relation $a = uv = vu$ we conclude by induction that u and v are powers of a common element.

LEMMA 4. *If* a *and* b *have powers* a^m *and* b^n *with a common initial segment of length* $|a| + |b|$, *then* a *and* b *are powers of a common element.*

Proof. The hypothesis implies that ba^m and b^{n+1} have a common initial segment of length $|a| + 2|b|$, whence ba^m and b^n have a common initial segment of length $|a| + |b|$. Similarly, ab^n and a^m have a common initial segment of length $|a| + |b|$. It follows that ab^n and ba^m have a common initial segment of length $|a| + |b|$, and hence that $ab = ba$. The conclusion now follows by Lemma 3.

COROLLARY 4.1. *If* $a^m = b^n$ *and* $m \geq 1$, *then* a *and* b *are powers of a common element.*

Proof. If $m = 1$ or $n \leq 1$ the conclusion is immediate; otherwise

$$|a^m| = |b^n| \geq |a| + |b|$$

and the conclusion follows by Lemma 4.

COROLLARY 4.2. *If* $a \neq 1$, *then there exists a unique primitive* b *and an integer* $k \geq 1$ *such that* $a = b^k$.

LEMMA 5. *If* a *and* b *are primitive and have powers* a^m *and* b^n *with a common initial segment of length* $|a| + |b|$, *then* $a = b$.

Proof. The conclusion immediately follows from Lemma 4 and Corollary 4.2.

LEMMA 6. *If* a *is a reduced word and* $\bar{a}$ *divides* ac *in the sense that* $ac = u\bar{a}v$ *for some* u *and* v, *then* $\bar{a}$ *divides* c.

Proof. Assume that $ac = u\bar{a}v$ and that $\bar{a}$ does not divide c. By Lemma 1 it follows, first, that $|c| < |\bar{a}v|$, whence $|u| < |a|$, and, second, that $a = uw$ and $wc = \bar{a}v$ for some $w \neq 1$. Now $wc = \bar{a}v = \bar{w}\bar{u}v$ implies that $w = \bar{w}$, which is impossible for $w \neq 1$ and w reduced.

COROLLARY 6.1. *If* a *is reduced and* $\bar{a}$ *divides a power of a word of the form* $a^m b$, *then* $\bar{a}$ *divides* b.

COROLLARY 6.2. *If* a *is a reduced word and* $\bar{a}$ *divides a power of* a, *then* $a = 1$.

LEMMA 7. *If* a *is a reduced word and both* a *and* $\bar{a}$ *divide some power of a cyclically reduced word* b, *then* b *is cyclically conjugate to a word of the form* $av\bar{a}u$.

Proof. Since a begins some cyclic conjugate of a power of b, and thus a begins some power of a cyclic conjugate b' of b, $a = b'^k b_1$, where $b' = b_1 b_2$ and $k \geq 0$. Since $\bar{a}$ is a factor of a power of b and hence of a power of $b'^{k+1} = ab_2$, it follows by Corollary 6.1 that $\bar{a}$ divides b_2. Thus b_2 has the form $b_2 = v\bar{a}u$. Since $a = b'^k b_1$ with $|a| \leq |b_2| < |b'|$, it follows that $k = 0$ and $a = b_1$. Consequently, $b' = b_1 b_2 = av\bar{a}u$.

3. REDUCTION OF THE PROBLEM

We consider now elements ϕa, ϕb, and ϕc of the free group G that satisfy an equation

$$(\phi a)^M = (\phi b)^N (\phi c)^P, \tag{3.1}$$

where $M, N, P \geq 2$, and we shall show that ϕa, ϕb, and ϕc are powers of a common element of G. Our aim in this section is to replace the hypothesis (3.1) by hypotheses on the elements a, b, and c of the monoid F. Clearly, we may suppose that a, b, and c are reduced and primitive. Under this assumption, it will suffice to show that one of a, b, and c is equal or inverse to another.

