

The Equations of Structure of an N -Linear Connection in the Bundle of Accelerations

Gheorghe Atanasiu

Abstract

The study of higher order Lagrange spaces founded on the notion of bundle of velocities of order k has been recently given by Radu Miron and author in [2]-[5]. The bundle of acceleration correspond in this study to $k = 2$.

In this paper we shall give the equations of structure of an N -linear connection in the bundle of accelerations.

Mathematics Subject Classification: 53C05

Key words: 2-osculator bundle, connections, equations of structure

1 The bundle of accelerations ([2],[3])

Let M be a real n -dimensional C^∞ -manifold and (Osc^2M, π, M) its 2-osculator bundle, or the bundle of accelerations. The canonical local coordinates on the total space $E = Osc^2M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$. A coordinate transformation $(x^i, y^{(1)i}, y^{(2)i}) \rightarrow (\tilde{x}^i, \tilde{y}^{(1)i}, \tilde{y}^{(2)i})$ on E is given by

$$(1.1) \quad \left\{ \begin{array}{l} \tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n), \quad \text{rank} \left\| \frac{\partial \tilde{x}^i}{\partial x^j} \right\| = n, \\ \tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j} \\ 2\tilde{y}^{(2)i} = \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2 \frac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j} \end{array} \right. .$$

If N is a nonlinear connection on E and J is the tangent structure of second order [2], then $N_0 = N$, $N_1 = J(N_0)$ are two distributions geometrically defined on E , everyone of local dimension n . Let us consider the distribution V_2 on E locally generated by the vector fields $\left\{ \frac{\partial}{\partial y^{(2)i}} \right\}$. Consequently, the tangent space to E at a point $u \in E$ is given by a direct sum of the vector spaces:

$$(1.2) \quad T_u(E) = N_0(u) \oplus N_1(u) \oplus V_2(u), \quad \forall u \in E.$$

An adapted basis to the direct decomposition (1.2) is given by

$$(1.3) \quad \left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}} \right\} \quad (i = 1, \dots, n),$$

where

$$(1.4) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)}^r{}_i \frac{\partial}{\partial y^{(1)r}} - N_{(2)}^r{}_i \frac{\partial}{\partial y^{(2)r}}$$

and

$$(1.4') \quad \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - N_{(1)}^r{}_i \frac{\partial}{\partial y^{(2)r}}.$$

The systems of functions $N_{(1)}^i{}_j, N_{(2)}^i{}_j$ are called the *coefficients* of the nonlinear connection N .

If we consider the projectors h, v_1, v_2 determined by (1.2) and denoting $v_\alpha X = X^{v_\alpha}$ ($\alpha = 1, 2$), we can uniquely write

$$(1.5) \quad X = X^H + X^{v_1} + X^{v_2}, \quad (\forall) X \in \mathcal{X}(\mathcal{E}).$$

Thus, we have

$$(1.5') \quad X^H = X^{(0)i} \frac{\delta}{\delta x^i}, \quad X^{v_1} = X^{(1)i} \frac{\delta}{\delta y^{(1)i}}, \quad X^{v_2} = X^{(2)i} \frac{\partial}{\partial y^{(2)i}}.$$

The coordinates $X^{(\alpha)i}$, ($\alpha = 0, 1, 2$), change under (1.1) as follows:

$$(1.5'') \quad \tilde{X}^{(\alpha)i} = \frac{\partial \tilde{x}^i}{\partial x^j} X^{(\alpha)j}, \quad (\alpha = 0, 1, 2).$$

Each one of them is called a *distinguished vector field*, shortly a *d-vector field*.

Let us consider the dual basis of (1.3):

$$(1.3') \quad \{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\} \quad (i = 1, \dots, n).$$

Then for a field of 1-form ω on E , we can put

$$(1.6) \quad \omega = \omega^H + \omega^{v_1} + \omega^{v_2},$$

where

$$(1.6') \quad \omega^H = \omega_i^{(0)} dx^i, \quad \omega^{v_1} = \omega_i^{(1)} \delta y^{(1)i}, \quad \omega^{v_2} = \omega_i^{(2)} \delta y^{(2)i},$$

and with respect to (1.1) we have

$$(1.6'') \quad \omega_i^{(\alpha)} = \frac{\partial \tilde{x}^j}{\partial x^i} \tilde{\omega}_j^{(\alpha)}, \quad (\alpha = 0, 1, 2).$$

Each one of them is called a *distinguished covector field*, shortly a *d-covector field*.

