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of multi-dimensional random walks**

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The Netherlands

THE EQUILIBRIUM DISTRIBUTION FOR A CLASS OF MULTI-DIMENSIONAL RANDOM WALKS

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Abstract

In previous papers, it has been proved that the equilibrium distribution of homogeneous, nearest-neighboring random walks on a two-dimensional grid can be constructed explicitly through a compensation procedure if and only if there are no transitions to the North, North-East and East for points in the interior. In the present paper the extension to N -dimensional random walks is investigated. It appears that for higher dimensions the same condition should be satisfied for each plane in the grid space.

Since induction with respect to the dimension is applied, the step from dimension 2 to dimension 3 is worked out in detail. For the proof of the if-part the condition is added that the random walk satisfies the so-called projection property on the boundaries. For 3-dimensional random walks, the equilibrium distribution appears to be the sum of six alternating series of binary trees of product forms. These analytic results make it possible to develop efficient numerical procedures. Such procedures are sketched in the paper. As a numerical illustration, the procedures are applied to the model of a 2×3 switch.

Keywords: Multi-dimensional random walk, Markov chain, equilibrium distribution, product forms, compensation approach.

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1. Introduction

Several queueing systems can be modeled as *multi-dimensional random walks* on an integer grid. Therefore, much effort has been put in investigating the equilibrium distribution of such random walks. For so-called *product-form networks of queues* (see, for instance, Baskett et al. [10]), it has been proved that the equilibrium distribution can be written as a product of powers of fixed factors. The random walks representing product-form networks of queues may have an arbitrary dimension. There is one other class of random walks which is known to provide explicit forms for the equilibrium distributions. This class consists of two-dimensional random walks on the integer grid in the positive quadrant of the plane. Apart from the obvious ergodicity requirements, one has to require that from points in the interior no transitions can be made to the North, North-East and East in order to obtain explicit expressions in the form of infinite series of products of powers of fixed factors (in [8] this has been proved for a class of homogeneous, nearest-neighboring random walks). The terms of these series may be constructed consecutively with the help of a *compensation procedure*.

It is quite natural, therefore, to investigate the possibilities of extending this compensation procedure to random walks with more than two dimensions. This is the topic of the present paper. Actually, some random walks with more than two dimensions have been solved explicitly. However, in these cases, the state space was only infinite in one direction (compare [2] for a treatment of the $E_k | E_r | c$ queue, see also Bertsimas [11]; compare [9] for a treatment of the shortest queue system with N servers and threshold jockeying).

The compensation procedure for two-dimensional random walks is a direct approach for solving the equilibrium equations without resorting to generating functions. As said before, the compensation approach only works in the two-dimensional case if there are no transitions to the North, North-East and East.

There are two indirect approaches via generating functions which work in principle for more general two-dimensional random walks on the integer grid in the positive quadrant.

The oldest of these approaches is the *uniformization technique*, which was developed by Kingman [27] and Flatto and McKean [22] for the symmetric shortest queue system with two servers. For the generating function $f(x,y)$ for the equilibrium distribution of the lengths of the two queues, they derive a functional equation with as unknown functions $f(x,y)$ on the one side and the generating functions $f(x,0)$ and $f(0,y)$ for the equilibrium probabilities on the axis on the other side of the equation. The functions $f(x,0)$ and $f(0,y)$ and, hence, $f(x,y)$ are shown to be meromorphic and explicit formulae are derived for the poles and their residues after having introduced a uniformizing variable. By decomposing the meromorphic function $f(x,y)$ into partial fractions it follows that the equilibrium probabilities may be written as infinite linear combinations of product forms. The same technique has been used by Hofri [23] for a multiprogramming queueing model (see also [7]) and by Jaffe [26] for the 2×2 clocked buffered switch. All three cases for which the uniformization technique has been worked out, have the property that there are no transitions from interior points to the North, North-East and East. In all three cases the generating function is meromorphic and partial fraction decomposition of this function yields expressions for the equilibrium probabilities in the form of infinite linear combinations of product forms, although it appears to be

difficult to give explicit formulae for the coefficients of the linear combinations.

The uniformization technique has also been employed by Flatto and Hahn [21] to analyze the fork and join model with two servers. They show that the generating functions $f(x, 0)$ and $f(0, y)$ can be extended to multiple-valued algebraic functions. However, partial fraction decomposition is not available for multiple-valued functions, so that it is no longer possible to derive exact formulae for the equilibrium probabilities via this decomposition. Recently, Wright [34] analyzed a generalization of the fork and join model by using the uniformization technique. In his analysis he encounters the same difficulties as Flatto and Hahn [21], i.e. multiple-valued functions. Until recently, there was no general result for two-dimensional random walks based on a derivation via the uniformization technique, although it seems possible to derive a general result for cases which satisfy the extra condition. Nevertheless, the results of [27] and [22] for the symmetric shortest queue problem largely inspired the compensation procedure (compare [5]), which indeed gives explicit formulae for the coefficients of the linear combination for all cases satisfying the extra condition.

Extension of the uniformization technique to random walks with more than two dimensions has never succeeded. One reason for this failure might be that the extra condition has never appeared as essential in these investigations, whereas the shortest queue problem with N servers ($N > 2$) appears not to satisfy the extension of the extra condition to higher dimensions (cf. Section 4). Recently, inspired by [8], Cohen [16] has shown that a technique, which is actually a direct generalization of the uniformization technique, may be used for the same class of problems as the class to which the compensation approach is applicable (which has to do with the resemblance between both methods described in [13]). Therefore we can conclude that for a two-dimensional, irreducible, positive recurrent, homogeneous, nearest-neighboring random walk the compensation approach and the uniformization technique are only usable, in the sense that they give explicit expressions for the equilibrium probabilities in the form of series of products of powers, if there are no transitions from interior points to the North, North-East and East. But, if this condition is satisfied, then the latter two methods are very suitable. Further, we believe that the compensation approach is preferable to the uniformization technique, since it leads to more explicit results (explicit formulae for *all* equilibrium probabilities, for example) and it avoids complex analysis.

A newer indirect method for solving the functional equation for the generating function of the equilibrium distribution, is the *boundary value method*. This method aims at reducing the functional equation to a standard problem of the theory of boundary value problems and integral equations for complex functions and has established itself as a powerful method for a large class of two-dimensional random walks in the first quadrant; see Cohen and Boxma [17]. Queueing problems solved by the boundary value method are the symmetric shortest queue model, the M/G/2 queue, a polling model with two queues and 1-limited service (see [17] for all these examples), the coupled processor model (see [17], the work of Fayolle and Iasnogorodski [18, 19, 24] and also Konheim et al. [28]), the longest queue model with nonpreemptive priority (see Cohen [14]; the longest queue model with preemptive priority has been treated by Zheng and Zipkin [35], who solve the equilibrium equations iteratively, and by Flatto [20], who explicitly solves the functional equation for the generating function) and the 2×2 clocked buffered switch (see Jaffe [25]). For a review of the boundary value method for two-dimensional problems, see Cohen [15]. Some examples mentioned show already that the boundary value method is not restricted to random walks without transitions

from interior points to the North, North-East and East. It seems to be the only really general method for two-dimensional random walks on the integer grid in the positive quadrant. However, the compensation method gives more complete results in the cases in which it works. About extension of the boundary value method to higher dimensions the review paper [15] states that it should be possible in principle, but the mathematical as well as the numerical analysis becomes very intricate.

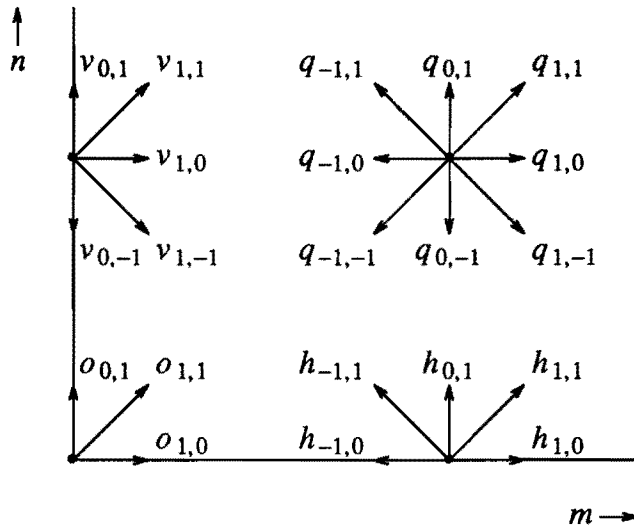


Figure 1. The transition rates for a two-dimensional, homogeneous, nearest-neighboring random walk on the lattice of the first quadrant; for all states the transitions to themselves have been left away.

As said before, the present paper is devoted to investigating the possibilities of extending the compensation method to higher dimensional random walks. Therefore, we now give a short characterization of the method. For simplicity, we only consider two-dimensional random walks with transitions to the nearest neighbors in the horizontal, vertical or diagonal sense as depicted in Figure 1. The first step is to characterize the product forms which satisfy the equilibrium equations in the interior points ($m \geq 2, n \geq 2$). Subsequently, it is tried to construct an infinite series of such solutions which also satisfies the boundary equations. The construction starts by taking a product form which satisfies the interior equations as well as the equations for one of the boundaries. It is then corrected by adding a product form which not only satisfies the interior equations, but also makes the sum satisfy the equations on the other boundary. Then a new correction term is added to make the solution again satisfy the equations on the first boundary, etc. Requirements for this method to work are:

1. In each step it should be possible to find a new correction term which satisfies the needs;
2. The resulting series should converge.

In [8], it appeared that these requirements are fulfilled if and only if the random walk is irreducible, positive recurrent and

$$q_{1,0} = q_{0,1} = q_{1,1} = 0. \tag{1.1}$$

The latter equations say that from interior states there are no transitions to the North, East and North-East respectively.

Condition (1.1) arises from convergence requirements for the infinite linear combinations of product forms constructed by the compensation approach, and certainly limits the applicability of this method. However, if this condition is satisfied, then the compensation approach is very powerful. Application of this method shows that the equilibrium distribution consists of a linear combination of at most four series of product-form solutions and explicit formulae are produced for all coefficients and factors.

There are a number of well-known queueing problems present in the class of random walks depicted in Figure 1. For these problems, condition (1.1) is satisfied by the symmetric shortest queue problem, Hofri's multiprogramming queues model and the 2×2 clocked buffered switch (see [3, 5, 13]), while the condition is violated by the coupled processor model, the longest queue model and the fork and join model. In [4], it has been shown that the restriction to nearest-neighbor transitions is not essential, however, it simplifies the arguments considerably. Particularly, the finding of good starting solutions becomes much more complex. In [6], it has been shown that the compensation approach can also be used for random walks in more complex areas.

Our approach for extending the compensation procedure to higher dimensions is based on induction with respect to the dimension of the state space. For the case of dimension N , it is the aim to compensate a starting solution on each of the boundary hyperplanes with dimension $N-1$, consecutively. For finding starting solutions, the solution for $(N-1)$ -dimensional problems is needed.

The main part of the paper will be devoted to a detailed treatment of the case $N=3$. Finally, it will be sketched how this treatment can be generalized to an induction step for an arbitrary value of N . The approach generates, in a natural way, the extension of condition (1.1) to N -dimensional random walks:

Let q_{t_1, \dots, t_N} with $t_i \in \{-1, 0, 1\}$ denote the transition rates for interior states in a similar way as in Figure 1 for $N=2$. Then the condition becomes

$$q_{t_1, \dots, t_N} = 0 \quad \text{if } t_i + t_j > 0 \text{ for some } i, j \in \{1, \dots, N\}, i \neq j. \quad (1.2)$$

This condition essentially restricts the applicability of the compensation approach for higher dimensions, however, when the condition is satisfied, the solution is very explicit and useful (see Section 9).

If one constructs the equilibrium equations for the random walk depicted in Figure 1, then each of the boundaries consists of two layers, viz. $m=0$ and $m=1$ for the vertical boundary and $n=0$ and $n=1$ for the horizontal boundary. This feature complicates the analysis. Therefore, we will simplify the model by assuming that, for all states (m_1, \dots, m_N) with $m_i \geq 1$ for all i , not only the rates for transitions starting in those states are equal, but also the rates for transitions ending in those states. In Figure 1, this would e.g. imply

$$h_{i,1} = q_{i,1} \text{ for all } i \in \{-1, 0, 1\}.$$

Note that also the restriction to nearest-neighbor transitions was made in order to avoid 'fat' boundaries. Another simplification, which will be made in the course of the paper is the

assumption that the transition rates on the boundaries of the state space satisfy a simple projection condition.

The organization of the paper is as follows. In Section 2 we give some examples of N -dimensional random walks and we describe the class of three-dimensional random walks to which we want to apply the compensation approach. In Section 3 the compensation approach itself is described for three-dimensional random walks in the positive octant. The formal solutions constructed by the compensation approach are required to be absolutely convergent, which leads to a necessary condition, condition (1.2), and to a reformulation of the formal solutions; these are the subject of Section 4. Hereafter, we restrict ourselves to random walks which also satisfy the projection property described in Section 5. For such a random walk, in Section 6, condition (1.2) is shown to be also sufficient for the absolute convergence. Subsequently, in Section 7, the Main Theorem is presented. In that section it is shown how for a random walk with the projection property the equilibrium distribution has to be constructed. Next, in Section 8, we present some numerical results for the 2×3 switch, which provides an example of a three-dimensional random walk satisfying condition (1.2). In Section 9, we extend the main results to the N -dimensional case. Finally, in Section 10, the conclusions and suggestions for future research are given.

2. The class of three-dimensional random walks and the equilibrium equations

This section opens with three examples of N -dimensional random walks. The first two examples satisfy the condition (1.1) for $N=2$. However, only the second one satisfies the condition (1.2) for arbitrary $N > 2$.

Example 2.1: The symmetric shortest queue system

This system consists of N parallel, identical servers, each with exponentially distributed service times with mean 1. Jobs arrive at the system according to a Poisson stream with intensity $N\rho$, $0 < \rho < 1$ (this implies the ergodicity of the system), and an arriving job always joins the shortest queue (ties are broken with equal probabilities). This system may be modeled by a continuous-time Markov chain with states (m_1, \dots, m_N) , where m_1 denotes the number of jobs at the shortest queue and m_k denotes the difference between the queue lengths of the k -th shortest queue and the $(k-1)$ -th shortest queue (for all queue lengths the jobs in service have to be included). For the case $N=2$, the positive rates for the interior points are

$$q_{1,-1} = 2\rho, \quad q_{-1,1} = q_{0,-1} = 1,$$

which shows that condition (1.1) is satisfied. Applying the compensation approach shows that the equilibrium distribution for the symmetric shortest queue system with two servers may be written as an infinite linear combination (or a series) of product-form solutions (see [5]). For $N > 2$, however, condition (1.2) is not satisfied (as we shall see in Section 4).