It is also clear that we may replace ϕa, ϕb, and ϕc by their conjugates under any element of G, and thus replace a, b, and c, by the corresponding reduced primitive words a', b', and c'. If $b = ub'\bar{u}$, where b' is cyclically reduced, then conjugation by ϕu replaces b by b', thereby justifying the assumption that

(3.2) b *is cyclically reduced.*

We next justify the additional assumption that

(3.3) bc *is reduced.*

If $b = \bar{c}$, the desired conclusion holds. Otherwise, by Lemma 5, there is a bound on the length of an initial segment common to powers of $\bar{b}$ and c. Therefore, for some m and n, neither $\bar{b}^m$ nor c^n is an initial segment of the other. It follows that there exist factorizations $b = b_1 b_2$ and $c = c_1 c_2$, with $b_1 \neq 1$, $c_2 \neq 1$, such that, $b_2 b^{m-1}$ and $c^{n-1} c_1$ are formal inverses and that $b_1 c_2$ is reduced. Conjugation by

$$\phi(c^{n-1} c_1) = \phi(\bar{b}_2 \bar{b}^{m-1})$$

now replaces b by $b' = b_2 b_1$ and c by $c' = c_2 c_1$. Now b' is cyclically reduced, since b is reduced. In addition, $b'c'$ is reduced, since $b_1 c_2$ is reduced and $b_1, c_2 \neq 1$.

We now show that we may assume further that one of the following two conditions holds: either

(Case I) a *is cyclically reduced,*

or

(Case II) c *is cyclically reduced.*

We argue by induction on $|a|$. The initial case that $|a| = 1$ falls under Case I. Suppose then that (3.2) and (3.3) hold, but that neither a nor c is cyclically reduced. Then a has the form $a = ua'\bar{u}$ for some letter $u \in X \cup \bar{X}$, and therefore the reduced word $ua'^M\bar{u}$ representing $(\phi a)^M$ begins with u and ends with $\bar{u}$. Since bc is reduced, it follows that the reduced word for $(\phi b)^N(\phi c)^P$ begins with the same letter as b and ends with the same letter as c. In view of (3.1), we see that b begins with u and that c ends with $\bar{u}$. Thus $b = ub_0$ for some b_0, and c, since it is not cyclically reduced, has the form $c = uc'\bar{u}$. Conjugation by ϕu now replaces a, b, and c by a', $b' = b_0 u$, and c'. Here b' is cyclically reduced since b is cyclically reduced. Further, b' c' is reduced, since $b' = b_0 u$ with $u \neq 1$, and, since $c = uc'\bar{u}$ is reduced, uc' is reduced. Thus (3.2) and (3.3) remain true, while a is replaced by a' with $|a'| = |a| - 2$. The desired conclusion therefore follows by induction.

In the remaining sections we treat Cases I and II in turn. We emphasize that the two cases are not disjoint; in fact, we treat Case II by reducing it to its intersection with Case I. For this common case, where both a and c are cyclically reduced, the argument of Section 4 could, of course, be substantially simplified.

4. CASE I

Write $c = \bar{g}c'g$, where c' is cyclically reduced and g is possibly 1. Since a and b are cyclically reduced and bc is reduced,

$$a^M = b^N \bar{g} c'^P g,$$

where both members are reduced, and thus represent the same word. Dropping primes, we show that if

$$a^M = b^N \bar{g} c^P g \tag{4.1}$$

in the monoid F, where the word represented by the two members is reduced and a, b, and c are primitive, then $g = 1$ and $a = b = c$. We proceed by induction on $|a|$, the initial case $|a| = 1$ being vacuous.

The equation (4.1) implies the analogous equation

$$a'^M = \bar{c}^P g \bar{b}^N \bar{g}, \tag{4.1'}$$

where a' is a cyclic conjugate of $\bar{a}$. If $|b^N| \geq |a| + |b|$, then a^M and b^N have a common initial segment of length $|a| + |b|$, and it follows by Lemma 5 that $a = b$. Thus we may assume that

$$|b^{N-1}| < |a| \tag{4.2}$$

and, symmetrically, in view of (4.1'), that

$$|c^{P-1}| < |a|. \tag{4.3}$$

By Lemma 7, from the occurrences of the factors $\bar{g}$ and g in a^M we may conclude that a has the form $a = u\bar{g}vg$ and that

$$b^N\bar{g} = a^{m_1}u\bar{g}, \quad c^P g = vga^{m_2}, \tag{4.4}$$

where $m_1 + m_2 + 1 = M$. Thus $|a^{m_1}| < |b^N|$. On the other hand (4.2) implies that $|b^N| \leq |a^2|$, and we conclude that $m_1 \leq 1$. Similarly, $m_2 \leq 1$, whence $M \leq 3$; that is, $M = 2$ or $M = 3$.