Analogously, we can define a *distinguished tensor field* on E of type (r, s) (shortly, a *d-tensor field*).

Now, we consider the 2-tangent structure J on $Osc^2M = E$ and a nonlinear connection N on E .

We define an *N-linear connection* on E as a linear connection D on E which preserves by parallelism the horizontal distribution N and which is compatible with the structure J (i.e., $D_X J = 0$, $\forall X \in \mathcal{X}(\mathcal{E})$).

In the adapted basis (1.3) it is sufficient to give

$$(1.7) \quad D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = L_{ij}^m \frac{\delta}{\delta y^{(\alpha)m}}, \quad D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} = C_{(\beta)ij}^m \frac{\delta}{\delta y^{(\alpha)m}}$$

$$(\alpha = 0, 1, 2, \quad \beta = 1, 2 \quad \text{and} \quad y^{(0)i} = x^i)$$

in order to obtain all the coefficients

$$D\Gamma(N) = \left(L_{jm}^i, C_{(1)jm}^i, C_{(2)jm}^i \right)$$

of an N -linear connection D .

In the algebra of the d -tensor field generated by

$$\left\{ 1, \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}, dx^i, \delta y^{(1)i}, \delta y^{(2)i} \right\},$$

the h -covariant derivatives will be noted with " | " and the v_α -covariant derivatives will be noted with " | ^(α) ", $\alpha = 1, 2$.

Applying (1.4) and (1.4') we obtain

Theorem 1.1. *If N is a nonlinear connection on E with the coefficients $N_{(1)j}^i, N_{(2)j}^i$, then the following relations hold*

$$(1.8) \quad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = \sum_{(i,j)} \left\{ \left(\frac{\partial N_{(1)i}^m}{\partial x^j} + N_{(1)i}^r \frac{\partial N_{(1)j}^m}{\partial y^{(1)r}} + N_{(2)i}^r \frac{\partial N_{(1)j}^m}{\partial y^{(2)r}} \right) \frac{\partial}{\partial y^{(1)m}} + \left(\frac{\partial N_{(2)i}^m}{\partial x^j} + N_{(1)i}^r \frac{\partial N_{(2)j}^m}{\partial y^{(1)r}} + N_{(2)i}^r \frac{\partial N_{(2)j}^m}{\partial y^{(2)r}} \right) \frac{\partial}{\partial y^{(2)m}} \right\},$$

$$(1.9) \quad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)j}} \right] = \left(\frac{\partial N_{(1)i}^m}{\partial y^{(1)j}} - N_{(1)j}^r \frac{\partial N_{(1)i}^m}{\partial y^{(2)r}} \right) \frac{\partial}{\partial y^{(1)m}} + \left[\left(\frac{\partial N_{(2)i}^m}{\partial y^{(1)j}} - \frac{\partial N_{(1)j}^m}{\partial x^i} \right) + N_{(1)i}^r \frac{\partial N_{(1)j}^m}{\partial y^{(1)r}} + N_{(2)i}^r \frac{\partial N_{(1)j}^m}{\partial y^{(2)r}} - N_{(1)j}^r \frac{\partial N_{(2)i}^m}{\partial y^{(2)r}} \right] \frac{\partial}{\partial y^{(2)m}},$$

$$(1.10) \quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^{(2)j}} \right] = \frac{\partial N_{(1)i}^m}{\partial y^{(2)j}} \frac{\partial}{\partial y^{(1)m}} + \frac{\partial N_{(2)i}^m}{\partial y^{(2)j}} \frac{\partial}{\partial y^{(2)m}},$$

$$(1.11) \quad \left[\frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(1)j}} \right] = \sum_{(i,j)} \left\{ \left(\frac{\partial N_{(1)i}^m}{\partial y^{(1)j}} + N_{(1)i}^r \frac{\partial N_{(1)j}^m}{\partial y^{(2)r}} \right) \frac{\partial}{\partial y^{(2)m}} \right\},$$

$$(1.12) \quad \left[\frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)j}} \right] = \frac{\partial N_{(1)i}^m}{\partial y^{(2)j}} \frac{\partial}{\partial y^{(2)m}},$$

where $\sum_{(i,j)}$ is the symbol of alternate sum.