Example 2.2: The $2 \times N$ switch

The $2 \times N$ clocked buffered switch is a discrete-time queueing system with N parallel servers and two types of arriving jobs. Jobs of type k , $k=1,2$, arrive according to a Bernoulli stream with rate r_k , $0 < r_k \leq 1$, i.e. every time unit (clock cycle) the number of arriving jobs of type k is one with probability r_k and zero with probability $1-r_k$. Upon arrival a job of type k joins the queue at server l , $l=1, \dots, N$, with probability $\hat{r}_{k,l}$, $\hat{r}_{k,l} > 0$, where $\sum_{l=1}^N \hat{r}_{k,l} = 1$ for $k=1,2$. As a result, every time unit the number of arriving jobs of type k at server l is one with probability $r_{k,l} = r_k \hat{r}_{k,l}$ and zero with probability $1-r_{k,l}$. Jobs are assumed to arrive at the beginning of a time unit and they are immediately candidates for service. Each server serves exactly one job per time unit, if one present. Since we want to have an ergodic system, it is assumed that $r_{1,l} + r_{2,l} < 1$ for all l . The $2 \times N$ switch is described by a discrete-time Markov chain with states (m_1, \dots, m_N) , where m_l denotes the number of waiting jobs at server l at the beginning of a time unit (just before the arrival instant). The 2×2 switch also satisfies condition (1.1), since for this system the only positive transition rates for the states in the interior are

$$\begin{aligned} q_{1,-1} &= r_{1,1}r_{2,1}, & q_{0,0} &= r_{1,1}r_{2,2} + r_{1,2}r_{2,1}, & q_{-1,1} &= r_{1,2}r_{2,2}, \\ q_{-1,0} &= (1-r_1)r_{2,2} + (1-r_2)r_{1,2}, & q_{0,-1} &= (1-r_1)r_{2,1} + (1-r_2)r_{1,1}, \\ q_{-1,-1} &= (1-r_1)(1-r_2). \end{aligned}$$

The 2×2 switch appears to belong to the class of two-dimensional, irreducible, positive recurrent, homogeneous, nearest-neighboring random walks which also have an extra property, called the *projection property*, and therefore its equilibrium distribution is equal to the sum of two alternating series of pure product-form distributions, see [13]. This last paper also contains a comparison between the compensation approach and the uniformization technique and the boundary value method used by Jaffe [25, 26]. Especially the resemblance between the compensation approach and the uniformization technique is very interesting. Also for $N > 2$, the $2 \times N$ switch satisfies the condition (1.2) as well as the projection property.

Example 2.3: The fork and join model

This system consists of N parallel servers, where customers arrive according to a Poisson stream with intensity λ , $\lambda > 0$. Each customer brings along N subjobs, one subjob for each server, and may leave the system if and only if all its subjobs have been served. Each server uses a FCFS service discipline and for server k the service times are assumed to be exponentially distributed with mean $1/\mu_k$, $\mu_k > \lambda$ (which implies the ergodicity of the system). This system may be described by a continuous-time Markov chain with states (m_1, \dots, m_N) , where m_k denotes the number of unfinished subjobs at server k . For the case $N=2$, the positive transition rates for the interior points are

$$q_{1,1} = \lambda, \quad q_{-1,0} = \mu_1, \quad q_{0,-1} = \mu_2,$$

which shows that the fork and join model with two queues violates condition (1.1). Hence, we may conclude that the compensation approach is not applicable. This conclusion could be expected, since the asymptotic formula derived by Flatto and Hahn (see [21], Theorem 7.1) for the equilibrium probability $p(m_1, m_2)$ as m_1 is fixed and $m_2 \rightarrow \infty$ involves a factor $m_2^{3/2}$ (if $\mu_1 < \mu_2$), which suggests that $p(m_1, m_2)$ does not consist of a linear combination of product forms $\alpha^{m_1} \beta^{m_2}$. Also for $N > 2$, condition (1.2) is violated.

We now formulate the class of random walks for the case $N=3$. Later on the general case will be treated. Consider a three-dimensional random walk, i.e. a Markov chain or a Markov process, with state space

$$M = \{ (m,n,r) \mid m,n,r \in \mathbb{N}_0 \},$$

where \mathbb{N}_0 is the set of nonnegative integers. This state space may be divided into a set of interior points and various sets of boundary points. Define

$$M_J = \{ (m_1, m_2, m_3) \in M \mid m_i = 0 \text{ for all } i \in J \text{ and } m_i > 0 \text{ for all } i \notin J \}, \quad J \subset I,$$

where $I := \{1,2,3\}$, then M_\emptyset is the interior of M ; $M_{\{1\}}$, $M_{\{2\}}$ and $M_{\{3\}}$ are the boundary planes; $M_{\{1,2\}}$, $M_{\{1,3\}}$ and $M_{\{2,3\}}$ are the axes and M_I is the origin (the subscript indicates which of the variables equals zero); see Figure 2.

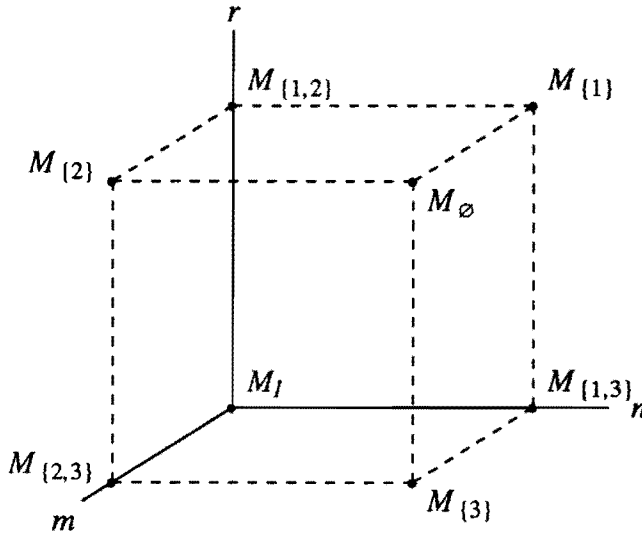


Figure 2. Eight states (m,n,r) of eight different subsets M_J of the state space M .

As stipulated in the introduction, we make some assumptions on the transition rates in order to obtain similar equilibrium equations for all states belonging to the same set M_J . In other words, we attempt to avoid 'fat' boundaries. The following assumption does this job.

Assumption 2.1

- (i) For all states only transitions to nearest neighbors are allowed;
- (ii) Strong homogeneity: all states belonging to the same subset M_J , $J \subset I$, have the same outgoing transition rates and the same incoming transition rates.

In technical terms Assumption 2.1(i) means: for all $J \subset I$ and all states $(m,n,r) \in M_J$, only transitions are allowed to the states $(m+t_1, n+t_2, r+t_3)$ with $(t_1, t_2, t_3) \in T_J$ and

$$T_J = \{ (t_1, t_2, t_3) \mid t_i \in \{0,1\} \text{ for all } i \in J \text{ and } t_i \in \{-1,0,1\} \text{ for all } i \notin J \}, \quad J \subset I.$$

Assumption 2.1(ii) consists of two parts. The first part, concerning the outgoing transition rates, means that for all $J \subset I$, the rate for a transition from any state $(m, n, r) \in M_J$ to any state $(m+t_1, n+t_2, r+t_3)$ with $(t_1, t_2, t_3) \in T_J$ is equal to the same variable, say q_{t_1, t_2, t_3}^J . The second part, concerning the incoming transition rates, means that each rate q_{t_1, t_2, t_3}^J , which corresponds to a transition from a state in M_J to a state in a higher-dimensional subset (this is the case if $t_i = 1$ for some $i \in J$), should be identical to the corresponding transition rate in that higher-dimensional subset. Hence, for all $J \subset I, J \neq \emptyset$, we have to require that

$$q_{t_1, t_2, t_3}^J = q_{t_1, t_2, t_3}^{J'} \quad \text{for all } (t_1, t_2, t_3) \in T_J,$$

where $J' = J'(J, (t_1, t_2, t_3)) = \{i \in J \mid t_i \neq 1\}$ (note that, if $t_i \neq 1$ for all $i \in J$, then $J' = J$ and the equality $q_{t_1, t_2, t_3}^J = q_{t_1, t_2, t_3}^{J'}$ reduces to an identity).

There are a number of queueing problems which can be represented by a random walk in the class under consideration. This is due to the fact that the assumptions 2.1(i) and 2.1(ii) are rather natural if the coordinates of the states (m, n, r) of a random walk represent the queue lengths of some queueing system. In that case, usually each transition represents an arrival or a departure of a customer or a job, by which only transitions to neighboring states occur (cf. 2.1(i)). Further, states in the same subset M_J have the same set of idle servers, by which one may expect that the same events or transitions occur in these states (the same outgoing transition rates, i.e. the first part of (ii)). Finally, transitions which lead to an increase in one or more coordinates usually represent arrivals. If these arrivals do not depend on the state of the system, then also the second part of (ii) (the same incoming transition rates) will be satisfied. Queueing systems satisfying (i) and (ii) are the 2×3 switch, the fork and join model and the coupled processor model. If the coordinates of the states (m, n, r) of a random walk do not represent queue lengths (but, for example, differences of queue lengths), then (i) and the first part of (ii) still may be satisfied, but the second part of (ii) should be expected to be violated. Queueing systems satisfying (i) and the first part of (ii), but violating the second part of (ii), are the symmetric shortest queue system and the symmetric longest queue system. However, the main results derived in this paper are still valid for these systems, since the second part of (ii), which mainly is introduced to avoid fat boundaries, is not essential for the main results and the analysis.

To simplify the notation, in the remainder of this paper we shall almost always write T instead of T_\emptyset and q_{t_1, t_2, t_3} instead of $q_{t_1, t_2, t_3}^\emptyset$. Further, we assume that we have discrete-time random walks (Markov chains), by which for each state the total rate of outgoing transitions adds up to 1, i.e.

$$\sum_{(t_1, t_2, t_3) \in T_J} q_{t_1, t_2, t_3}^J = 1, \quad J \subset I.$$

In case of a continuous-time random walk this equation can be satisfied after rescaling time.

The objective of the analysis in the next few sections is to find out for which part of the class of three-dimensional, irreducible, positive recurrent, strongly homogeneous, nearest-neighboring random walks we are able to determine the equilibrium distribution $\{p_{m, n, r}\}$ with the help of the compensation approach. This equilibrium distribution is characterized as the unique normalized solution of the equilibrium equations. For the time being, we only need

the equilibrium equations for the interior and the three boundary planes:

$$p_{m,n,r} = \sum_{(t_1,t_2,t_3) \in T} q_{t_1,t_2,t_3} p_{m-t_1,n-t_2,r-t_3} , \quad (m,n,r) \in M_{\emptyset}, \quad (2.1)$$

$$p_{0,n,r} = \sum_{(-1,t_2,t_3) \in T} q_{-1,t_2,t_3} p_{1,n-t_2,r-t_3} + \sum_{(0,t_2,t_3) \in T} q_{0,t_2,t_3}^{[1]} p_{0,n-t_2,r-t_3} , \quad (0,n,r) \in M_{\{1\}}, \quad (2.2)$$

$$p_{m,0,r} = \sum_{(t_1,-1,t_3) \in T} q_{t_1,-1,t_3} p_{m-t_1,1,r-t_3} + \sum_{(t_1,0,t_3) \in T} q_{t_1,0,t_3}^{[2]} p_{m-t_1,0,r-t_3} , \quad (m,0,r) \in M_{\{2\}}, \quad (2.3)$$

$$p_{m,n,0} = \sum_{(t_1,t_2,-1) \in T} q_{t_1,t_2,-1} p_{m-t_1,n-t_2,1} + \sum_{(t_1,t_2,0) \in T} q_{t_1,t_2,0}^{[3]} p_{m-t_1,n-t_2,0} , \quad (m,n,0) \in M_{\{3\}}. \quad (2.4)$$

3. The compensation approach.

When extending the compensation approach to 3 dimensions, one builds a linear combination of (possibly complex) product forms $\alpha^m \beta^n \gamma^r$ which are solutions of the equilibrium equation (2.1) for the interior of the state space. Substituting $\alpha^m \beta^n \gamma^r$ in (2.1) and dividing both sides of the equation by $\alpha^{m-1} \beta^{n-1} \gamma^{r-1}$ leads to the following characterization:

Lemma 3.1

The product form $\alpha^m \beta^n \gamma^r$ is a solution of equation (2.1) if and only if (α, β, γ) satisfies

$$\alpha \beta \gamma = \sum_{(t_1,t_2,t_3) \in T} q_{t_1,t_2,t_3} \alpha^{1-t_1} \beta^{1-t_2} \gamma^{1-t_3} . \quad (3.1)$$

Any linear combination $\sum_i c_i \alpha_i^m \beta_i^n \gamma_i^r$ of solutions $(\alpha_i, \beta_i, \gamma_i)$ of (3.1) also satisfies (2.1). By using compensation arguments, we try to build linear combinations which also satisfy the equilibrium equations (2.2)-(2.4) for the boundary planes. Product forms with one or more factors equal to zero lead to special, non-relevant cases and, since later on the final solution has to be normalized, also product forms with one of the factors larger than or equal to 1 in modulus are not relevant. Hence, we are only interested in solutions $(\alpha_i, \beta_i, \gamma_i)$ that are elements of

$$P = \{ (\alpha, \beta, \gamma) \in \mathcal{C}^3 \mid (\alpha, \beta, \gamma) \text{ satisfies (3.1), } \alpha, \beta, \gamma \neq 0 \text{ and } |\alpha|, |\beta|, |\gamma| < 1 \} .$$

In this section we shall describe the construction of the linear combinations mentioned above, which in fact are trees of product forms. These linear combinations will be called *formal solutions*, since we do not pay attention to the convergence of these trees. The

convergence and the question how and which formal solutions should be combined to get a solution of *all* equilibrium equations, are discussed in the next sections.

The main idea of the compensation approach is that an error of a solution $\alpha^m \beta^n \gamma^r$ of (2.1) on one of the boundary planes (i.e. a violation of one of the equilibrium equations (2.2)-(2.4) for the boundary planes by $\alpha^m \beta^n \gamma^r$) may be compensated by adding another product form. The addition of such a product form is called a *compensation step*. Starting with a solution of (2.1) and successively performing compensation steps to compensate errors of previous product forms should lead to a solution of (2.1)-(2.4).

Let us start with describing a compensation step on the boundary plane $m=0$ (i.e. on $M_{\{1\}}$). Let $(\alpha, \beta, \gamma) \in P$, i.e. $\alpha^m \beta^n \gamma^r$ is a solution of (2.1), and suppose that this product form violates the equilibrium equation (2.2) for the boundary plane $m=0$. To correct the error of $\alpha^m \beta^n \gamma^r$ on $m=0$, we try to add a compensation term $\hat{\alpha}^m \hat{\beta}^n \hat{\gamma}^r$ such that $\alpha^m \beta^n \gamma^r + \hat{\alpha}^m \hat{\beta}^n \hat{\gamma}^r$ is a solution of (2.1) and (2.2). Substitution of this linear combination in (2.2) leads to the equation

$$K(\alpha, \beta, \gamma) \beta^{n-1} \gamma^{r-1} + \hat{\alpha} K(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) \hat{\beta}^{n-1} \hat{\gamma}^{r-1} = 0 \text{ for all } n \geq 1, r \geq 1, \quad (3.2)$$

where the function $K(\alpha, \beta, \gamma)$ is defined by

$$K(\alpha, \beta, \gamma) = \alpha \sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3} + \sum_{(0, t_2, t_3) \in T} q_{0, t_2, t_3}^{(1)} \beta^{1-t_2} \gamma^{1-t_3} - \beta \gamma.$$

Here, $K(\alpha, \beta, \gamma) = 0$ if and only if $\alpha^m \beta^n \gamma^r$ is a solution of (2.2). Because $\alpha^m \beta^n \gamma^r$ has been supposed to violate (2.2), we have to choose $\hat{\beta} = \beta$ and $\hat{\gamma} = \gamma$ to satisfy (3.2).