If $M = 2$, we may suppose by symmetry that $m_1 = 1$ and $m_2 = 0$. Now (4.4) implies that $b^N = au = u\bar{g}vgu$ and $c^P = v$, whence $b^N = u\bar{g}c^P gu$. Therefore there exists a cyclic conjugate b' of b for which $b'^N = u^2\bar{g}c^P g$. Since

$$|b'| = |b| \leq |b^{N-1}| < |a|,$$

we conclude by induction that $g = 1$ and $b' = u = c$, and therefore that $b = b' = c$.

If $M = 3$, then $m_1 = m_2 = 1$, and (4.4) implies that

$$b^N = au = u\bar{g}vgu \quad \text{and} \quad c^P = vgu\bar{g}v.$$

By symmetry we may suppose that $|c^P| \leq |b^N|$, and therefore $|v| \leq |u|$. By (4.2), $|b^{N-1}| < |a|$, and therefore $b^N = au$ implies that $|u| < |b|$. It follows from $b^N = u\bar{g}vgu$ that b both begins and ends with u. From Lemma 2 it follows that u and b have the forms $u = (pq)^k p$ and $b = (pq)^{k+1} p$ where, since b is primitive, $q \neq 1$. Now $b = uqp = pqu$, whence $b^N = u\bar{g}vgu$ implies that

$$\bar{g}vg = qpb^{N-2}pq.$$

If $|q| \leq |g|$, then g ends with both q and $\bar{q}$, which is impossible for $q \neq 1$. Therefore $|g| < |q|$, $q = \bar{g}q_1 g$ for some q_1, and

$$v = q_1 gpb^{N-2}p\bar{g}q_1.$$

Because $|v| \leq |u| < |b|$, this implies that $N = 2$ and $v = q_1 gp^2\bar{g}q_1$. It follows that

$$2|p| + |q| < |v| \leq |u| = (k+1)|p| + k|q|,$$

and therefore that $k \geq 2$.

The relation

$$c^P = vgu\bar{g}v = q_1 gp^2\bar{g}q_1 g(pq)^k pgq_1 gp^2\bar{g}q_1 = q_1 gp(pq)^{k+2} p^2\bar{g}q_1$$

implies that

$$c'^P = (pq)^{k+2} p^2\bar{g}q_1^2 gp$$

where c' is a cyclic conjugate of c. Now the inequality $k \geq 2$ implies that

$$|qp^2\bar{g}q_1^2gp| < 4|p| + 3|q| \leq |(pq)^{k+1}p|.$$

Therefore

$$|c'^P| = |(pq)^{k+1}p| + |qp^2\bar{g}q_1^2gp| < 2|(pq)^{k+1}p|$$

and $|c'| < |(pq)^{k+1}p|$. Thus c'^P has an initial segment $(pq)^{k+2}p$ of length greater than $|c'| + |pq|$ in common with $(pq)^{k+3}$, and it follows by Lemma 5 and Corollary 4.1, since c' is primitive, that pq is a power of c'. From the fact that $(pq)^{k+2}$ and $(pq)^{k+2}p^2\bar{g}q_1^2gp$ are both powers of c' it follows that $p^2\bar{g}q_1^2gp$ is a power of c', and therefore $p^3\bar{g}q_1^2g$ is a power of a cyclic conjugate c'' of c'. From the relation

$$|c''| = |c| \leq |pq| < |p^3\bar{g}q_1^2g|$$

it follows that $c''^Q = p^3\bar{g}q_1^2g$ for some integer $Q \geq 2$. Finally, since

$$|c''| = |c| \leq |c^{P-1}| < |a|,$$

we may conclude by induction that $g = 1$ and that p and $q_1 = q$ are powers of a common element, which contradicts the hypothesis that $b = (pq)^{k+1}p$ is primitive.

5. CASE II

Write $a = \bar{g}a'g$, where a' is cyclically reduced and possibly $g = 1$. Dropping primes, we obtain an equation

$$\bar{g}a^M g = b^N c^P \tag{5.1}$$

in the monoid F. Here the word represented by the two members is reduced, and a, b, and c are primitive. We shall show that $g = 1$ and $a = b = c$. In view of Case I, it will suffice to show that $g = 1$.