2 The torsion and curvature d -tensor fields

The torsion tensor of the N -linear connection D on $E = Osc^2 M$,

$$(2.1) \quad T(X, Y) = D_X Y - D_Y X - [X, Y], \quad \forall X, Y \in \mathcal{X}(\mathcal{E})$$

has a number of horizontal and vertical components corresponding to D^H , D^{v_1} and D^{v_2} .

We put

$$(2.2) \quad \begin{aligned} T \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) &= \overset{(0)}{T}{}^m{}_{ij} \frac{\delta}{\delta x^m} + \overset{(1)}{T}{}^m{}_{ij} \frac{\delta}{\delta y^{(1)m}} + \overset{(2)}{T}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}} \\ T \left(\frac{\delta}{\delta y^{(1)j}}, \frac{\delta}{\delta x^i} \right) &= \overset{(0)}{P}{}^m{}_{ij} \frac{\delta}{\delta x^m} + \overset{(1)}{P}{}^m{}_{ij} \frac{\delta}{\delta y^{(1)m}} + \overset{(2)}{P}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}} \\ T \left(\frac{\partial}{\partial y^{(2)j}}, \frac{\delta}{\delta x^i} \right) &= \overset{(0)}{P}{}^m{}_{ij} \frac{\delta}{\delta x^m} + \overset{(1)}{P}{}^m{}_{ij} \frac{\delta}{\delta y^{(1)m}} + \overset{(2)}{P}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}} \\ T \left(\frac{\partial}{\partial y^{(2)j}}, \frac{\delta}{\delta y^{(1)i}} \right) &= \overset{(0)}{P}{}^m{}_{ij} \frac{\delta}{\delta x^m} + \overset{(1)}{P}{}^m{}_{ij} \frac{\delta}{\delta y^{(1)m}} + \overset{(2)}{P}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}} \\ T \left(\frac{\delta}{\delta y^{(1)j}}, \frac{\delta}{\delta y^{(1)i}} \right) &= \overset{(0)}{S}{}^m{}_{ij} \frac{\delta}{\delta x^m} + \overset{(1)}{S}{}^m{}_{ij} \frac{\delta}{\delta y^{(1)m}} + \overset{(2)}{S}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}} \\ T \left(\frac{\partial}{\partial y^{(2)j}}, \frac{\partial}{\partial y^{(2)i}} \right) &= \overset{(0)}{S}{}^m{}_{ij} \frac{\delta}{\delta x^m} + \overset{(1)}{S}{}^m{}_{ij} \frac{\delta}{\delta y^{(1)m}} + \overset{(2)}{S}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}} \end{aligned}$$

Then, by (2.1), (1.7) and Theorem 1.1, we have

Theorem 2.1. *The torsion tensor of an N -linear connection $D\Gamma(N) = \left(L_{ij}^m, C_{(1)ij}^m, C_{(2)ij}^m \right)$ is characterized by the d -tensor fields with local components:*