Now, requiring that $\alpha^m \beta^n \gamma^r + \hat{\alpha}^m \hat{\beta}^n \hat{\gamma}^r$ is a solution of (2.1) leads to the condition that $(\hat{\alpha}, \beta, \gamma)$ also has to be a solution of equation (3.1), which is a quadratic equation in α for fixed β and γ :

$$\left[\sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3} \right] \alpha^2 - \left[\beta \gamma - \sum_{(0, t_2, t_3) \in T} q_{0, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3} \right] \alpha + \left[\sum_{(1, t_2, t_3) \in T} q_{1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3} \right] = 0. \quad (3.3)$$

Choosing $\hat{\alpha} = \alpha$ would not lead to compensation and therefore we have to take $\hat{\alpha}$ equal to the companion solution to α of (3.3), i.e.

$$\hat{\alpha} = \frac{f_1(\beta, \gamma)}{\alpha},$$

where $f_1(\beta, \gamma)$ is the product of the roots of the quadratic equation (3.3):

$$f_1(\beta, \gamma) = \frac{\sum_{(1, t_2, t_3) \in T} q_{1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3}}{\sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3}}. \quad (3.4)$$

Finally, again substituting $\alpha^m \beta^n \gamma^r + \hat{\alpha}^m \hat{\beta}^n \hat{\gamma}^r$ in (2.2) leads to the determination of $\hat{\alpha}$ (see (3.2)):

$$\hat{a} = -\frac{K(\alpha, \beta, \gamma)}{K(\hat{\alpha}, \beta, \gamma)}.$$

So, we get $\hat{a}=0$ if and only if $\alpha^m \beta^n \gamma^r$ already satisfied (2.2). By using the formula for the sum of the roots of the quadratic equation (3.3), the expression for \hat{a} may be rewritten to

$$\hat{a} = -\frac{g_1(\beta, \gamma) - \hat{\alpha}}{g_1(\beta, \gamma) - \alpha}$$

with

$$g_1(\beta, \gamma) = \frac{\sum_{(0, t_2, t_3) \in T} (q_{0, t_2, t_3}^{(1)} - q_{0, t_2, t_3}) \beta^{1-t_2} \gamma^{1-t_3}}{\sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3}}. \quad (3.5)$$

The compensation step on the boundary plane $m=0$ fails if the denominator of $f_1(\beta, \gamma)$ vanishes (in that case (3.3) has only one solution). Another special case occurs if the numerator of $f_1(\beta, \gamma)$ is equal to zero, since then $\hat{\alpha}=0$. To avoid these special cases, we make Assumption 3.1(i). Naturally, we need the analogous assumptions for the compensation steps on the boundary planes $n=0$ and $r=0$ (see (ii) and (iii)).

Assumption 3.1

- (i) $\sum_{(1, t_2, t_3) \in T} q_{1, t_2, t_3} > 0$ and $\sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} > 0$
(there is a forward and a backward rate component);
- (ii) $\sum_{(t_1, 1, t_3) \in T} q_{t_1, 1, t_3} > 0$ and $\sum_{(t_1, -1, t_3) \in T} q_{t_1, -1, t_3} > 0$
(there is a rate component to the east and to the west);
- (iii) $\sum_{(t_1, t_2, 1) \in T} q_{t_1, t_2, 1} > 0$ and $\sum_{(t_1, t_2, -1) \in T} q_{t_1, t_2, -1} > 0$
(there is a rate component to the north and to the south).

Of course, under this assumption, it may still incidentally happen that the numerator or denominator of $f_1(\beta, \gamma)$ is equal to zero. Further, it may still happen that the denominator of \hat{a} vanishes (recall that $K(\hat{\alpha}, \beta, \gamma)=0$ if and only if $\hat{\alpha}^m \beta^n \gamma^r$ is a solution of (2.2)). However, these three cases are not likely to occur. Therefore, for the time being we shall neglect them, but at the end of Section 4 we shall discuss what to do if these special cases do occur (see Remark 4.1).

For a compensation step on the boundary planes $n=0$ and $r=0$, similar results can be derived as for the compensation step on the boundary plane $m=0$. These results are summarized in the following lemma.

Lemma 3.2

- (i) Let $\alpha^m \beta^n \gamma^r$ with $\alpha, \beta, \gamma \neq 0$ satisfy (2.1) and let $a \in \mathbb{C}$. Then $a\alpha^m \beta^n \gamma^r + \hat{a}\hat{\alpha}^m \beta^n \gamma^r$ satisfies (2.1) and (2.2), if and only if $\hat{\alpha}$ and \hat{a} are taken equal to

$$\hat{\alpha} = \frac{f_1(\beta, \gamma)}{\alpha} \quad \text{and} \quad \hat{a} = - \frac{g_1(\beta, \gamma) - \hat{\alpha}}{g_1(\beta, \gamma) - \alpha} a ,$$

where $f_1(\beta, \gamma)$ and $g_1(\beta, \gamma)$ are defined by (3.4) and (3.5);

(ii) Let $\alpha^m \beta^n \gamma^r$ with $\alpha, \beta, \gamma \neq 0$ satisfy (2.1) and let $b \in \mathbb{C}$. Then $b\alpha^m \beta^n \gamma^r + \hat{b}\alpha^m \hat{\beta}^n \hat{\gamma}^r$ satisfies (2.1) and (2.3), if and only if $\hat{\beta}$ and \hat{b} are taken equal to

$$\hat{\beta} = \frac{f_2(\alpha, \gamma)}{\beta} \quad \text{and} \quad \hat{b} = - \frac{g_2(\alpha, \gamma) - \hat{\beta}}{g_2(\alpha, \gamma) - \beta} b ,$$

where $f_2(\alpha, \gamma)$ and $g_2(\alpha, \gamma)$ are defined in the same way as $f_1(\beta, \gamma)$ and $g_1(\beta, \gamma)$, but with the combinations t, t_2, t_3 replaced by t_1, t, t_3 for all $t \in \{-1, 0, 1\}$, the powers β^{1-t_2} by α^{1-t_1} and the rates $q_{0, t_2, t_3}^{[1]}$ by $q_{t_1, 0, t_3}^{[2]}$;

(iii) Let $\alpha^m \beta^n \gamma^r$ with $\alpha, \beta, \gamma \neq 0$ satisfy (2.1) and let $c \in \mathbb{C}$. Then $c\alpha^m \beta^n \gamma^r + \hat{c}\alpha^m \hat{\beta}^n \hat{\gamma}^r$ satisfies (2.1) and (2.4), if and only if $\hat{\gamma}$ and \hat{c} are taken equal to

$$\hat{\gamma} = \frac{f_3(\alpha, \beta)}{\gamma} \quad \text{and} \quad \hat{c} = - \frac{g_3(\alpha, \beta) - \hat{\gamma}}{g_3(\alpha, \beta) - \gamma} c ,$$

where $f_3(\alpha, \beta)$ and $g_3(\alpha, \beta)$ are defined in the same way as $f_1(\beta, \gamma)$ and $g_1(\beta, \gamma)$, but with the combinations t, t_2, t_3 replaced by t_1, t_2, t for all $t \in \{-1, 0, 1\}$, the powers γ^{1-t_3} by α^{1-t_1} and the rates $q_{0, t_2, t_3}^{[1]}$ by $q_{t_1, t_2, 0}^{[3]}$.

The Lemmas 3.1 and 3.2 provide the tools for the compensation approach to construct a solution of (2.1)-(2.4). Let $(\alpha, \beta, \gamma) \in P$, i.e. $\alpha^m \beta^n \gamma^r$ is a solution of (2.1). Most likely, this solution, which we call the *starting solution*, is not a solution of the equilibrium equations (2.2)-(2.4) for the boundary planes. Therefore *compensation terms* have to be added to correct the errors of $\alpha^m \beta^n \gamma^r$ on these boundary planes. To correct the error on the boundary plane $m=0$ for example, we have to add a product form $a_{(1)} \alpha_{(1)}^m \beta_{(1)}^n \gamma_{(1)}^r$ with $\beta_{(1)} = \beta$, $\gamma_{(1)} = \gamma$ and $\alpha_{(1)}$ and $a_{(1)}$ defined according to Lemma 3.2(i). Unfortunately, this compensation term introduces two new errors on the other two boundary planes. To compensate these new errors, which are hoped to be smaller than the initial error of $\alpha^m \beta^n \gamma^r$ on $m=0$, two more compensation terms have to be added. To compensate the new error of the term $a_{(1)} \alpha_{(1)}^m \beta_{(1)}^n \gamma_{(1)}^r$ on $n=0$, we add a product form $a_{(1,2)} b_{(1,2)} \alpha_{(1,2)}^m \beta_{(1,2)}^n \gamma_{(1,2)}^r$ with $\alpha_{(1,2)} = \alpha_{(1)}$, $\gamma_{(1,2)} = \gamma_{(1)}$, $a_{(1,2)} = a_{(1)}$ and $\beta_{(1,2)}$ and $b_{(1,2)}$ defined according to Lemma 3.2(ii). To compensate the new error on $r=0$ a product form $a_{(1,3)} c_{(1,3)} \alpha_{(1,3)}^m \beta_{(1,3)}^n \gamma_{(1,3)}^r$ is added.

Continuing the above procedure leads to the generation of a tree or a network of product forms; see Figure 3. The product forms are labeled as follows. For each vector v out of the set

$$V = \{ (v_1, \dots, v_l) \mid l \in \mathbb{N}_0, v_1 \in I \text{ and } v_k \in I \setminus \{v_{k-1}\} \text{ for all } k \geq 2 \} ,$$

we get a product form $a_v b_v c_v \alpha_v^m \beta_v^n \gamma_v^r$. The empty vector \emptyset , which we get for $l=0$, is used as subscript for the starting solution. For all other elements $v = (v_1, \dots, v_l) \in V \setminus \{\emptyset\}$ the product form $a_v b_v c_v \alpha_v^m \beta_v^n \gamma_v^r$ is the compensation term which compensates an error of $a_{p(v)} b_{p(v)} c_{p(v)} \alpha_{p(v)}^m \beta_{p(v)}^n \gamma_{p(v)}^r$, where $p(v) = (v_1, \dots, v_{l-1})$ is the *parent* of v . The last component of v denotes on which boundary an error of $a_{p(v)} b_{p(v)} c_{p(v)} \alpha_{p(v)}^m \beta_{p(v)}^n \gamma_{p(v)}^r$ is compensated: on $m=0$ if $v_l=1$, on $n=0$ if $v_l=2$ and on $r=0$ if $v_l=3$. In Figure 3, the factors α, β

and γ denote which new factor one gets for each compensation step. When compensating on $m=0$ we get a compensation term with a new α -factor, on $n=0$ we get a new β -factor and on $r=0$ a new γ -factor.

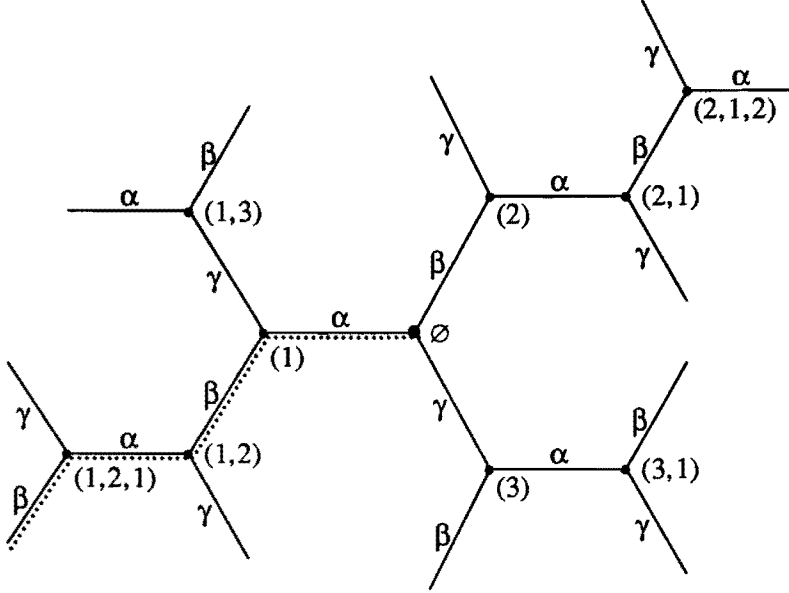


Figure 3. The construction process of a formal solution. Node v represents the product form $a_v b_v c_v \alpha_v^m \beta_v^n \gamma_v^r$.

The sum of the starting solution $(\alpha, \beta, \gamma) \in P$ and all compensation terms is denoted by $x_{m,n,r}(\alpha, \beta, \gamma)$. So,

$$x_{m,n,r}(\alpha, \beta, \gamma) = \sum_{v \in V} a_v b_v c_v \alpha_v^m \beta_v^n \gamma_v^r, \quad (3.6)$$

where $\alpha_\emptyset = \alpha$, $\beta_\emptyset = \beta$, $\gamma_\emptyset = \gamma$ and for all $v \in V \setminus \{\emptyset\}$ we have (see Lemma 3.2 and the previous paragraph):

$$\begin{aligned} \beta_v &= \beta_{p(v)}, \quad \gamma_v = \gamma_{p(v)}, \quad b_v = b_{p(v)}, \quad c_v = c_{p(v)}, \\ \alpha_v &= \frac{f_1(\beta_{p(v)}, \gamma_{p(v)})}{\alpha_{p(v)}}, \quad a_v = -\frac{g_1(\beta_{p(v)}, \gamma_{p(v)}) - \alpha_v}{g_1(\beta_{p(v)}, \gamma_{p(v)}) - \alpha_{p(v)}} a_{p(v)} \quad \text{if } v_{l(v)} = 1; \\ \alpha_v &= \alpha_{p(v)}, \quad \gamma_v = \gamma_{p(v)}, \quad a_v = a_{p(v)}, \quad c_v = c_{p(v)}, \\ \beta_v &= \frac{f_2(\alpha_{p(v)}, \gamma_{p(v)})}{\beta_{p(v)}}, \quad b_v = -\frac{g_2(\alpha_{p(v)}, \gamma_{p(v)}) - \beta_v}{g_2(\alpha_{p(v)}, \gamma_{p(v)}) - \beta_{p(v)}} b_{p(v)} \quad \text{if } v_{l(v)} = 2; \\ \alpha_v &= \alpha_{p(v)}, \quad \beta_v = \beta_{p(v)}, \quad a_v = a_{p(v)}, \quad b_v = b_{p(v)}, \\ \gamma_v &= \frac{f_3(\alpha_{p(v)}, \beta_{p(v)})}{\gamma_{p(v)}}, \quad c_v = -\frac{g_3(\alpha_{p(v)}, \beta_{p(v)}) - \gamma_v}{g_3(\alpha_{p(v)}, \beta_{p(v)}) - \gamma_{p(v)}} c_{p(v)} \quad \text{if } v_{l(v)} = 3. \end{aligned}$$

Here, $l(v)$ is defined as the *length* (i.e. the number of components) of a vector $v \in V$ and $v_{l(v)}$

is the last component of v . For the initial factors a_\emptyset , b_\emptyset and c_\emptyset any choice is allowed. To get a starting solution $a_\emptyset b_\emptyset c_\emptyset \alpha_\emptyset^m \beta_\emptyset^n \gamma_\emptyset^r$ which is a product-form *distribution*, we choose $a_\emptyset = 1-\alpha$, $b_\emptyset = 1-\beta$ and $c_\emptyset = 1-\gamma$ (so, the sum of that solution over all m , n and r is equal to 1).