If $|c^P| \leq |g|$, then $g = hc^P$ for some h; and after cancelling a factor c^P from each member, we see that $\bar{c}^P\bar{h}a^Mh = b^N$. This equation falls under Case I and has a solution only if $\bar{c} = b$, which is contrary to the hypothesis that b^Nc^P is reduced. Thus we may assume that $|g| < |c^P|$, and, symmetrically, that $|g| < |b^N|$. It follows by Lemma 1 that

$$\begin{aligned} g &= \overline{b^{n_1}b_1} = c_2c^{p_2},\\ a^{m_1}a_1 &= b_2b^{n_2}, \\ c^{p_1}c_1 &= a_2a^{m_2}, \end{aligned} \tag{5.2}$$

where $a = a_1a_2$, $b = b_1b_2$, and $c = c_1c_2$, and where $m_1 + m_2 + 1 = M$, $n_1 + n_2 + 1 = N$, and $p_1 + p_2 + 1 = P$. We remark that (5.2) implies (5.1).

The system (5.2) leads to systems

$$g' = \overline{a^{m_1}a_1} = \bar{b}_2(\bar{b}_1\bar{b}_2)^{n_2}, \quad c^{p_1}c_1 = a_2a^{m_2}, \quad (\bar{b}_1\bar{b}_2)^{n_1}\bar{b}_1 = c_2c^{p_2}, \tag{5.2'}$$

and

$$g'' = \overline{\bar{c}^{p_2}\bar{c}_2} = \bar{b}_1 \bar{b}^{n_1}, \quad \bar{a}_1^{m_2}\bar{a}_2 = \bar{c}_1\bar{c}^{p_1}, \quad \bar{b}^{n_2}\bar{b}_2 = \bar{a}_1\bar{a}^{m_1}. \tag{5.3''}$$

Thus the hypothesis on the six exponents that there exist an element g and factorizations of three primitive words a, b, and c such that (5.2) holds is symmetric under cyclic permutation of the pairs (n_1, n_2), (m_1, m_2), (p_1, p_2), and also under interchange of (n_1, n_2) with (p_1, p_2) coupled with reversal of all three pairs.

We exploit this symmetry to reduce the discussion of (5.2) to three cases as follows. First, we choose Case A to be that where the two exponents in some one of the equations (5.2) both vanish. We choose Case B to be that where all of m_1, n_1, and p_1, or else all of m_2, n_2, and p_2, are positive. For Case C we may, in view of Case B, assume that some exponent vanishes, say $n_1 = 0$. Then $n_2 \neq 0$ in view of the equation $n_1 + n_2 + 1 = N$, while we may assume that $p_2 \neq 0$ in view of Case A. In view of Case B, we may now assume that $m_2 = 0$, while, in view of Case A, we may take $p_1 \neq 0$. We now treat these three cases in turn.

Case A. Here we suppose that the two exponents in the first of equations (5.2) vanish. This equation becomes $g = \bar{b}_1 = c_2$. Thus $b = \bar{g}b_2$, $c = c_1\bar{g}$, and substituting these right-hand members in (5.1) and cancelling the initial $\bar{g}$ and the final g, we obtain the equation

$$a^M = (b_2\bar{g})^{N-1} b_2 c_1 (gc_1)^{P-1}. \tag{5.3}$$

If $|(b_2\bar{g})^{N-1}b_2| \geq |b_2\bar{g}| + |a|$, it follows by Lemma 5 that $a = b_2\bar{g}$, and since $\bar{g}ag$ is reduced, that $g = 1$, as required. Thus we may assume that

$$|(b_2\bar{g})^{N-2} b_2| < |a|.$$

Also $|(b_2\bar{g})^{N-1}| = |a|$ would imply that $a = (b_2\bar{g})^{N-1}$, and, since $\bar{g}ag$ is reduced, g would equal 1; therefore we may assume that $a \neq (b_2\bar{g})^{N-1}$. We show that the inequality $|a| < |(b_2\bar{g})^{N-1}|$ is impossible. This inequality implies that the first factor a in the product (5.1) begins with b_2 and that the second begins with $h_2 b_2$, for some factorization $\bar{g} = h_1 h_2$ with $h_1, h_2 \neq 1$. By Lemma 2, there exist u, v, and k such that $b_2 = (uv)^k u$ divides a power of $h_2 = uv$. From (5.3), there exists a cyclic conjugate a' of a for which

$$a'^M = (\bar{g}b_2)^{N-1} b_2 c_1 (gc_1)^{P-1} b_2.$$

By Lemma 7, a' has the form $a' = \bar{g}pgq$; and from the relations

$$|a'| = |a| < |(b_2\bar{g})^{N-1}| = |(\bar{g}b_2)^{N-1}|$$

we conclude that g divides b_2. Moreover, $\bar{h}_2$ divides $g = \bar{h}_2\bar{h}_1$. It follows that $\bar{h}_2$ divides a power of h_2, which, by Corollary 6.2, contradicts the fact that $h_2 \neq 1$. Thus we may assume, symmetrically, that