$$(2.3) \quad \begin{cases} \begin{matrix} \overset{(0)}{T}{}^m{}_{ij} \\ \underset{(0)}{T}{}^m{}_{ij} \end{matrix} = L_{ij}^m - L_{ji}^m, \begin{matrix} \overset{(0)}{P}{}^m{}_{ij} \\ \underset{(1)}{P}{}^m{}_{ij} \end{matrix} = C_{ij}^m, \begin{matrix} \overset{(0)}{P}{}^m{}_{ij} \\ \underset{(2)}{P}{}^m{}_{ij} \end{matrix} = C_{ij}^m \\ \begin{matrix} \overset{(1)}{T}{}^m{}_{ij} \\ \underset{(0)}{T}{}^m{}_{ij} \end{matrix} = \frac{\delta N^m{}_i}{\delta x^j} - \frac{\delta N^m{}_j}{\delta x^i}, \\ \begin{matrix} \overset{(2)}{T}{}^m{}_{ij} \\ \underset{(0)}{T}{}^m{}_{ij} \end{matrix} = \frac{\delta N^m{}_i}{\delta x^j} - \frac{\delta N^m{}_j}{\delta x^i} + N^m{}_r \left(\frac{\delta N^r{}_i}{\delta x^j} - \frac{\delta N^r{}_j}{\delta x^i} \right), \end{cases}$$

$$(2.4) \quad \begin{cases} \begin{matrix} \overset{(1)}{P}{}^m{}_{ij} \\ \underset{(1)}{P}{}^m{}_{ij} \end{matrix} = \frac{\delta N^m{}_i}{\delta y^{(1)j}} - L_{ji}^m, \begin{matrix} \overset{(2)}{P}{}^m{}_{ij} \\ \underset{(1)}{P}{}^m{}_{ij} \end{matrix} = \frac{\delta N^m{}_i}{\delta y^{(1)j}} + N^m{}_r \frac{\delta N^r{}_i}{\delta y^{(1)j}} - \frac{\delta N^m{}_j}{\delta y^{(1)i}}, \\ \begin{matrix} \overset{(1)}{P}{}^m{}_{ij} \\ \underset{(2)}{P}{}^m{}_{ij} \end{matrix} = \frac{\partial N^m{}_i}{\partial y^{(2)j}}, \begin{matrix} \overset{(2)}{P}{}^m{}_{ij} \\ \underset{(2)}{P}{}^m{}_{ij} \end{matrix} = \frac{\partial N^m{}_i}{\partial y^{(2)j}} + N^m{}_r \frac{\partial N^r{}_i}{\partial y^{(2)j}} - L_{ji}^m, \\ \begin{matrix} \overset{(0)}{P}{}^m{}_{ij} \\ \underset{(12)}{P}{}^m{}_{ij} \end{matrix} = 0, \begin{matrix} \overset{(1)}{P}{}^m{}_{ij} \\ \underset{(12)}{P}{}^m{}_{ij} \end{matrix} = C_{ij}^m, \begin{matrix} \overset{(2)}{P}{}^m{}_{ij} \\ \underset{(12)}{P}{}^m{}_{ij} \end{matrix} = \frac{\partial N^m{}_i}{\partial y^{(2)j}} - C_{ji}^m, \end{cases}$$

$$(2.5) \quad \begin{cases} \begin{matrix} \overset{(0)}{S}{}^m{}_{ij} \\ \underset{(1)}{S}{}^m{}_{ij} \end{matrix} = 0, \begin{matrix} \overset{(1)}{S}{}^m{}_{ij} \\ \underset{(1)}{S}{}^m{}_{ij} \end{matrix} = C_{ij}^m - C_{ji}^m, \begin{matrix} \overset{(2)}{S}{}^m{}_{ij} \\ \underset{(1)}{S}{}^m{}_{ij} \end{matrix} = \frac{\delta N^m{}_i}{\delta y^{(1)j}} - \frac{\delta N^m{}_j}{\delta y^{(1)i}}, \\ \begin{matrix} \overset{(0)}{S}{}^m{}_{ij} \\ \underset{(2)}{S}{}^m{}_{ij} \end{matrix} = 0, \begin{matrix} \overset{(1)}{S}{}^m{}_{ij} \\ \underset{(2)}{S}{}^m{}_{ij} \end{matrix} = 0, \begin{matrix} \overset{(2)}{S}{}^m{}_{ij} \\ \underset{(2)}{S}{}^m{}_{ij} \end{matrix} = C_{ij}^m - C_{ji}^m \end{cases}$$