For each $(\alpha, \beta, \gamma) \in P$ the solution $\{x_{m,n,r}(\alpha, \beta, \gamma)\}$ is well-defined by (3.6) if the sum in (3.6) converges absolutely, i.e. if

$$\sum_{v \in V} |a_v b_v c_v \alpha_v^m \beta_v^n \gamma_v^r| < \infty. \quad (3.7)$$

In principle, this should hold for all states $(m, n, r) \in M$. Since we do not know for which starting solutions (α, β, γ) , and for which states (m, n, r) , condition (3.7) holds, we call each solution $\{x_{m,n,r}(\alpha, \beta, \gamma)\}$ a *formal solution*. If (3.7) holds, then $\{x_{m,n,r}(\alpha, \beta, \gamma)\}$ will be a solution of the equilibrium equations (2.1)-(2.4) for the interior and the boundary planes. Since each term of the sum in (3.6) is a solution of (2.1), it is obvious that the whole sum also satisfies (2.1). By taking connected pairs of product forms in the network pictured in Figure 3, it is shown that $\{x_{m,n,r}(\alpha, \beta, \gamma)\}$ is also a solution of the equations for the boundary planes. If (3.7) holds, then $x_{m,n,r}(\alpha, \beta, \gamma)$ may be rewritten as a sum of pairs of product forms with the same β - and γ -factor (i.e. pairs of product forms which are connected by an α -edge in Figure 3),

$$x_{m,n,r}(\alpha, \beta, \gamma) = \sum_{\substack{v \in V \setminus \{\emptyset\} \\ v_l(v)=1}} (a_{p(v)} \alpha_{p(v)}^m + a_v \alpha_v^m) b_{p(v)} c_{p(v)} \beta_{p(v)}^n \gamma_{p(v)}^r, \quad (3.8)$$

from which it immediately follows that $\{x_{m,n,r}(\alpha, \beta, \gamma)\}$ is a solution of the equilibrium equation (2.2) for the boundary plane $m=0$, since each pair of product forms in the above sum is a solution of (2.2). Taking pairs with the same α -factor and γ -factor and pairs with the same α -factor and β -factor shows that $\{x_{m,n,r}(\alpha, \beta, \gamma)\}$ satisfies the equilibrium equations (2.3) and (2.4) for the boundary planes $n=0$ and $r=0$. In the next section we shall investigate whether condition (3.7) is satisfied. We shall also investigate if for a formal solution it holds that

$$|\alpha_v| < 1, \quad |\beta_v| < 1, \quad |\gamma_v| < 1 \quad \text{for all } v \in V, \quad (3.9)$$

since solutions of all equilibrium equations have to be normalized to produce the equilibrium distribution. Because each starting solution is required to be an element of P , (3.9) is satisfied for $v = \emptyset$ by definition; for all other $v \in V$ this has to be verified yet.

4. Two necessary conditions

In this section we show that the requirements (3.7) and (3.9) lead to necessary conditions for the transition rates q_{t_1, t_2, t_3} in the interior of the state space and for the starting solution (α, β, γ) . As we shall see, the resulting condition for the rates q_{t_1, t_2, t_3} is rather severe, but it is satisfied by the 2×3 switch (the other examples of the queueing systems mentioned in Section 2 violate this condition). The resulting condition for the starting solution (α, β, γ) will lead to a small renovation of the definition of the formal solutions, but fortunately does not further restrict the applicability of the compensation approach.

Under certain circumstances a formal solution $\{x_{m,n,r}(\alpha, \beta, \gamma)\}$ reduces to a *finite* sum. This may happen for example if one has a Jackson network by which it is possible to choose a starting solution that is also a solution of the equations for the boundary planes. In that case $a_{(1)} = b_{(2)} = c_{(3)} = 0$ and, due to the recursions that we have for the coefficients a_v , b_v and c_v , $a_v b_v c_v = 0$ for all $v \in V \setminus \{\emptyset\}$. In general, however, a formal solution $\{x_{m,n,r}(\alpha, \beta, \gamma)\}$ will have an infinite number of nonnull terms and in that case it is required that $|a_v b_v c_v \alpha_v^m \beta_v^n \gamma_v^r|$ gets sufficiently small for vectors $v \in V$ with large lengths $l(v)$.

Define a *path* as a sequence $\{v^{(k)}\}$ of vectors of V with $v^{(0)} = \emptyset$ and $v^{(k)} \in O(v^{(k-1)})$ for all $k \geq 1$, where $O(v)$ is the *offspring* of a vector v :

$$O(v) = \{v' \in V \mid p(v') = v\}, \quad v \in V.$$

In Figure 3, the dotted line denotes an example of a path. Requiring (3.7) implies that

$$\sum_{k=0}^{\infty} |a_{v^{(k)}} b_{v^{(k)}} c_{v^{(k)}} \alpha_{v^{(k)}}^m \beta_{v^{(k)}}^n \gamma_{v^{(k)}}^r| < \infty \quad (4.1)$$

for all paths $\{v^{(k)}\}$. It is interesting to verify this condition for the two paths for which we have product forms with alternately new α - and β -factors (i.e. the paths with $v_{(v)}^{(k)} \in \{1, 2\}$ for all $k \geq 1$, see Figure 3). In general, for at least one of these two paths all terms in the sum of (4.1) will be nonnull. Without loss of generality we may assume that this is at least the case for the path with $v^{(0)} = \emptyset$ and $v^{(k)} = (1, 2, \dots, 1)$ if $k \geq 1$ and k odd, $v^{(k)} = (1, 2, \dots, 1, 2)$ if $k \geq 1$ and k even (which is just the path denoted by the dotted line in Figure 3). For this path condition (4.1) reduces to

$$\sum_{k=0}^{\infty} |a_{v^{(k)}} b_{v^{(k)}} \alpha_{v^{(k)}}^m \beta_{v^{(k)}}^n| < \infty,$$

since $\gamma_{v^{(k)}} = \gamma$ and $c_{v^{(k)}} = 1 - \gamma$ for all k . Because we want this condition to hold for all m and n and the coefficients $a_{v^{(k)}}$ and $b_{v^{(k)}}$ may not be expected to go to zero as k goes to infinity, it seems reasonable to require that

$$\alpha_{v^{(k)}} \rightarrow 0 \text{ and } \beta_{v^{(k)}} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (4.2)$$

For odd k the factors $\alpha_{v^{(k-1)}}$ and $\alpha_{v^{(k)}}$ are the roots of the quadratic equation (3.1) (see also (3.3)) for fixed $\beta = \beta_{v^{(k-1)}}$ and γ and therefore

$$\alpha_{v^{(k-1)}} \alpha_{v^{(k)}} = f_1(\beta_{v^{(k-1)}}, \gamma), \quad \alpha_{v^{(k-1)}} + \alpha_{v^{(k)}} = \frac{\beta_{v^{(k-1)}} \gamma - \sum_{(0, t_2, t_3) \in T} q_{0, t_2, t_3} \beta_{v^{(k-1)}}^{1-t_2} \gamma^{1-t_3}}{\sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} \beta_{v^{(k-1)}}^{1-t_2} \gamma^{1-t_3}}.$$

According to (4.2), for both equations the parts on the left-hand side go to zero as $k \rightarrow \infty$, whereas the parts on the right-hand side go to

$$\frac{\sum_{(1, 1, t_3) \in T} q_{1, 1, t_3} \gamma^{1-t_3}}{\sum_{(-1, 1, t_3) \in T} q_{-1, 1, t_3} \gamma^{1-t_3}} \quad \text{and} \quad - \frac{\sum_{(0, 1, t_3) \in T} q_{0, 1, t_3} \gamma^{1-t_3}}{\sum_{(-1, 1, t_3) \in T} q_{-1, 1, t_3} \gamma^{1-t_3}}$$

respectively. Of course both sides of an equation have to go to the same limit and therefore (4.2) results in the condition that $q_{0, 1, t_3} = q_{1, 1, t_3} = 0$ for all t_3 . In the same way, considering

the sum and the product for $\beta_{v^{(k-1)}}$ and $\beta_{v^{(k)}}$ for even k , leads to the condition that $q_{1,0,t_3} = q_{1,1,t_3} = 0$ for all t_3 . So, summarizing, the requirement in (4.1) for paths $\{v^{(k)}\}$ with $v_{(v)}^{(k)} \in \{1,2\}$ for all k leads to

$$q_{0,1,t_3} = q_{1,0,t_3} = q_{1,1,t_3} = 0 \text{ for all } t_3 \in \{-1,0,1\}.$$

Similar conditions are derived by considering paths with $v_{(v)}^{(k)} \in \{1,3\}$ or $v_{(v)}^{(k)} \in \{2,3\}$ for all values of k . Combining these conditions results in:

Condition 4.1 (*necessary condition arising from (3.7)*)

For all $(t_1, t_2, t_3) \in T$,

$$q_{t_1, t_2, t_3} = 0 \text{ if } t_i + t_j > 0 \text{ for some } i, j \in I, i \neq j. \quad (4.3)$$

Unfortunately, this condition, which is an extension of (1.1), is rather severe: for all states in the interior, transitions may only have positive rates, if a positive step in one coordinate is always accompanied by negative steps in the other two coordinates, i.e. $q_{t_1, t_2, t_3} > 0$ and $t_i = 1$ for some $i \in I$ implies $t_j = -1$ for all $j \in I \setminus \{i\}$. In case the coordinates m, n and r represent queue lengths, Condition 4.1 implies that for all states (m, n, r) in the interior only transitions are possible to themselves or to states with a lower total number of jobs. Although Condition 4.1 only has been derived for the class of strongly homogeneous random walks described in Section 2, the condition is also necessary for random walks which are only homogeneous, i.e. for random walks for which all states belonging to the same subset M_j have the same outgoing transition rates but not necessarily also the same incoming transition rates. For these random walks the condition may be derived along the same lines. The 2×3 switch satisfies the condition, but the other queueing problems mentioned in the last paragraph of Section 2 violate the condition; below we show this for the three problems described as Example 2.1-2.3 in Section 1. As a consequence, we can conclude that the compensation approach possibly works for the 2×3 switch, whereas this method certainly does not work for the other problems. For the 2×3 switch and the three-dimensional symmetric shortest queue system the same was already concluded in [32] from numerical results.

Example 2.1: The symmetric shortest queue system (*continued*)

For the three-dimensional shortest queue problem the only positive transition rates q_{t_1, t_2, t_3} for the interior are:

$$q_{1,-1,0} = 3\rho, \quad q_{-1,1,0} = q_{0,-1,1} = q_{0,0,-1} = 1.$$

The first rate is due to an arrival of a job and the other three rates come from departures of jobs. Condition 4.1 is obviously violated in this case and therefore we conclude that, contrary to the two-dimensional case, the compensation approach will not work for the three-dimensional shortest queue problem.

Example 2.2: The $2 \times N$ switch (*continued*)

For the 2×3 switch the only positive transition rates q_{t_1, t_2, t_3} for the interior are:

$$\begin{aligned}
 q_{1,-1,-1} &= r_{1,1}r_{2,1}, & q_{-1,1,-1} &= r_{1,2}r_{2,2}, & q_{-1,-1,1} &= r_{1,3}r_{2,3}, \\
 q_{-1,0,0} &= r_{1,2}r_{2,3} + r_{1,3}r_{2,2}, & q_{0,-1,0} &= r_{1,1}r_{2,3} + r_{1,3}r_{2,1}, \\
 q_{0,0,-1} &= r_{1,1}r_{2,2} + r_{1,2}r_{2,1}, & q_{-1,-1,0} &= (1-r_1)r_{2,3} + (1-r_2)r_{1,3}, \\
 q_{-1,0,-1} &= (1-r_1)r_{2,2} + (1-r_2)r_{1,2}, & q_{0,-1,-1} &= (1-r_1)r_{2,1} + (1-r_2)r_{1,1}, \\
 q_{-1,-1,-1} &= (1-r_1)(1-r_2).
 \end{aligned}$$

As we see, Condition 4.1 is satisfied for this system. This perfectly corresponds with the intuitive interpretation described above. If for the 2×3 switch at the beginning of a time unit all servers have jobs available (i.e. we are in a state of the interior), then at the next discrete time event three jobs will leave the system while at most two jobs arrive.

Example 2.3: The fork and join model (continued)

For the three-dimensional version of the fork and join model we have

$$q_{1,1,1} = \lambda, \quad q_{-1,0,0} = \mu_1, \quad q_{0,-1,0} = \mu_2, \quad q_{0,0,-1} = \mu_3,$$

by which it is concluded that the compensation approach is also unsuitable for this case (as we saw in Section 1, the same holds for the two-dimensional fork and join model).

Condition 4.1, which in the remainder of the analysis is assumed to be satisfied, together with Assumption 3.1 implies that

$$q_{1,-1,-1} > 0, \quad q_{-1,1,-1} > 0, \quad q_{-1,-1,1} > 0.$$

A more important consequence is that assuming (4.3) leads to a considerable simplification of the quadratic equation (3.1). Due to this simplification we can prove some useful properties for the factors α_v , β_v and γ_v with the help of *Rouché's Theorem* (cf. Titchmarsh [31]). Together with the condition stated in (3.9), this will lead to the second necessary condition. To simplify the expressions in the following lemma we introduce the variables

$$q_t^{(i)} = \sum_{\substack{(s_1, s_2, s_3) \in T \\ s_i = t}} q_{s_1, s_2, s_3}, \quad t \in \{-1, 0, 1\}, \quad i \in I. \tag{4.4}$$

The rate $q_t^{(1)}$ denotes the sum of the rates of all transitions causing a step t in the m -direction; and a similar interpretation holds for the rates $q_t^{(2)}$ and $q_t^{(3)}$. Note that, due to Condition 4.1,

$$q_1^{(1)} = q_{1,-1,-1}, \quad q_1^{(2)} = q_{-1,1,-1}, \quad q_1^{(3)} = q_{-1,-1,1}.$$

Lemma 4.1

(i) For fixed β and γ , $0 < |\beta| < 1$ and $0 < |\gamma| < 1$, the quadratic equation (3.1) has exactly one root α with $0 < |\alpha| < C_1 |\beta\gamma|$ and $C_1 = \min[1, q_1^{(1)}/q_-^{(1)}]$. The second root α , which only exists if

$$\sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3} \neq 0,$$

satisfies $|\alpha| > \hat{C}_1 |\beta\gamma|$ with $\hat{C}_1 = \max[1, q_1^{(1)}/q_-^{(1)}]$.

(ii) For fixed α and γ , $0 < |\alpha| < 1$ and $0 < |\gamma| < 1$, the quadratic equation (3.1) has exactly one root β with $0 < |\beta| < C_2 |\alpha\gamma|$ and $C_2 = \min[1, q_1^{(2)}/q_{-1}^{(2)}]$. The second root β , which only exists if

$$\sum_{(t_1, -1, t_3) \in T} q_{t_1, -1, t_3} \alpha^{1-t_1} \gamma^{1-t_3} \neq 0,$$

satisfies $|\beta| > \hat{C}_2 |\alpha\gamma|$ with $\hat{C}_2 = \max[1, q_1^{(2)}/q_{-1}^{(2)}]$.

(iii) For fixed α and β , $0 < |\alpha| < 1$ and $0 < |\beta| < 1$, the quadratic equation (3.1) has exactly one root γ with $0 < |\gamma| < C_3 |\alpha\beta|$ and $C_3 = \min[1, q_1^{(3)}/q_{-1}^{(3)}]$. The second root γ , which only exists if

$$\sum_{(t_1, t_2, -1) \in T} q_{t_1, t_2, -1} \alpha^{1-t_1} \beta^{1-t_2} \neq 0,$$

satisfies $|\gamma| > \hat{C}_3 |\alpha\beta|$ with $\hat{C}_3 = \max[1, q_1^{(3)}/q_{-1}^{(3)}]$.

Proof.

We shall only prove part (i); the parts (ii) and (iii) may be proved along the same lines.