$$|(b_2\bar{g})^{N-1}| < |a|, \quad |(gc_1)^{P-1}| < |a|. \tag{5.4}$$

In view of (5.4), cancelling the first and last factors a from (5.3) shows that a^{M-2} divides $b_2 c_1$. The inequalities (5.4) imply that $|b_2|$, $|c_1| < |a|$; hence $|b_2 c_1| < 2|a|$. It follows that $M - 2 < 2$, that is, $M = 2$ or $M = 3$.

If $M = 2$, we may suppose by symmetry that

$$|(b_2\bar{g})^{N-1}b_2| \geq |c_1(gc_1)^{P-1}|.$$

It follows that $(b_2\bar{g})^{N-1}b_2 = a_1a_2a_1$ and $a_2 = c_1(gc_1)^{P-1}$, whence

$$(b_2\bar{g})^N = a_1c_1(gc_1)^{P-1}a_1g \quad \text{and} \quad b'^N = (a_1\bar{g})^1 a_1c_1(gc_1)^{P-1},$$

where b' is a cyclic conjugate of b. Since this equation is of the form (5.3) with $|b'| = |b_2g| < |a|$, we may conclude by induction on $|a|$ that $g = 1$.

If $M = 3$, then not both N and P can exceed 2, since then (5.4) would imply that $2|b_2|$, $2|c_1| < |a|$, and that

$$|b_2| + |c_1| < |a| \leq |b_2c_1|.$$

By symmetry, we assume that $N = 2$. Equation (5.3) now becomes

$$a^3 = b_2gb_2c_1(gc_1)^{P-1}.$$

It follows that there exist factorizations $b_2 = b_3b_4$ and $c_1 = c_3c_4$ with $b_3 \neq 1$ and $c_4 \neq 1$ such that

$$a = b_2\bar{g}b_3 = b_4c_3 = c_4(gc_1)^{P-1}.$$

Since a begins both with $b_2 = b_3b_4$ and with b_4, it follows by Lemma 2 that $b_3 = uv$ and $b_4 = (uv)^k u$ for some u, v and some k. Now $b_2\bar{g}b_3 = b_4c_3$ implies that $c_3 = vu\bar{g}uv$, while $b_4c_3 = c_4(gc_1)^{P-1}$ implies that

$$c_3b_4c_3 = c_1(gc_1)^{P-1} \quad \text{and} \quad c_1(gc_1)^{P-1} = vu\bar{g}(uv)^{k+2}u\bar{g}uv.$$

Since $|c_1| > |c_3| = vu\bar{g}uv$, there exist an $h \geq 1$ and a factorization $uv = w_1w_2$ with $w_1 \neq 1$ such that the initial occurrence of c_1 in this expression for $c_1(gc_1)^{P-1}$ has the form $c_1 = vu\bar{g}(uv)^h w_1$. Now c_1 ends both with $w_1w_2w_1$ and with $\bar{g}uv$. Since $|w_2w_1| = |uv|$, it follows that unless $g = 1$, $\bar{g}$ ends with the same letter as w_1, and hence as c_1, which is contrary to the hypothesis that c_1g is reduced. Therefore $g = 1$, as required.

Case B. Here we may assume that m_1, n_1, $p_1 \neq 0$ and, using cyclic symmetry, that $|c| \leq |a|$. From the equality

$$c^{p_1}c_1 = a_2a^{m_2} \qquad (p_1 \neq 0),$$

we conclude that c begins $(a_2a_1)^{m_2+1}$ and since $|c| \leq |a| = |a_2a_1|$, that c begins, and therefore divides, a_2a_1. From the equality

$$a^{m_1}a_1 = b_2b^{n_2} \qquad (m_1 \neq 0)$$

we conclude that a_2a_1 divides b^{n_2+1}, and hence that c divides b^{n_2+1}. It follows from the relation

$$b^{n_1}b_1 = \bar{c}^{p_2}\bar{c}_2 \qquad (n \;\; \neq 0)$$

that b begins $\bar{c}^{p_2+1}$. Thus b, and with it b^{n_2+1}, is a product of initial segments of $\bar{c}$. Now the factor c of b^{n_2+1} must end with a part of some initial segment $d \neq 1$ of

$\bar{c} = de$, and since $|d| \leq |\bar{c}| = |c|$, c must end with all of d, that is, $c = fd$. But then $c = fd = \bar{e}\bar{d}$ and $d = \bar{d}$, which is impossible for a part $d \neq 1$ of the reduced word c. This shows that Case B is impossible.