Also, we can use the notations

$$\begin{aligned} \begin{matrix} \overset{(0)}{T}{}^m{}_{ij} \\ \underset{(0)}{T}{}^m{}_{ij} \end{matrix} &= T^m{}_{ij}, \quad \begin{matrix} \overset{(1)}{T}{}^m{}_{ij} \\ \underset{(0)}{T}{}^m{}_{ij} \end{matrix} = \overset{(1)}{R}{}^m{}_{ij}, \quad \begin{matrix} \overset{(2)}{T}{}^m{}_{ij} \\ \underset{(0)}{T}{}^m{}_{ij} \end{matrix} = \overset{(2)}{R}{}^m{}_{ij}, \\ \begin{matrix} \overset{(1)}{S}{}^m{}_{ij} \\ \underset{(1)}{S}{}^m{}_{ij} \end{matrix} &= S^m{}_{ij}, \quad \begin{matrix} \overset{(2)}{S}{}^m{}_{ij} \\ \underset{(1)}{S}{}^m{}_{ij} \end{matrix} = \overset{(1)}{R}{}^m{}_{ij}, \quad \begin{matrix} \overset{(2)}{S}{}^m{}_{ij} \\ \underset{(2)}{S}{}^m{}_{ij} \end{matrix} = S_{(2)}^m{}_{ij} \end{aligned}$$

Theorem 2.2. *An N -linear connection $DF(N) = (L_{ij}^m, C_{(1)}^m{}_{ij}, C_{(2)}^m{}_{ij})$ is without torsion if and only if*

$$(2.6) \quad \begin{cases} L_{ij}^m = L_{ji}^m, \quad C_{ij}^m = 0, \quad \overset{(\gamma)}{R}{}^m{}_{ij} = 0, \\ \overset{(\gamma)}{P}{}^m{}_{ij} = 0, \quad \overset{(\gamma)}{P}{}^m{}_{(\alpha\beta)} = 0, \quad \overset{(\gamma)}{S}{}^m{}_{ij} = 0, \quad \alpha < \beta; \alpha, \beta, \gamma = 1, 2. \end{cases}$$

It is to notice the fact that an N -linear connection D is called semi-symmetric if

$$(2.7) \quad \begin{aligned} [T(X^H, Y^H)]^H &= X^H \eta(Y^H) - Y^H \eta(X^H) \\ [T(X^{v\alpha}, Y^{v\alpha})]^{v\alpha} &= X^{v\alpha} \sigma_\alpha(Y^{v\alpha}) - Y^{v\alpha} \sigma_\alpha(X^{v\alpha}), \\ \forall X, Y \in \mathcal{X}(\mathcal{E}), \quad \eta, \sigma_\alpha &\in \mathcal{X}^*(\mathcal{E}), \quad \alpha = \infty, \in. \end{aligned}$$

Denoting by

$$\begin{aligned} R\left(\frac{\delta}{\delta x^q}, \frac{\delta}{\delta x^p}\right) \frac{\delta}{\delta x^r} &= R_{r \quad pq}^m \frac{\delta}{\delta x^m}, & R\left(\frac{\delta}{\delta y^{(\beta)q}}, \frac{\delta}{\delta x^p}\right) \frac{\delta}{\delta x^r} &= P_{r \quad pq}^m \frac{\delta}{\delta x^m}, \\ R\left(\frac{\delta}{\delta y^{(\beta)q}}, \frac{\delta}{\delta y^{(\alpha)p}}\right) \frac{\delta}{\delta x^r} &= P_{r \quad pq}^m \frac{\delta}{\delta x^m}, & R\left(\frac{\delta}{\delta y^{(\alpha)q}}, \frac{\delta}{\delta y^{(\alpha)p}}\right) \frac{\delta}{\delta x^r} &= S_{(\alpha)r \quad pq}^m \frac{\delta}{\delta x^m} \end{aligned}$$

we have

Theorem 2.3. *The curvature tensor field R of an N -linear connection $D\Gamma(N) = (L_{ij}^m, C_{(1)ij}^m, C_{(2)ij}^m)$ is characterized by the following d -tensor fields on E :*