Consider the quadratic equation (3.1) for fixed β and γ , $0 < |\beta| < 1$ and $0 < |\gamma| < 1$. After rewriting (3.1) to (3.3), using (4.3), dividing by $\beta^2 \gamma^2$ and substituting $z = \alpha/(\beta\gamma)$, we get the quadratic equation

$$\left[\sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3} \right] z^2 - (1 - [q_{0,0,0} + q_{0,-1,0}\beta + q_{0,0,-1}\gamma + q_{0,-1,-1}\beta\gamma])z + q_{1,-1,-1} = 0. \quad (4.5)$$

Let $f(z)$ be the first term of the quadratic function in (4.5) and let $g(z)$ be the remaining part. Then we have the following bounds for $f(z)$ and $g(z)$ (for $z \neq 0$):

$$\begin{aligned} |f(z)| &\leq \left[\sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} |\beta^{1-t_2} \gamma^{1-t_3}| \right] |z|^2 < q_{-1}^{(1)} |z|^2, \\ |g(z)| &\geq |z| - [q_{0,0,0} + q_{0,-1,0}\beta + q_{0,0,-1}\gamma + q_{0,-1,-1}\beta\gamma] |z| - q_{1,-1,-1} \\ &\geq (1 - [q_{0,0,0} + q_{0,-1,0} + q_{0,0,-1} + q_{0,-1,-1}]) |z| - q_{1,-1,-1} \\ &= (1 - q_0^{(1)}) |z| - q_1^{(1)}. \end{aligned}$$

As we see, these bounds only depend on the absolute value of z . For all z on the circle $C = \{z \mid |z| = r\}$ with radius $r > 0$, we have $|f(z)| - |g(z)| < h(r)$, where

$$h(r) = q_{-1}^{(1)} r^2 - (1 - q_0^{(1)}) r + q_1^{(1)}.$$

Obviously, $|f(z)| < |g(z)|$ for all z on C , if r is chosen such that $h(r) \leq 0$. Since $h(r)$ is a convex quadratic function for which $h(0) = q_{1,-1,-1} > 0$ and $h(1) = 0$, the function $h(r)$ has two positive zeros, namely 1 and $\hat{r} = q_1^{(1)}/q_{-1}^{(1)}$ (use the rule for the product of the two roots of a quadratic equation), and $h(r) \leq 0$ for all r in the closed interval between these two zeros. So, Rouché's theorem may be applied for all $r \in [C_1, \hat{C}_1]$, where $C_1 = \min[1, \hat{r}]$ and $\hat{C}_1 = \max[1, \hat{r}]$. This theorem tells that the number of solutions of (4.5) in the region $|z| < r$ is equal to the number of zeros of $g(z)$ in this region. The linear function $g(z)$ has one zero

z_0 , which is located in the region $|z| < C_1$, since

$$\begin{aligned} |z_0| &= \frac{q_{1,-1,-1}}{|1 - [q_{0,0,0} + q_{0,-1,0}\beta + q_{0,0,-1}\gamma + q_{0,-1,-1}\beta\gamma]|} \\ &\leq \frac{q_{1,-1,-1}}{1 - [q_{0,0,0} + q_{0,-1,0} + q_{0,0,-1} + q_{0,-1,-1}]} = \frac{q_1^{(1)}}{q_1^{(1)} + q_{-1}^{(1)}} < C_1. \end{aligned}$$

As a result, applying Rouché's theorem for $r = C_1$ proves that (4.5) has exactly one solution with $|z| < C_1$, i.e. $|\alpha| < C_1|\beta\gamma|$ (since $q_{1,-1,-1} > 0$, this solution is a nonnull solution, so we also know that $|\alpha| > 0$). Next, applying Rouché's theorem for $r = \hat{C}_1$ shows that if (4.5) has a second root, which is the case if and only if the coefficient of z^2 is not equal to zero, then this root must be in the region $|z| > \hat{C}_1$, i.e. $|\alpha| > \hat{C}_1|\beta\gamma|$. \square

When reading Lemma 4.1, it is important to note that by definition $C_i \leq 1$ and $\hat{C}_i \geq 1$ for all $i \in I$. So, part (i) of Lemma 4.1 implies that if the quadratic equation (3.1) for fixed β and γ , $0 < |\beta| < 1$ and $0 < |\gamma| < 1$, has two roots α , then one root satisfies $0 < |\alpha| < |\beta\gamma|$ and the other root satisfies $|\alpha| > |\beta\gamma|$; and similarly for the parts (ii) and (iii). For later reference in this section, we state that at least two of the three constants C_i are smaller than 1, since

$$C_i = 1 \text{ for some } i \in I \Rightarrow C_j < 1 \text{ for all } j \in \Lambda\{i\}. \quad (4.6)$$

This property is proved by analyzing the transition rates q_{i_1, i_2, i_3} . For example, $C_1 = 1$ implies that $q_1^{(1)}/q_{-1}^{(1)} \geq 1$, and, hence, $q_{1,-1,-1} \geq q_{-1,1,-1} + q_{-1,-1,1}$ and therefore

$$\frac{q_1^{(2)}}{q_{-1}^{(2)}} \leq \frac{q_{-1,1,-1}}{q_{1,-1,-1} + q_{-1,-1,1}} \leq \frac{q_{-1,1,-1}}{q_{-1,1,-1} + 2q_{-1,-1,1}} < 1,$$

i.e. $C_2 < 1$; and similarly one shows that $C_3 < 1$. Finally, with the help of (4.6) and the fact that by definition $C_i = 1$ or $\hat{C}_i = 1$ for all $i \in I$, one can show that

$$\hat{C}_i > 1 \text{ for some } i \in I \Rightarrow \hat{C}_j = 1 \text{ for all } j \in \Lambda\{i\},$$

by which at most one of the three constants \hat{C}_i is larger than 1.

The properties stated in Lemma 4.1 are used to prove Lemma 4.2, which states that for each relevant solution of the quadratic equation (3.1), i.e. for each $(\alpha, \beta, \gamma) \in P$, exactly one factor is smaller than the product of (some constant $C_i \leq 1$ and) the other two factors. Thereafter, with the help of the Lemmas 4.1 and 4.2 and (4.6), we are able to define a path that always leads to a vector v for which one of the factors α_v , β_v and γ_v is larger than or equal to one in absolute value (i.e. for which (3.9) is violated).

Lemma 4.2

Each solution $(\alpha, \beta, \gamma) \in P$ possesses exactly one of the following three properties:

- (i) $|\alpha| < C_1|\beta\gamma|$, $|\beta| > \hat{C}_2|\alpha\gamma|$ and $|\gamma| > \hat{C}_3|\alpha\beta|$;
- (ii) $|\beta| < C_2|\alpha\gamma|$, $|\alpha| > \hat{C}_1|\beta\gamma|$ and $|\gamma| > \hat{C}_3|\alpha\beta|$;
- (iii) $|\gamma| < C_3|\alpha\beta|$, $|\alpha| > \hat{C}_1|\beta\gamma|$ and $|\beta| > \hat{C}_2|\alpha\gamma|$.

Proof.

By Lemma 4.1, each solution $(\alpha, \beta, \gamma) \in P$ satisfies

$$\begin{aligned} & (|\alpha| < C_1 |\beta\gamma| \text{ or } |\alpha| > \hat{C}_1 |\beta\gamma|) \\ & \text{and } (|\beta| < C_2 |\alpha\gamma| \text{ or } |\beta| > \hat{C}_2 |\alpha\gamma|) \\ & \text{and } (|\gamma| < C_3 |\alpha\beta| \text{ or } |\gamma| > \hat{C}_3 |\alpha\beta|). \end{aligned} \quad (4.7)$$

Since $|\alpha| < C_1 |\beta\gamma|$ implies that $|\alpha| < |\beta|$ and $|\alpha| < |\gamma|$, and similarly for $|\beta| < C_2 |\alpha\gamma|$ and $|\gamma| < C_3 |\alpha\beta|$, (α, β, γ) satisfies at most one of the "<"-inequalities in (4.7). Further, since (α, β, γ) is a solution of (3.1), (α, β, γ) satisfies

$$\begin{aligned} 0 &= \left| \alpha\beta\gamma - \sum_{(t_1, t_2, t_3) \in T} q_{t_1, t_2, t_3} \alpha^{1-t_1} \beta^{1-t_2} \gamma^{1-t_3} \right| \\ &\geq |\alpha\beta\gamma| - \sum_{(t_1, t_2, t_3) \in T} q_{t_1, t_2, t_3} |\alpha^{1-t_1} \beta^{1-t_2} \gamma^{1-t_3}| \\ &= \sum_{(t_1, t_2, t_3) \in T} q_{t_1, t_2, t_3} (|\alpha\beta\gamma| - |\alpha^{1-t_1} \beta^{1-t_2} \gamma^{1-t_3}|) \\ &= q_{1, -1, -1} (|\alpha| - |\beta\gamma|) + q_{-1, 1, -1} (|\beta| - |\alpha\gamma|) + q_{-1, -1, 1} (|\gamma| - |\alpha\beta|) \\ &\quad + \sum_{t_1, t_2, t_3 \in \{-1, 0\}} q_{t_1, t_2, t_3} (|\alpha\beta\gamma| - |\alpha^{1-t_1} \beta^{1-t_2} \gamma^{1-t_3}|) \\ &\geq q_{1, -1, -1} (|\alpha| - |\beta\gamma|) + q_{-1, 1, -1} (|\beta| - |\alpha\gamma|) + q_{-1, -1, 1} (|\gamma| - |\alpha\beta|), \end{aligned}$$

which shows that (α, β, γ) cannot satisfy all three ">"-inequalities in (4.7). So, (α, β, γ) has to satisfy at least one of the "<"-inequalities. This proves that exactly one of the "<"-inequalities in (4.7) is satisfied, which completes the proof. \square

Lemma 4.3

For each starting solution $(\alpha, \beta, \gamma) \in P$, there exists a vector $v \in V$ such that $|\alpha_v| \geq 1$, $|\beta_v| \geq 1$ or $|\gamma_v| \geq 1$.

Proof.

Let $(\alpha, \beta, \gamma) \in P$ be a starting solution. Due to the properties stated in the Lemmas 4.1 and 4.2, we are able to construct a path $\{v^{(k)}\}$ for which the absolute values of the factors $\alpha_{v^{(k)}}$, $\beta_{v^{(k)}}$ and $\gamma_{v^{(k)}}$ are monotonically increasing for increasing k . The path starts with the empty vector \emptyset , for which the corresponding solution $(\alpha_\emptyset, \beta_\emptyset, \gamma_\emptyset) = (\alpha, \beta, \gamma) \in P$ and thus satisfies exactly one of the three property stated in Lemma 4.2. Suppose that property (i) is satisfied, i.e. $|\alpha_\emptyset| < C_1 |\beta_\emptyset \gamma_\emptyset|$, $|\beta_\emptyset| > \hat{C}_2 |\alpha_\emptyset \gamma_\emptyset|$ and $|\gamma_\emptyset| > \hat{C}_3 |\alpha_\emptyset \beta_\emptyset|$. As we know, the vector (1) has the same β - and γ -factor as \emptyset , but a new α -factor $\alpha_{(1)}$, which is the companion solution to α_\emptyset of the quadratic equation (3.1) for fixed $\beta = \beta_\emptyset$ and $\gamma = \gamma_\emptyset$. By Lemma 4.1, $\alpha_{(1)}$ has to be the root which satisfies $|\alpha| > \hat{C}_1 |\beta_\emptyset \gamma_\emptyset|$ and thus $\alpha_{(1)}$ is larger than α_\emptyset in absolute value.

We find

$$|\alpha_{(1)}| > \hat{C}_1 |\beta_{(1)}\gamma_{(1)}| = \hat{C}_1 |\beta_{\emptyset}\gamma_{\emptyset}| > \frac{\hat{C}_1}{C_1} |\alpha_{\emptyset}| = \frac{\hat{C}_1}{C_1} |\alpha|.$$

If $|\alpha_{(1)}| < 1$, then $(\alpha_{(1)}, \beta_{(1)}, \gamma_{(1)})$ is also an element of P and therefore also satisfies one of the properties stated in Lemma 4.2. Since $|\alpha_{(1)}| > \hat{C}_1 |\beta_{(1)}\gamma_{(1)}|$, it satisfies property (ii) or property (iii). Suppose that (ii) is satisfied, then it is useful to consider the vector (1,2). This vector has the same α - and γ -factor as (1), but a new and larger β -factor:

$$|\beta_{(1,2)}| > \hat{C}_2 |\alpha_{(1,2)}\gamma_{(1,2)}| = \hat{C}_2 |\alpha_{(1)}\gamma_{(1)}| > \frac{\hat{C}_2}{C_2} |\beta_{(1)}|.$$

When comparing the factors of vector (1,2) with the factors of the starting solution, we find

$$|\alpha_{(1,2)}| = |\alpha_{(1)}| > \frac{\hat{C}_1}{C_1} |\alpha|, \quad |\beta_{(1,2)}| > \frac{\hat{C}_2}{C_2} |\beta_{(1)}| = \frac{\hat{C}_2}{C_2} |\beta|,$$

$$|\gamma_{(1,2)}| = |\gamma_{(1)}| = |\gamma|.$$

If $|\beta_{(1,2)}| < 1$, then $(\alpha_{(1,2)}, \beta_{(1,2)}, \gamma_{(1,2)}) \in P$ and again the construction process may be continued.

In general we construct a path $\{v^{(k)}\}$ with $v^{(0)} = \emptyset$ and for all $k = 1, 2, \dots$ the vector $v^{(k)}$ is an element of the offspring of $v^{(k-1)}$, i.e. $p(v^{(k)}) = v^{(k-1)}$, and the last element $v_{\{v^{(k)}\}}^{(k)}$ of $v^{(k)}$ is taken equal to

$$v_{\{v^{(k)}\}}^{(k)} = \begin{cases} 1 & \text{if } |\alpha_{v^{(k-1)}}| < |\beta_{v^{(k-1)}}\gamma_{v^{(k-1)}}|; \\ 2 & \text{if } |\beta_{v^{(k-1)}}| < |\alpha_{v^{(k-1)}}\gamma_{v^{(k-1)}}|; \\ 3 & \text{if } |\gamma_{v^{(k-1)}}| < |\alpha_{v^{(k-1)}}\beta_{v^{(k-1)}}|. \end{cases}$$

Here, the construction process is stopped as soon as

$$|\alpha_{v^{(k)}}| \geq 1, \quad |\beta_{v^{(k)}}| \geq 1 \quad \text{or} \quad |\gamma_{v^{(k)}}| \geq 1 \tag{4.8}$$

for some $k \geq 1$. In that case $(\alpha_{v^{(k)}}, \beta_{v^{(k)}}, \gamma_{v^{(k)}})$ is not an element of P , by which the essential properties of Lemma 4.2 cannot be used anymore. To complete the proof of Lemma 4.3, it suffices to prove that (4.8) always occurs for some k .