Case C. Here $n_1 = m_2 = 0$ with the remaining exponents positive. The equations (5.2) now take the form

$$b_1 = \bar{c}^{p_2}\bar{c}_2,$$

$$a^{m_1}a_1 = b_2 b^{n_2}, \tag{5.5}$$

$$a_2 = c^{p_1}c_1.$$

If a_2 divided b_1, then c, which divides a_2, would divide b_1; and hence c would divide a power of $\bar{c}$. By Corollary 6.2, this would imply that $c = 1$. We conclude, symmetrically, that neither of a_2 and b_1 divides the other. Now the inequality

$$|a^{m_1}a_1| \geq |a| + |b|$$

would imply by Lemma 5 that $a_1 a_2 = b_2 b_1$, whence one of a_2 and b_1 would divide the other. We conclude that $|a^{m_1}a_1| < |a| + |b|$. By symmetry, we assume that $|b| \leq |a|$, whence $|a^{m_1}a_1| < 2|a|$ and $m_1 = 1$. We now see that $|a_1 a_2 a_1| < |a| + |b|$, whence $|a_1| < |b|$. If $n_2 = 1$, it follows from the relation $a_1 a_2 a_1 = b_2 b_1 b_2$ that either a_2 divides b_1 or b_1 divides a_2. If $n_2 > 2$, from the equality $a_1 a_2 a_1 = b_2 b^{n_2}$, we see by cancelling the factors a_1 that $(b_2 b_1)^{n_2 - 2} b_2$ divides a_2, and therefore b_1 divides a_2. We conclude that $n_2 = 2$.

The second of the equations (5.2) now takes the form

$$a_1 a_2 a_1 = b_2 b_1 b_2 b_1 b_2. \tag{5.6}$$

Since it was established that $|a_1| < |b| = |b_2 b_1|$, it follows from (5.6) that $|a_2| > |b_2|$. Consequently b_2 is in the middle of a_2, that is, there exist u and v, with $|u| = |v|$, such that $a_2 = ub_2 v$. It follows from (5.6) that $a_1 u = b_2 b_1$ and $va_1 = b_1 b_2$. Since b_1 does not divide $a_2 = ub_2 v$, b_1 does not divide u, and, from $a_1 u = b_2 b_1$ we see that $|u| < |b_1|$. From the relations $|v| = |u| < |b_1|$ and $va_1 = b_1 b_2$ it follows that v divides $b_1 = \bar{c}^{p_2}\bar{c}_2$. If $|c|$ were no greater than $|v|$, then v would contain a cyclic conjugate of $\bar{c}$, which, since v divides $a_2 = c^{p_1}c_1$, would contradict Corollary 6.2. We conclude that $|u| = |v| < |c|$. But now $a_2 = ub_2 v$, and therefore also $ub_2 b_1$, begins with c, while

$$vb_2 b_1 = va_1 u = b_1 b_2 u$$

begins with $\bar{c}$. Since $|u| = |v| < |c|$ there exists a $d \neq 1$ for which $c = ud$ and $\bar{c} = vd$. This implies that $c = ud = \bar{d}\bar{v}$, which contradicts the hypothesis that c is cyclically reduced.

REFERENCES

1. G. Baumslag, *On a problem of Lyndon,* J. London Math. Soc. 35 (1960), 30-32.
2. F. W. Levi, *On semigroups,* Bull. Calcutta Math. Soc. 36 (1944), 141-146.
3. R. C. Lyndon, *The equation* $a^2 b^2 = c^2$ *in free groups,* Michigan Math. J. 6 (1956), 89-95.
4. E. Schenkman, *The equation* $a^n b^n = c^n$ *in a free group,* Ann. of Math. 70 (1959), 562-564.
5. M. P. Schützenberger, *Sur l'équation* $a^{2+n} = b^{2+m} c^{2+p}$ *dans un groupe libre,* C. R. Acad. Sci. Paris 248 (1959), 2435-2436.
6. J. Stallings, *On certain relations in free groups* (Abstract 559-166) Notices Amer. Math. Soc. 6 (1959), 532.

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