$$(2.8) \quad \begin{aligned} R_{r \quad pq}^m &= \frac{\delta L_{rp}^m}{\delta x^q} - \frac{\delta L_{rq}^m}{\delta x^p} + L_{rp}^t L_{tq}^m - L_{rq}^t L_{tp}^m + \\ &\quad + R_{(0) \quad pq}^{(1)t} C_{(1)rt}^m + R_{(0) \quad pq}^{(2)t} C_{(2)rt}^m, \end{aligned}$$

$$(2.9) \quad P_{(1) \quad pq}^m = \frac{\delta L_{rp}^m}{\delta y^{(1)q}} - C_{(1)rq|p}^m + P_{(1) \quad pq}^{(1)t} C_{(1)rt}^m + P_{(1) \quad pq}^{(2)t} C_{(2)rt}^m,$$

$$(2.10) \quad P_{(2) \quad pq}^m = \frac{\partial L_{rp}^m}{\partial y^{(2)q}} - C_{(2)rq|p}^m + P_{(2) \quad pq}^{(1)t} C_{(1)rt}^m + P_{(2) \quad pq}^{(2)t} C_{(2)rt}^m,$$

$$(2.11) \quad P_{(12) \quad pq}^m = \frac{\partial C_{rp}^m}{\partial y^{(2)q}} - C_{(2)rq|p}^{(1)} + P_{(12) \quad pq}^{(2)t} C_{(2)rt}^m,$$

$$(2.12) \quad S_{(1) \quad pq}^m = \frac{\delta C_{rp}^m}{\delta y^{(1)q}} - \frac{\delta C_{rq}^m}{\delta y^{(1)p}} + C_{(1)rp}^t C_{(1) \quad tq}^m - C_{(1) \quad rq}^t C_{(1) \quad tp}^m + R_{(1) \quad pq}^{(1)t} C_{(1)rt}^m,$$

$$(2.13) \quad S_{(2) \quad pq}^m = \frac{\partial C_{rp}^m}{\partial y^{(2)q}} - \frac{\partial C_{rq}^m}{\partial y^{(2)p}} + C_{(2)rp}^t C_{(2) \quad tq}^m - C_{(2) \quad rq}^t C_{(2) \quad tp}^m.$$

Theorem 2.4. *The curvature tensor field R of an N -linear connection D becomes zero if and only if*

$$(2.14) \quad R_{r \quad pq}^m = P_{(\alpha) \quad pq}^m = P_{(12) \quad pq}^m = S_{r \quad pq}^m = 0, \quad \alpha = 1, 2.$$

3 The equations of structure

Let (C, c) , $c : I \rightarrow Osc^2M$, $C = Im c$ be a smooth parametrized curve on Osc^2M and let \dot{c} be the tangent vector field

$$(3.1) \quad \dot{c} = \dot{c}^H + \dot{c}^{v_1} + \dot{c}^{v_2}$$

We consider the vector field $dc = \dot{c}dt$ on the curve c .
According to (3.1) we have

$$(3.2) \quad dc = (dc)^H + (dc)^{v_1} + (dc)^{v_2}$$

and by the adapted bases, we get

$$(3.2') \quad dc = dx^i \frac{\delta}{\delta x^i} + \delta y^{(1)i} \frac{\delta}{\delta y^{(1)i}} + \delta y^{(2)i} \frac{\partial}{\partial y^{(2)i}}.$$

Let D be an N -linear connection and $Y \in \mathcal{X}(\mathcal{O}f]^\epsilon \mathcal{M})$. We denote $D_{dc}Y$ with DY . DY is the covariant differential of the vector field Y on the curve c .

We put

$$D_{(dc)^H}Y = D^HY, \quad D_{(dc)^{v_1}}Y = D^{v_1}Y, \quad D_{(dc)^{v_2}}Y = D^{v_2}Y$$

and we can write the covariant differential of Y in the form

$$(3.3) \quad DY = D^HY + D^{v_1}Y + D^{v_2}Y.$$

Now, we take $Y = Y^H = Y^i \frac{\delta}{\delta x^i}$ and we obtain

$$(3.4) \quad DY = (Y^i|_m dx^m + Y^i|_m^{(1)} \delta y^{(1)m} + Y^i|_m^{(2)} \delta y^{(2)m}) \frac{\delta}{\delta x^i}.$$

The equality (3.4) is changes correspondingly if $Y = Y^{v_1}$ or $Y = Y^{v_2}$.