For each vector $v^{(k)}$, $k \geq 1$, two of the factors $\alpha_{v^{(k)}}$, $\beta_{v^{(k)}}$ and $\gamma_{v^{(k)}}$ are equal to the corresponding factors for $v^{(k-1)}$, whereas the third factor is new and may be proved to be larger in absolute value (by using the Lemmas 4.1 and 4.2):

$$|\alpha_{v^{(k)}}| > \frac{\hat{C}_1}{C_1} |\alpha_{v^{(k-1)}}|, \quad |\beta_{v^{(k)}}| = |\beta_{v^{(k-1)}}|, \quad |\gamma_{v^{(k)}}| = |\gamma_{v^{(k-1)}}| \quad \text{if } v_{\{v^{(k)}\}}^{(k)} = 1;$$

$$|\alpha_{v^{(k)}}| = |\alpha_{v^{(k-1)}}|, \quad |\beta_{v^{(k)}}| > \frac{\hat{C}_2}{C_2} |\beta_{v^{(k-1)}}|, \quad |\gamma_{v^{(k)}}| = |\gamma_{v^{(k-1)}}| \quad \text{if } v_{\{v^{(k)}\}}^{(k)} = 2;$$

$$|\alpha_{v^{(k)}}| = |\alpha_{v^{(k-1)}}|, \quad |\beta_{v^{(k)}}| = |\beta_{v^{(k-1)}}|, \quad |\gamma_{v^{(k)}}| > \frac{\hat{C}_3}{C_3} |\gamma_{v^{(k-1)}}| \quad \text{if } v_{\{v^{(k)}\}}^{(k)} = 3.$$

Let $n_i(k)$ denote the number of i -s in the sequence $v_{\{v^{(1)}\}}^{(1)}, \dots, v_{\{v^{(k)}\}}^{(k)}$, i.e. in the vector $v^{(k)}$:

$$n_i(k) = |\{l \mid l \in \{1, \dots, k\} \text{ and } v_l^{(k)} = i\}|, \quad i \in I, k \geq 0.$$

Then, by induction, it is easily proved that

$$|\alpha_{v^{(k)}}| \geq \left[\frac{\hat{C}_1}{C_1} \right]^{n_1(k)} |\alpha|, \quad |\beta_{v^{(k)}}| \geq \left[\frac{\hat{C}_2}{C_2} \right]^{n_2(k)} |\beta|, \quad |\gamma_{v^{(k)}}| \geq \left[\frac{\hat{C}_3}{C_3} \right]^{n_3(k)} |\gamma| \quad (4.9)$$

for all $k \geq 0$. Since at least two of the three constants C_i are smaller than 1 (see (4.6)), we have

$$\frac{\hat{C}_i}{C_i} > \frac{1}{C_i} > 1$$

for at least two $i \in I$, say for i_1 and i_2 . Further, since by definition for all vectors $v^{(k)}$ two succeeding coordinates are always different, at least one of the powers $n_{i_1}(k)$ and $n_{i_2}(k)$ is increased by 1 if k is increased by 2. So, the sequence $n_{i_1}(k) + n_{i_2}(k)$, $k = 0, 2, 4, \dots$, is strictly increasing. As a result, at least one of the right-hand sides in (4.9) has to become larger than or equal to 1 for some k , which proves that (4.8) always occurs for some k . \square

Lemma 4.3 shows that (3.9) is never satisfied. However, (3.9) should only be satisfied for those vectors v for which $a_v b_v c_v \neq 0$. So, if for some vector v one of the factors α_v , β_v and γ_v is not smaller than one in absolute value, then we want $a_v b_v c_v$ to be equal to zero. This implies that there has to be a vector v' on the path from \emptyset to v for which $\alpha_{v'}^m \beta_{v'}^n \gamma_{v'}^r$ also satisfies the equilibrium equations for one of the boundary planes. After renumbering the terms this is equivalent with the requirement that the *starting solution* (α, β, γ) also satisfies the equations for one of the boundary planes, i.e. with the requirement that $a_{(1)} = 0$, $b_{(2)} = 0$ or $c_{(3)} = 0$. The proof of Lemma 4.3 shows that we have to require $a_{(1)} = 0$ if $|\alpha| < |\beta\gamma|$, $b_{(2)} = 0$ if $|\beta| < |\alpha\gamma|$ and $c_{(3)} = 0$ if $|\gamma| < |\alpha\beta|$ (if $|\alpha| < |\beta\gamma|$ for example, then there is a path via (1) that leads to a vector v for which $|\alpha_v| \geq 1$, $|\beta_v| \geq 1$ or $|\gamma_v| \geq 1$). This results in the second necessary condition for a formal solution. Under the assumption of Condition 4.1, this condition is also sufficient to meet (3.9) for all vectors v with $a_v b_v c_v \neq 0$.

Condition 4.2 (necessary condition arising from (3.7) and (3.9))

A starting solution $(\alpha, \beta, \gamma) \in P$ also has to be a solution of the equilibrium equations for one of the boundary planes. It has to satisfy:

- equilibrium equation (2.2) for the boundary plane $m = 0$ if $|\alpha| < |\beta\gamma|$;
- equilibrium equation (2.3) for the boundary plane $n = 0$ if $|\beta| < |\alpha\gamma|$;
- equilibrium equation (2.4) for the boundary plane $r = 0$ if $|\gamma| < |\alpha\beta|$.

From now on we are only interested in formal solutions $\{x_{m,n,r}(\alpha, \beta, \gamma)\}$ for which the starting solution (α, β, γ) satisfies Condition 4.2. All these starting solutions belong to one of the sets P_i , $i \in I$, where P_1 is defined as the set of appropriate starting solutions on the boundary plane $m = 0$, i.e.

$$P_1 = \{(\alpha, \beta, \gamma) \in P \mid \alpha^m \beta^n \gamma^r \text{ is also a solution of (2.2) and } |\alpha| < |\beta\gamma| \},$$

and P_2 and P_3 are defined as the sets of appropriate starting solutions on $n = 0$ and $r = 0$,

respectively. For each starting solution $(\alpha, \beta, \gamma) \in P_i$, $i \in I$, the corresponding formal solution reduces to a *binary tree* of product forms and it is denoted by $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$, where the superscript (i) denotes on which boundary it starts:

$$x_{m,n,r}^{(i)}(\alpha, \beta, \gamma) = \sum_{v \in V_i} a_v b_v c_v \alpha_v^m \beta_v^n \gamma_v^r \quad (4.10)$$

with

$$V_i = \{(v_1, \dots, v_l) \in V \mid \text{if } v \neq \emptyset \text{ then } v_1 \neq i\}.$$

The following lemma states that for each formal solution $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$ all factors α_v , β_v and γ_v are well-defined (i.e. one always has nonnull denominators for the terms $f_i(\cdot, \cdot)$ in the definitions of these factors) and all $(\alpha_v, \beta_v, \gamma_v)$ are elements of P . The coefficients a_v , b_v and c_v cannot be guaranteed to be well-defined. Part (i) of the following lemma is easily proved with the help of Lemmas 4.1 and 4.2; the other parts follow from part (i).

Lemma 4.4

Let $i \in I$ and $(\alpha, \beta, \gamma) \in P_i$. Then all factors α_v , β_v and γ_v of the formal solution $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$ are well-defined and they have the following properties:

(i) For all $v \in V_i$, we have

$$\begin{cases} 0 < |\alpha_v| < C_1 |\beta_v \gamma_v| & \text{if } v_{l(v)} = 1; \\ 0 < |\beta_v| < C_2 |\alpha_v \gamma_v| & \text{if } v_{l(v)} = 2; \\ 0 < |\gamma_v| < C_3 |\alpha_v \beta_v| & \text{if } v_{l(v)} = 3; \end{cases}$$

(ii) For each path $\{v^{(k)}\}$ in V_i , all three factors $\alpha_{v^{(k)}}$, $\beta_{v^{(k)}}$ and $\gamma_{v^{(k)}}$ decrease monotonically in absolute value for increasing k ;

(iii) $(\alpha_v, \beta_v, \gamma_v) \in P$ for all $v \in V_i$;

(iv) For each path $\{v^{(k)}\}$ in V_i , at least two of the three factors $\alpha_{v^{(k)}}$, $\beta_{v^{(k)}}$ and $\gamma_{v^{(k)}}$ go to zero exponentially fast as $k \rightarrow \infty$.

We have derived two necessary conditions arising from (3.7) and (3.9), however we still have the following questions about the convergence of the redefined formal solutions $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$. Are the sums $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ absolutely convergent for all $(m, n, r) \in M$? And, is it possible to normalize all formal solutions? We know that all factors α_v , β_v and γ_v are smaller than one in absolute value, which is necessary for the normalization. Furthermore, according to Lemma 4.4(iv), for each path $\{v^{(k)}\}$ at least two of the three factors $\alpha_{v^{(k)}}$, $\beta_{v^{(k)}}$ and $\gamma_{v^{(k)}}$ go exponentially fast to zero as $k \rightarrow \infty$. Since not all three factors are guaranteed to go to zero, different paths may have different limits for the coefficients a_v , b_v and c_v , which is one of the difficulties arising when one tries to prove the absolute convergence. Another difficulty is the fact that each formal solution is a binary tree instead of a series, by which it is no longer sufficient for the convergence to prove that the ratio of two successive terms has a limit smaller than one.

If the questions raised above are answered, then the question arises which formal solutions should be linearly combined to get the equilibrium distribution $\{p_{m,n,r}\}$, i.e. to get a

solution which also satisfies the equilibrium equations for the axes and the origin. For the general case we are not able yet to answer all these questions, but for random walks with the projection property we are. The $2 \times N$ switch has this property.

Remark 4.1 (*about Assumption 3.1 and special cases*)

In each compensation step it may happen that a special case occurs when computing the new product-form factor or the coefficient of the corresponding compensation term. For example, as we have seen at the beginning of Section 3, for a compensation term $\hat{a}\hat{\alpha}^m\beta^n\gamma^r$ added to compensate for an error of a product form $\alpha^m\beta^n\gamma^r$ on the boundary plane $m=0$, a special case occurs if the numerator or denominator of $f_1(\beta, \gamma)$ is equal to zero or if the denominator of the coefficient \hat{a} vanishes. To avoid that such special cases have to be taken into account we introduced Assumption 3.1. However, if this assumption is not satisfied, then one could try to determine the equilibrium distribution in an alternative way (because of symmetry, it suffices to treat only the case that part (i) is not satisfied):

- If $\sum_{(1,t_2,t_3) \in T} q_{1,t_2,t_3} = 0$, then the numerator of $f_1(\beta, \gamma)$ is identical to zero and each compensation term on the boundary plane $m=0$ gets an α -factor equal to zero. Such compensation terms do not introduce new errors on the other two boundary planes and therefore they should not lead to the generation of more compensation terms. In this case, constructing series of terms $\alpha^m\beta^n\gamma^r + \hat{a}0^m\beta^n\gamma^r$ by alternately compensating on the boundary planes $n=0$ and $r=0$ could lead to finding the equilibrium distribution.
- If $\sum_{(-1,t_2,t_3) \in T} q_{-1,t_2,t_3} = 0$ (which, as one can easily check, is the case for the symmetric longest queue system; see Zheng and Zipkin [35] and Flatto [20]), then the denominator of $f_1(\beta, \gamma)$ is identical to zero and the compensation approach cannot be used at all. In this case one could try to determine the equilibrium probabilities $p_{m,n,r}$ iteratively: first for $m=0$, next for $m=1$, etc.

If Assumption 3.1 is satisfied, then only incidentally a special case may occur. In that case, in general the formal solutions will not reduce to finite sums and one has to require that Condition 4.1 is satisfied. Then for each formal solution starting with an appropriate starting solution (see Condition 4.2) all factors of its product forms will be well-defined and nonnull, i.e. the denominator and the numerator of the functions $f_i(\dots)$ never vanish for the relevant values of the domain variables (see Lemma 4.4). The only special case that still might occur is that the denominator in the definition of one of the coefficients vanishes. If this happens for one of the formal solutions needed for the equilibrium distribution, then this indicates that the equilibrium distribution does not only consist of product forms $\alpha^m\beta^n\gamma^r$ but also of alternative terms such as $m\alpha^m\beta^n\gamma^r$ (see also Section 7 of [8]). Fortunately this is never the case for random walks with the projection property.

5. The projection property

To be able to continue the analysis which we performed for the class of strongly homogeneous, nearest-neighboring random walks in the Sections 3 and 4, we would like to have some extra information. This extra information is obtained by the introduction of the so-called *projection property*, which is described in this section. This property is defined in such a way

that the marginal distributions of the equilibrium distribution $\{p_{m,n,r}\}$ may be characterized as the equilibrium distributions of one- and two-dimensional random walks. This characterization enables us to derive explicit formulae for the marginal distributions. Besides, as we shall see in the next section, the projection property also leads to an important simplification for the coefficients a_v , b_v and c_v of a formal solution.

Consider the class of three-dimensional, irreducible, positive recurrent, strongly homogeneous, nearest-neighboring random walks; see Assumption 2.1 of Section 2. In the next sections we shall restrict ourselves to the subclass of random walks which also have the projection property.

Since the formal definition of the projection property is rather complex, we first describe this property for the class of two-dimensional, homogeneous, nearest-neighboring random walks; see Figure 1. In general words, for a random walk of this class the projection property is described as follows. For all states at the vertical boundary the set of outgoing transitions is a kind of projection of the set of transitions for the interior points, and similarly for the horizontal boundary; for the origin the set of transitions is the projection of the set of transitions of both the vertical boundary and the horizontal boundary. For the vertical boundary for example, this means: the rates v_{1,t_2} are the same as the rates q_{1,t_2} and the rates q_{0,t_2} are equal to the sums of the rates q_{0,t_2} and q_{-1,t_2} . So to speak, the set of transitions for the vertical boundary is obtained by pushing the set of transitions for the interior against this vertical boundary. For the origin the impact of the projection property is a little bit more complex. This set of transitions for the origin is obtained by pushing the set for the vertical boundary against the horizontal boundary. As a result, we find that the rates $o_{1,1}$, $o_{0,1}$, $o_{1,0}$ and $o_{0,0}$ are equal to $q_{1,1}$ ($=v_{1,1}$), $q_{0,1}+q_{-1,1}$ ($=v_{0,1}$), $q_{1,0}+q_{-1,-1}$ ($=v_{1,0}+v_{-1,-1}$) and $q_{0,0}+q_{-1,0}+q_{0,-1}+q_{-1,-1}$ ($=v_{0,0}+v_{0,-1}$) respectively. As one can easily verify, the same rates for the origin are obtained by pushing the set of transitions for the horizontal boundary against the vertical boundary.

Let us now describe the projection property for the class of three-dimensional, homogeneous, nearest-neighboring random walks. In this case the projection property may be described as follows. For all $J \subset I$, $J \neq \emptyset$, the set of outgoing transitions for the states $(m,n,r) \in M_J$ is a kind of projection of the set of transitions for the states $(m,n,r) \in M_{J \setminus \{i\}}$, where $i \in J$. For example, the set of transitions for the boundary plane $m=0$, i.e. for $M_{\{1\}}$, is the projection of the set of transitions for the interior, i.e. for M_{\emptyset} , which means: the rates $q_{1,t_2,t_3}^{\{1\}}$ are the same as the rates q_{1,t_2,t_3} and the rates $q_{0,t_2,t_3}^{\{1\}}$ are equal to the sums of the rates q_{0,t_2,t_3} and q_{-1,t_2,t_3} . So to speak, the set of transitions for $M_{\{1\}}$ is got by pushing the set of transitions for the interior against the boundary plane $m=0$. Just like for the origin in the two-dimensional case, for the axes and the origin the impact of the projection property is more complex. This also follows from the following mathematical description of the projection property.

Assumption 5.1

Projection property: for all $J \subset I$, $J \neq \emptyset$, and all transitions $(t_1, t_2, t_3) \in T_J$, we have

$$q_{t_1, t_2, t_3}^J = \sum_{(u_1, u_2, u_3) \in U_J(t_1, t_2, t_3)} q_{u_1, u_2, u_3} ,$$

where $U_J(t_1, t_2, t_3) = \{(u_1, u_2, u_3) \in T \mid u_i = 0 \text{ if } t_i = 0 \text{ and } i \in J; u_i = t_i \text{ else}\}$.

Note that for a random walk with the projection property all transition rates are uniquely determined by the transition rates q_{t_1, t_2, t_2} for the interior of the state space.

A random walk with the projection property has the nice feature that the transitions for a subset of all components are independent of the state of the whole system. This does not mean that the marginal distribution for a subset of all components is independent of the distribution for the other components, but it does mean that all marginal distributions can be characterized as equilibrium distributions of lower-dimensional, homogeneous, nearest-neighboring random walks with the projection property. Queueing systems satisfying the projection property are the $2 \times N$ switch and the fork and join model. Of course, the shortest queue system violates this property (the random walk describing this system is not even strongly homogeneous).