If we consider the h - and v_α -covariant derivatives of Y^i , we have

$$(3.5) \quad DY = (dY^i + Y^j \omega^i_j) \frac{\delta}{\delta x^i}$$

where dY^i is the usual differential of the functions $Y^i(x, y^{(1)}, y^{(2)})$ in the adapted bases

$$(3.6) \quad dY^i = \frac{\delta Y^i}{\delta x^m} dx^m + \frac{\delta Y^i}{\delta y^{(1)m}} \delta y^{(1)m} + \frac{\partial Y^i}{\partial y^{(2)m}} \delta y^{(2)m},$$

and ω^i_j are the notations of the following covector fields:

$$(3.7) \quad \omega^i_j = L^i_{jm} dx^m + C^i_{jm} \delta y^{(1)m} + C^i_{jm} \delta y^{(2)m}$$

The covector fields ω^i_j are not dependent on the choice of the vector field $Y = Y^H$ or $Y = Y^{v_\alpha}$ ($\alpha = 1, 2$). They are determined by the N -linear connection, only.

We shall call ω^i_j , the *connection forms* of $D\Gamma(N)$.

To deduce the equations of structure of an N -linear connection $D\Gamma(N) = (L^i_{jm}, C^i_{jm}, C^i_{jm})$ we consider the exterior differentials of the 1-form fields $dx^i, \delta y^{(1)i}, \delta y^{(2)i}$ and of ω^i_j in the adapted bases $dx^i, \delta y^{(1)i}, \delta y^{(2)i}$.

Firstly, we obtain:

Theorem 3.1. *The exterior differentials of $\delta y^{(1)i}, \delta y^{(2)i}$ are given by*

$$(3.8) \quad d(\delta y^{(1)m}) = -\frac{1}{2} \overset{(1)}{R}_{(0)}^m{}_{ij} dx^i \wedge dx^j - (\overset{(1)}{P}_{(1)}^m{}_{ij} + L_{ij}^m) dx^i \wedge \delta y^{(1)j} - \\ - \overset{(1)}{P}_{(2)}^m{}_{ij} dx^i \wedge \delta y^{(2)j},$$

$$(3.9) \quad d(\delta y^{(2)m}) = -\frac{1}{2} \overset{(2)}{R}_{(0)}^m{}_{ij} dx^i \wedge dx^j - \overset{(2)}{P}_{(1)}^m{}_{ij} dx^i \wedge \delta y^{(1)j} - \\ (\overset{(2)}{P}_{(2)}^m{}_{ij} + L_{ji}^m) dx^i \wedge \delta y^{(2)j} - \frac{1}{2} \overset{(1)}{R}_{(1)}^m{}_{ij} \delta y^{(1)i} \wedge \delta y^{(1)j} - \overset{(1)}{P}_{(2)}^m{}_{ij} \delta y^{(1)i} \wedge \delta y^{(2)j}.$$

Using (3.8), (3.9) and (3.7) we have

Theorem 3.2. *The equations of structure of an N -linear connection $D\Gamma(N)$ are given by*

$$(3.10) \quad \begin{cases} d(dx^m) - dx^i \wedge \omega^m{}_j = -\Omega^{(0)m}, \\ d(\delta y^{(1)m}) - \delta y^{(1)j} \wedge \omega^m{}_j = -\Omega^{(1)m}, \\ d(\delta y^{(2)m}) - \delta y^{(2)j} \wedge \omega^m{}_j = -\Omega^{(2)m}, \end{cases}$$

and by

$$(3.11) \quad d\omega^m{}_r - \omega^s{}_r \wedge \omega^m{}_s = -\Omega^m{}_r,$$

where the 2-forms of torsion $\overset{(0)}{\Omega}^m$, $\overset{(1)}{\Omega}^m$, $\overset{(2)}{\Omega}^m$ are given by

$$(3.12) \quad \overset{(0)}{\Omega}^m = \frac{1}{2} T^m{}_{ij} dx^i \wedge dx^j + C_{(1)}^m{}_{ij} dx^i \wedge \delta y^{(1)j} + C_{(2)}^m{}_{ij} dx^i \wedge \delta y^{(2)j},$$

$$(3.13) \quad \overset{(1)}{\Omega}^m = \frac{1}{2} \overset{(1)}{R}_{(0)}^m{}_{ij} dx^i \wedge dx^j + \overset{(1)}{P}_{(1)}^m{}_{ij} dx^i \wedge \delta y^{(1)j} + \\ + \overset{(1)}{P}_{(2)}^m{}_{ij} dx^i \wedge \delta y^{(2)j} + \frac{1}{2} S_{(1)}^m{}_{ij} \delta y^{(1)i} \wedge \delta y^{(1)j} + C_{(2)}^m{}_{ij} \delta y^{(1)i} \wedge \delta y^{(2)j},$$

$$(3.14) \quad \overset{(2)}{\Omega}^m = \frac{1}{2} \overset{(2)}{R}_{(0)}^m{}_{ij} dx^i \wedge dx^j + \overset{(2)}{P}_{(1)}^m{}_{ij} dx^i \wedge \delta y^{(1)j} + \\ + \overset{(2)}{P}_{(2)}^m{}_{ij} dx^i \wedge \delta y^{(2)j} + \frac{1}{2} \overset{(1)}{R}_{(1)}^m{}_{ij} \delta y^{(1)i} \wedge \delta y^{(1)j} + \\ + \overset{(2)}{P}_{(12)}^m{}_{ij} \delta y^{(1)i} \wedge \delta y^{(2)j} + \frac{1}{2} S_{(2)}^m{}_{ij} \delta y^{(2)i} \wedge \delta y^{(2)j},$$

and the 2-form of curvature $\Omega^m{}_r$ is given by

$$(3.15) \quad \Omega^m{}_r = \frac{1}{2} R_r{}^m{}_{pq} dx^p \wedge dx^q + P_{(1)r}{}^m{}_{pq} dx^p \wedge \delta y^{(1)q} + \\ + P_{(2)r}{}^m{}_{pq} dx^p \wedge \delta y^{(2)q} + \frac{1}{2} S_{(1)r}{}^m{}_{pq} \delta y^{(1)p} \wedge \delta y^{(1)q} + \\ + P_{(12)r}{}^m{}_{pq} \delta y^{(1)p} \wedge \delta y^{(2)q} + \frac{1}{2} S_{(2)r}{}^m{}_{pq} \delta y^{(2)p} \wedge \delta y^{(2)q}.$$

The equations of structure of $D\Gamma(N)$ allow us to get some remarkable geometrical interpretations for the torsion and curvature d -tensor fields of the N -linear connection $D\Gamma(N)$.

References

- [1] C.T.J. Dodson and M.S.Radivoioivici, *Tangent and frame bundles of order two*, Anal. Şt. Univ. "Al.I.Cuza" Iaşi, 28(1982), 65-71.
- [2] R.Miron and Gh.Atanasiu, *Lagrange Geometry of Second Order*, Math. Comput. Modelling, Pergamon, 20, 4/5 (1994), 41-56.
- [3] R.Miron and Gh.Atanasiu, *Differential geometry of the k -osculator bundle*, Rev.Roumaine Math.Pures Appl., 41, 3/4 (1996), 205-236.
- [4] R.Miron and Gh.Atanasiu, *Prolongation of Riemannian, Finslerian and Lagrangian structures*, Rev.Roumaine Math.Pures.Appl., 41, 3/4(1996), 237-249.
- [5] R.Miron and Gh.Atanasiu, *Higher-order Lagrange spaces*, Rev.Roumaine Math.Pures Appl., 41, 3/4(1996), 251-262.

Department of Geometry
Transilvania University
2200 Braşov, Romania