Since we need the marginal distributions in later sections, we shall now derive formulae for them. Let us start with considering the one-dimensional marginal distributions $\{p_m^{(1)}\}$, $\{p_n^{(2)}\}$ and $\{p_r^{(3)}\}$:

$$p_m^{(i)} = \sum_{\substack{(n_1, n_2, n_3) \in M \\ n_i = m}} p_{n_1, n_2, n_3} , \quad m \geq 0, i \in I.$$

Analyzing these distributions for the component chains/random walks of the full Markov chain/random walk does not only lead to explicit formulae for these distributions, but it also leads to a simple, necessary and sufficient condition for the ergodicity. Due to the projection property, for all states (m, n, r) with $m \geq 1$ the total rate to states (m', n', r') with $m' = m + t$ equals $q_t^{(1)} = \sum_{(t_1, t_2, t_3) \in T} q_{t_1, t_2, t_3}$, where t is fixed and $t \in \{-1, 0, 1\}$, and for all states $(0, n, r)$ the total rate to states (t, n', r') equals $q_1^{(1)}$ for $t = 1$ and $q_0^{(1)} + q_{-1}^{(1)}$ for $t = 0$; similarly for the n - and r -direction. This shows that the distributions $\{p_m^{(i)}\}$ may be characterized as the equilibrium distributions of one-dimensional, homogeneous, nearest-neighboring random walks with the projection property; here, the transition rates for the interior are given by the variables $q_t^{(i)}$ defined by (4.4).

The full random walk will be positive recurrent (=ergodic) if and only if all component random walks are positive recurrent, i.e. if and only if the component random walks have negative drifts. So, for a random walk with the projection property the ergodicity condition is:

$$q_{-1}^{(i)} > q_1^{(i)} \quad \text{for all } i \in I. \tag{5.1}$$

This condition is assumed to be satisfied, which implies that we have the following geometric distributions for the one-dimensional marginal distributions $\{p_m^{(i)}\}$:

$$p_m^{(i)} = \left[1 - \frac{q_1^{(i)}}{q_{-1}^{(i)}} \right] \left[\frac{q_1^{(i)}}{q_{-1}^{(i)}} \right]^m , \quad m \geq 0, i \in I. \tag{5.2}$$

Here, $q_1^{(i)}/q_{-1}^{(i)}$ is the companion solution to 1 of the quadratic equation which is obtained by substituting a power in the equilibrium equation for the interior of the i -th component random walk. Since these quadratic equations can also be derived from the quadratic equation (3.1) by taking two of the three factors α , β and γ equal to 1, we have $q_1^{(i)}/q_{-1}^{(i)} = f_i(1,1)$ for all $i \in I$.

If $q_1^{(i)} = 0$ for some i , then all states (m_1, m_2, m_3) with $m_i > 0$ are transient and we can restrict ourselves to a lower-dimensional problem. To exclude this special case, we shall require that $q_1^{(i)} > 0$ for all i . Together with (5.1) this leads to the assumption that

$$q_{-1}^{(i)} > q_1^{(i)} > 0 \quad \text{for all } i \in I. \quad (5.3)$$

This condition implies Assumption 3.1. Given Condition 4.1, condition (5.3) is necessary and sufficient for having an irreducible and positive recurrent random walk, since then $q_{1,-1,-1} > 0$, $q_{-1,1,-1} > 0$ and $q_{-1,-1,1} > 0$ (without Condition 4.1, (5.3) does not guarantee the irreducibility, since then one might have the situation with $q_{-1,-1,-1} > q_{1,1,1} > 0$ and $q_{t_1, t_2, t_3} = 0$ for all other $(t_1, t_2, t_3) \in T$, for example).

Let us now consider the two-dimensional marginal distributions, which we denote by $\{p_{m,n}^{(1,2)}\}$, $\{p_{m,r}^{(1,3)}\}$ and $\{p_{n,r}^{(2,3)}\}$:

$$p_{m_1, m_2}^{(i,j)} = \sum_{\substack{(n_1, n_2, n_3) \in M \\ n_i = m_1, n_j = m_2}} p_{n_1, n_2, n_3}, \quad m_1, m_2 \geq 0, i, j \in I, i > j.$$

For all $i, j \in I, i > j$, $\{p_{m_1, m_2}^{(i,j)}\}$ is the equilibrium distribution of the two-dimensional random walk with the projection property for which the transition rates are given by

$$q_{t_1, t_2}^{(i,j)} = \sum_{\substack{(s_1, s_2, s_3) \in T \\ s_i = t_1, s_j = t_2}} q_{s_1, s_2, s_3}, \quad t_1, t_2 \geq 0, i, j \in I, i > j.$$

In general it is not possible to derive explicit formulae for the distributions $\{p_{m_1, m_2}^{(i,j)}\}$. However, in order to ensure the convergence of the formal solutions, we are only interested in three-dimensional random walks for which Condition 4.1 holds. Remarkably enough, this condition is just strong enough to let the rates $q_{t_1, t_2}^{(i,j)}$ satisfy condition (1.1) for all $i, j \in I, i > j$. As a result, all $\{p_{m_1, m_2}^{(i,j)}\}$ can be determined by using the two-dimensional version of the compensation approach.

The compensation approach, which has been developed in [8] for the class of two-dimensional, irreducible, positive recurrent, homogeneous, nearest-neighboring random walks satisfying condition (1.1), has been worked out in more detail in [13] for random walks having the projection property. For such a random walk the equilibrium distribution may be represented as a sum of two alternating series of product-form distributions. We shall further explain this main result on the basis of the random walk describing the behavior of the components n and r ; see Figure 4. For the equilibrium distribution $\{p_{n,r}^{(2,3)}\}$ of this random walk, the use of the compensation approach shows that

$$p_{n,r}^{(2,3)} = \sum_{k=0}^{\infty} (-1)^k (1-\beta_k^{(1)}) (\beta_k^{(1)})^n (1-\gamma_k^{(1)}) (\gamma_k^{(1)})^r + \sum_{k=0}^{\infty} (-1)^k (1-\hat{\beta}_k^{(1)}) (\hat{\beta}_k^{(1)})^n (1-\hat{\gamma}_k^{(1)}) (\hat{\gamma}_k^{(1)})^r, \quad n, r \geq 0, n+r \geq 1, \quad (5.4)$$

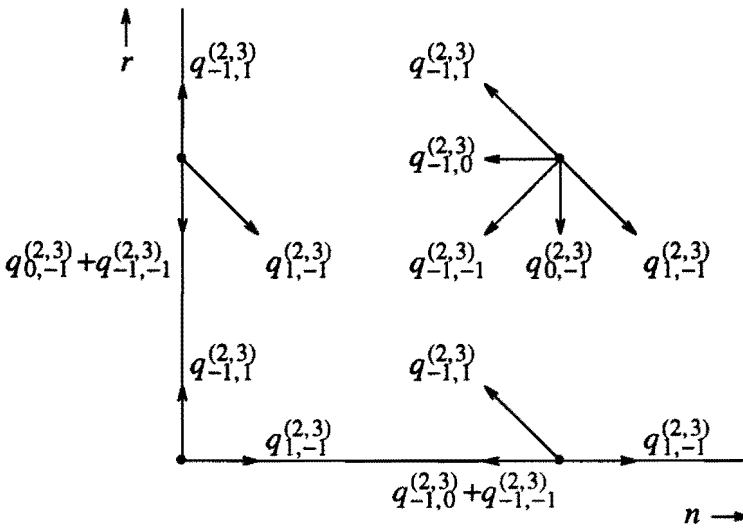


Figure 4. The transition rates for the random walk which describes the behavior for the components n and r ; for all states the transitions to themselves have been left out.

and $p_{0,0}^{(2,3)}$ follows from the normalization:

$$p_{0,0}^{(2,3)} = 1 - \sum_{\substack{n,r \geq 0 \\ n+r \geq 1}} p_{m,n}^{(2,3)}. \quad (5.5)$$

Here, the first series in (5.4) is a formal solution starting on the vertical boundary $n=0$ and it satisfies the following properties:

- * All $(\beta_k^{(1)}, \gamma_k^{(1)})$ are solutions of the quadratic equation

$$\beta\gamma = \sum_{t_2, t_3 \in \{-1, 0, 1\}} q_{t_2, t_3}^{(2,3)} \beta^{1-t_2} \gamma^{1-t_3}, \quad (5.6)$$

which is obtained after substituting the product form $\beta^n \gamma^r$ in the equilibrium equation for the interior. This quadratic equation is equivalent to the quadratic equation (3.1) for fixed $\alpha=1$, which implies that the product of the roots β of (5.6) for fixed γ is equal to $f_2(1, \gamma)$ and the product of the roots γ of (5.6) for fixed β is equal to $f_3(1, \beta)$;

- * $(\beta_0^{(1)}, \gamma_0^{(1)})$ is the unique solution of the equilibrium equations for the interior and the vertical boundary $n=0$: $\gamma_0^{(1)}$ is equal to the geometric factor of $\{p_r^{(3)}\}$, i.e. $\gamma_0^{(1)} = q_1^{(3)}/q_{-1}^{(3)} = f_3(1, 1)$, and $\beta_0^{(1)}$ is the companion solution to 1 of the quadratic equation (5.6) for fixed $\gamma = \gamma_0^{(1)}$, i.e. $\beta_0^{(1)} = f_2(1, \gamma_0^{(1)})$;
- * For all even k , the factors $\beta_{k+1}^{(1)}$ and $\gamma_{k+1}^{(1)}$ are chosen such that the sum of the k -th and $(k+1)$ -th term satisfies the equilibrium equations for the interior and the horizontal boundary $r=0$: $\beta_{k+1}^{(1)} = \beta_k^{(1)}$ and $\gamma_{k+1}^{(1)}$ is the companion solution to $\gamma_k^{(1)}$ of (5.6) for fixed $\beta = \beta_k^{(1)}$, i.e. $\gamma_{k+1}^{(1)} = f_3(1, \beta_k^{(1)})/\gamma_k^{(1)}$;
- * For all odd k , the factors $\beta_{k+1}^{(1)}$ and $\gamma_{k+1}^{(1)}$ are chosen such that the sum of the k -th and $(k+1)$ -th term satisfies the equilibrium equations for the interior and the vertical

boundary $n=0$: $\gamma_{k+1}^{(1)} = \gamma_k^{(1)}$ and $\beta_{k+1}^{(1)}$ is the companion solution to $\beta_k^{(1)}$ of (5.6) for fixed $\gamma = \gamma_k^{(1)}$, i.e. $\beta_{k+1}^{(1)} = f_2(1, \gamma_k^{(1)}) \beta_k^{(1)}$.

For the factors $\beta_k^{(1)}$ and $\gamma_k^{(1)}$ one may prove that

$$1 > \gamma_0^{(1)} > \beta_0^{(1)} = \beta_1^{(1)} > \gamma_1^{(1)} = \gamma_2^{(1)} > \beta_2^{(1)} = \dots$$

and the ratios $\gamma_{k+1}^{(1)}/\beta_k^{(1)}$ for even k and $\beta_{k+1}^{(1)}/\gamma_k^{(1)}$ for odd k decrease monotonically to

$$\frac{1}{A_2} = \frac{2q_{-1,1}^{(2,3)}}{q^{(2,3)} + \sqrt{(q^{(2,3)})^2 - 4q_{1,-1}^{(2,3)}q_{-1,1}^{(2,3)}}} \quad \text{and} \quad A_1 = \frac{q^{(2,3)} - \sqrt{(q^{(2,3)})^2 - 4q_{1,-1}^{(2,3)}q_{-1,1}^{(2,3)}}}{2q_{-1,1}^{(2,3)}}$$

respectively, where $q^{(2,3)} = 1 - q_{0,0}^{(2,3)}$ (see [13], Lemma 2 and the remarks after the Main Theorem). As a result,

$$\frac{\beta_{k+2}^{(1)}}{\beta_k^{(1)}} \downarrow \frac{A_1}{A_2} \quad \text{and} \quad \frac{\gamma_{k+2}^{(1)}}{\gamma_k^{(1)}} \downarrow \frac{A_1}{A_2} \quad \text{as } k \rightarrow \infty. \quad (5.7)$$

Hence, since $A_1/A_2 < 1$, the factors $\beta_k^{(1)}$ and $\gamma_k^{(1)}$ decrease exponentially fast to zero. Similar results hold for the second series in (5.4), which is a formal solution starting on the horizontal boundary $r=0$. For this solution the factors $\hat{\beta}_k^{(1)}$ and $\hat{\gamma}_k^{(1)}$ are defined by $\hat{\beta}_0^{(1)} = q_1^{(2)}/q_{-1}^{(2)} = f_2(1, 1)$, $\hat{\gamma}_0^{(1)} = f_3(1, \hat{\beta}_0^{(1)})$ and for all $k \geq 0$:

$$\begin{aligned} \hat{\gamma}_{k+1}^{(1)} &= \hat{\gamma}_k^{(1)}, \quad \hat{\beta}_{k+1}^{(1)} = f_2(1, \hat{\gamma}_k^{(1)}) \hat{\beta}_k^{(1)} \quad \text{if } k \text{ is even;} \\ \hat{\beta}_{k+1}^{(1)} &= \hat{\beta}_k^{(1)}, \quad \hat{\gamma}_{k+1}^{(1)} = f_3(1, \hat{\beta}_k^{(1)}) \hat{\gamma}_k^{(1)} \quad \text{if } k \text{ is odd.} \end{aligned}$$

By (5.7) and a similar result for the factors $\hat{\beta}_k^{(1)}$ and $\hat{\gamma}_k^{(1)}$, both series in (5.4) are absolutely convergent in all states except for the origin, by which the sum of both series satisfies all equilibrium equations except for the ones for the states (0,0), (0,1) and (1,0). By using the marginal distributions $\{p_n^{(2)}\}$ and $\{p_r^{(3)}\}$, it is shown that the solution given in (5.4) and (5.5) also satisfies the equilibrium equations in these remaining states.

For the sake of completeness, we also give the formulae for the other two-dimensional marginal distributions $\{p_{m,r}^{(1,3)}\}$ and $\{p_{m,n}^{(1,2)}\}$. For $\{p_{m,r}^{(1,3)}\}$ we have

$$\begin{aligned} p_{m,r}^{(1,3)} &= \sum_{k=0}^{\infty} (-1)^k (1 - \alpha_k^{(2)}) (\alpha_k^{(2)})^m (1 - \gamma_k^{(2)}) (\gamma_k^{(2)})^r \\ &\quad + \sum_{k=0}^{\infty} (-1)^k (1 - \hat{\alpha}_k^{(2)}) (\hat{\alpha}_k^{(2)})^m (1 - \hat{\gamma}_k^{(2)}) (\hat{\gamma}_k^{(2)})^r, \quad m, r \geq 0, \quad m+r \geq 1, \end{aligned} \quad (5.8)$$

where the first series represents the formal solution starting on the boundary $m=0$ and the second series is the formal solution starting on the boundary $r=0$. The factors $\alpha_k^{(2)}$, $\gamma_k^{(2)}$, $\hat{\alpha}_k^{(2)}$ and $\hat{\gamma}_k^{(2)}$ are defined by $\gamma_0^{(2)} = f_3(1, 1)$, $\alpha_0^{(2)} = f_1(1, \gamma_0^{(2)})$, $\hat{\alpha}_0^{(2)} = f_1(1, 1)$, $\hat{\gamma}_0^{(2)} = f_3(\hat{\alpha}_0^{(2)}, 1)$ and for all $k \geq 0$:

$$\begin{aligned} \alpha_{k+1}^{(2)} &= \alpha_k^{(2)}, \quad \gamma_{k+1}^{(2)} = f_3(\alpha_k^{(2)}, 1) \gamma_k^{(2)}, \quad \hat{\gamma}_{k+1}^{(2)} = \hat{\gamma}_k^{(2)}, \quad \hat{\alpha}_{k+1}^{(2)} = f_1(1, \hat{\gamma}_k^{(2)}) \hat{\alpha}_k^{(2)} \quad \text{if } k \text{ even;} \\ \gamma_{k+1}^{(2)} &= \gamma_k^{(2)}, \quad \alpha_{k+1}^{(2)} = f_1(1, \gamma_k^{(2)}) \alpha_k^{(2)}, \quad \hat{\alpha}_{k+1}^{(2)} = \hat{\alpha}_k^{(2)}, \quad \hat{\gamma}_{k+1}^{(2)} = f_3(\hat{\alpha}_k^{(2)}, 1) \hat{\gamma}_k^{(2)} \quad \text{if } k \text{ odd.} \end{aligned}$$

Finally, for $\{p_{m,n}^{(1,2)}\}$ we get

$$p_{m,n}^{(1,2)} = \sum_{k=0}^{\infty} (-1)^k (1-\alpha_k^{(3)}) (\alpha_k^{(3)})^m (1-\beta_k^{(3)}) (\beta_k^{(3)})^n + \sum_{k=0}^{\infty} (-1)^k (1-\hat{\alpha}_k^{(3)}) (\hat{\alpha}_k^{(3)})^m (1-\hat{\beta}_k^{(3)}) (\hat{\beta}_k^{(3)})^n, \quad m, n \geq 0, m+n \geq 1, \quad (5.9)$$

with $\beta_0^{(3)} = f_2(1, 1)$, $\alpha_0^{(3)} = f_1(\beta_0^{(3)}, 1)$, $\hat{\alpha}_0^{(3)} = f_1(1, 1)$, $\hat{\beta}_0^{(3)} = f_2(\hat{\alpha}_0^{(3)}, 1)$ and for all $k \geq 0$:

$$\alpha_{k+1}^{(3)} = \alpha_k^{(3)}, \quad \beta_{k+1}^{(3)} = f_2(\alpha_k^{(3)}, 1) / \beta_k^{(3)}, \quad \hat{\beta}_{k+1}^{(3)} = \hat{\beta}_k^{(3)}, \quad \hat{\alpha}_{k+1}^{(3)} = f_1(\hat{\beta}_k^{(3)}, 1) / \hat{\alpha}_k^{(3)} \quad \text{if } k \text{ even;}$$

$$\beta_{k+1}^{(3)} = \beta_k^{(3)}, \quad \alpha_{k+1}^{(3)} = f_1(\beta_k^{(3)}, 1) / \alpha_k^{(3)}, \quad \hat{\alpha}_{k+1}^{(3)} = \hat{\alpha}_k^{(3)}, \quad \hat{\beta}_{k+1}^{(3)} = f_2(\hat{\alpha}_k^{(3)}, 1) / \hat{\beta}_k^{(3)} \quad \text{if } k \text{ odd.}$$

Just like $p_{0,0}^{(2,3)}$, the remaining probabilities $p_{0,0}^{(1,3)}$ and $p_{0,0}^{(1,2)}$ follow from the normalization equation; see (5.5).

6. The convergence of the formal solutions

Due to the projection property, we get a considerable simplification in the formula for the formal solutions $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$, by which we are able to answer the convergence questions put at the end of Section 4.

By the projection property, the transition rates $q_{0,t_2,t_3}^{(1)}$ for the boundary plane $m=0$ are equal to $q_{0,t_2,t_3} + q_{-1,t_2,t_3}$, by which the function $g_1(\beta, \gamma)$ defined in (3.5) is identical to one; and similarly for $g_2(\alpha, \gamma)$ and $g_3(\alpha, \beta)$. As a consequence, the formulae for \hat{a} , \hat{b} and \hat{c} in Lemma 3.2 simplify to

$$\hat{a} = -\frac{1-\hat{\alpha}}{1-\alpha} a, \quad \hat{b} = -\frac{1-\hat{\beta}}{1-\beta} b \quad \text{and} \quad \hat{c} = -\frac{1-\hat{\gamma}}{1-\gamma} c \quad (6.1)$$

and similar simplifications are obtained for the coefficients a_v , b_v and c_v of each formal solution $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$ (see the definition of $x_{m,n,r}(\alpha, \beta, \gamma)$ in (3.6)). Using induction with respect to $l(v)$ shows that this results in

$$a_v b_v c_v = (-1)^{l(v)} (1-\alpha_v) (1-\beta_v) (1-\gamma_v) \quad \text{for all } v \in V, \quad (6.2)$$

by which formula (4.10) simplifies to

$$x_{m,n,r}^{(i)}(\alpha, \beta, \gamma) = \sum_{v \in V_i} (-1)^{l(v)} (1-\alpha_v) \alpha_v^m (1-\beta_v) \beta_v^n (1-\gamma_v) \gamma_v^r, \quad (\alpha, \beta, \gamma) \in P_i, \quad i \in I. \quad (6.3)$$

So, each formal solution is a kind of alternating sum of pure product-form distributions.

The most important consequence of the projection property, with regard to the convergence, is the boundedness of the product of the coefficients a_v , b_v and c_v :

$$|a_v b_v c_v| \leq 8 \quad \text{for all } v.$$

As a result, to prove that $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ is absolutely convergent, i.e. to prove that the absolute sum

$$\text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)) = \sum_{v \in V_i} |(-1)^{l(v)} (1-\alpha_v) \alpha_v^m (1-\beta_v) \beta_v^n (1-\gamma_v) \gamma_v^r|$$

converges, it suffices to show that $\sum_{v \in V_i} |\alpha_v^m \beta_v^n \gamma_v^r| < \infty$. This property, together with Lemma

4.4, forms the basis of the proof of the following theorem.

Theorem 6.1

For all $i \in I$ and $(\alpha, \beta, \gamma) \in P_i$:

(i) $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ is absolutely convergent for all states $(m, n, r) \in M_c$, where

$$M_c = \{ (m, n, r) \in M \mid (m, n, r) \in M_\emptyset \text{ or } (m, n, r) \in M_{\{j\}} \text{ for some } j \in I \};$$

(ii) $\sum_{(m,n,r) \in M_c} |x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)| \leq \sum_{(m,n,r) \in M_c} \text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma))$

$$\begin{aligned} &\leq \frac{\text{abs}(x_{0,1,1}^{(i)}(\alpha, \beta, \gamma))}{(1-|\beta|)(1-|\gamma|)} + \frac{\text{abs}(x_{1,0,1}^{(i)}(\alpha, \beta, \gamma))}{(1-|\alpha|)(1-|\gamma|)} \\ &\quad + \frac{\text{abs}(x_{1,1,0}^{(i)}(\alpha, \beta, \gamma))}{(1-|\alpha|)(1-|\beta|)} + \frac{\text{abs}(x_{1,1,1}^{(i)}(\alpha, \beta, \gamma))}{(1-|\alpha|)(1-|\beta|)(1-|\gamma|)} < \infty. \end{aligned}$$

Part (i) of this theorem states that $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ is absolutely convergent for all states in the interior and on the boundary planes. The set of these states is called the *convergence region* M_c . It is easily shown that $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ is not absolutely convergent, i.e. $\text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma))$ diverges, on the axes and in the origin. For example, $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ is shown to be not absolutely convergent for all states $(m, 0, 0)$ by considering the terms of a path $\{v^{(k)}\}$ in V_i with $v_{(v)}^{(k)} \in \{2, 3\}$ for all k . For this path $\alpha_{v^{(k)}} = \alpha$ for all k and $|\beta_{v^{(k)}}|$ and $|\gamma_{v^{(k)}}|$ decrease monotonically (see Lemma 4.4), so

$$\begin{aligned} \text{abs}(x_{m,0,0}^{(i)}(\alpha, \beta, \gamma)) &\geq \sum_{k=0}^{\infty} |(1-\alpha_{v^{(k)}})\alpha_{v^{(k)}}^m (1-\beta_{v^{(k)}})(1-\gamma_{v^{(k)}})| \\ &\geq (1-|\beta|)(1-|\gamma|) \sum_{k=0}^{\infty} |(1-\alpha)\alpha^m| = \infty. \end{aligned}$$

Part (ii) of Theorem 6.1 is needed in the next section; this part gives a useful upper bound for the summation of $\text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma))$ over all $(m, n, r) \in M_c$ and it states that $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$ can be normalized.

For the proof of Theorem 6.1, we shall use a recurrence relation for the sums $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$, which, as we know, are *binary trees* of product forms. Therefore, we temporarily have to extend the domains for the formal solutions $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$, which at the end of Section 4 only have been defined for starting solutions $(\alpha, \beta, \gamma) \in P_i$; see (4.10) and the definitions of the sets P_i . In the remainder of this section, we ignore the condition that $(\alpha, \beta, \gamma) \in P_i$ has to satisfy the equilibrium equation for the i -th boundary plane and we let $\{x_{m,n,r}^{(1)}(\alpha, \beta, \gamma)\}$ be defined for all solutions $(\alpha, \beta, \gamma) \in P'_i$, where

$$P'_1 = \{ (\alpha, \beta, \gamma) \in P \mid |\alpha| < |\beta\gamma| \}$$

and P'_2 and P'_3 are defined similarly, but with the condition $|\alpha| < |\beta\gamma|$ replaced by $|\beta| < |\alpha\gamma|$ and $|\gamma| < |\alpha\beta|$ respectively. As one can easily verify, then the following recurrence relation holds for all $i \in I$ and $(\alpha, \beta, \gamma) \in P'_i$:

$$x_{m,n,r}^{(i)}(\alpha, \beta, \gamma) = (1-\alpha)\alpha^m (1-\beta)\beta^n (1-\gamma)\gamma^r - \sum_{\substack{v \in V_i \\ l(v)=1}} x_{m,n,r}^{(v)}(\alpha_v, \beta_v, \gamma_v). \quad (6.4)$$

We shall use this recurrence relation to prove Theorem 6.1 for all $(\alpha, \beta, \gamma) \in P'_i$. Remark that for all these (α, β, γ) the properties for the factors α_v , β_v and γ_v given in Lemma 4.4 still hold; this lemma will be used to derive two preliminary results.

If one wants to prove the absolute convergence of a series, i.e. a unary tree, then one can try to do this by proving that for all $k \geq 0$ the k -th term is in absolute value smaller than C^k for some constant $C < 1$; in that case the sum of all terms is smaller than $1/(1-C)$. The analogue of this concept for a binary tree is proving that for all $k \geq 0$ all terms at distance k from the origin are in absolute value smaller than C^k for some constant $C < 1/2$; in that case the sum of the terms at distance k is smaller than $2^k C^k = (2C)^k$ and the sum of all terms is bounded by $1/(1-2C)$. This concept is used to derive a bound for the binary trees $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ with $|\alpha|, |\beta|, |\gamma| < 1/2$.

Consider a formal solution $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$ with $(\alpha, \beta, \gamma) \in P'_i$. Let the real constant C satisfy $C \geq \max[|\alpha|, |\beta|, |\gamma|]$ Remark that if $i=1$ then $|\alpha| < |\beta\gamma|$ and we have $\max[|\alpha|, |\beta|, |\gamma|] = \max[|\beta|, |\gamma|]$; and similarly for the cases $i=2$ and $i=3$. With the help of (i) and (ii) of Lemma 4.2 (property (ii) implies that $|\alpha_v|, |\beta_v|, |\gamma_v| < C$ for all v) and by using induction with respect to $l(v)$ one can show that

$$\begin{aligned} |\beta_v \gamma_v| &\leq C^{l(v)+2} \text{ if } v_{l(v)} = 1, & |\beta_v \gamma_v| &\leq C^{l(v)+3} \text{ if } v_{l(v)} \in \Lambda\{1\}; \\ |\alpha_v \gamma_v| &\leq C^{l(v)+2} \text{ if } v_{l(v)} = 2, & |\alpha_v \gamma_v| &\leq C^{l(v)+3} \text{ if } v_{l(v)} \in \Lambda\{2\}; \\ |\alpha_v \beta_v| &\leq C^{l(v)+2} \text{ if } v_{l(v)} = 3, & |\alpha_v \beta_v| &\leq C^{l(v)+3} \text{ if } v_{l(v)} \in \Lambda\{3\}. \end{aligned}$$

for all $v \in V_i$, where $v_{l(v)} := i$ for $v = \emptyset$. This implies that

$$|\beta_v \gamma_v|, |\alpha_v \gamma_v|, |\alpha_v \beta_v| \leq C^{l(v)+2} \text{ for all } v \in V_i. \quad (6.5)$$

As a consequence, $|\alpha_v^m \beta_v^n \gamma_v^r| \leq C^{l(v)+2}$ for all states (m, n, r) with at most one coordinate equal to zero, i.e. for all states $(m, n, r) \in M_c$, and, if $C < 1/2$, then we find

$$\text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)) \leq 8 \sum_{v \in V_i} |\alpha_v^m \beta_v^n \gamma_v^r| \leq 8C^2 \sum_{v \in V_i} C^{l(v)} = \frac{8C^2}{1-2C},$$

which proves the following lemma.

Lemma 6.1

Let $i \in I$ and $(\alpha, \beta, \gamma) \in P'_i$. Further, let the constant C satisfy $C \geq \max[|\alpha|, |\beta|, |\gamma|]$ and assume $C < 1/2$. Then $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ is absolutely convergent in all states $(m, n, r) \in M_c$ and

$$\text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)) \leq \frac{8C^2}{1-2C}.$$

The upper bound for $\text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma))$ given in Lemma 6.1, together with Lemma 4.4(i) and the recursion in (6.4), is used to prove the second preliminary result, which is needed to prove part (i) of Theorem 6.1.

Lemma 6.2

Let $i \in I$ and $(\alpha, \beta, \gamma) \in P'_i$. Further, assume that $\min[|\beta|, |\gamma|] < 1/2$ if $i=1$, $\min[|\alpha|, |\gamma|] < 1/2$ if $i=2$ and $\min[|\alpha|, |\beta|] < 1/2$ if $i=3$. Then $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ is absolutely convergent in all states $(m, n, r) \in M_C$.

Proof.

It suffices to prove the lemma for a formal solution $x_{m,n,r}^{(1)}(\alpha, \beta, \gamma)$, i.e. for the case $i=1$. W.l.o.g. we may assume $|\beta| \leq |\gamma|$; so $|\beta| < 1/2$. Define the path $\{v^{(k)}\}$ by $v^{(0)} = \emptyset$ and $v^{(k)} = (2, 1, \dots, 2)$ if $k \geq 1$ and k odd, $v^{(k)} = (2, 1, \dots, 2, 1)$ if $k \geq 1$ and k even. Further let the vectors $w^{(k)}$ for all $k \geq 1$ be defined by $w^{(k)} = (2, 1, \dots, 2, 1, 3)$ if k odd, $w^{(k)} = (2, 1, \dots, 2, 3)$ if k even ($w^{(k)}$ follows from $v^{(k-1)}$ by adding a 3). Then by using (6.4) one can show that

$$\begin{aligned} \text{abs}(x_{m,n,r}^{(1)}(\alpha, \beta, \gamma)) &= \sum_{k=0}^{\infty} |(1-\alpha_{v^{(k)}})\alpha_{v^{(k)}}^m (1-\beta_{v^{(k)}})\beta_{v^{(k)}}^n (1-\gamma_{v^{(k)}})\gamma_{v^{(k)}}^r| \\ &\quad + \sum_{k=1}^{\infty} \text{abs}(x_{m,n,r}^{(3)}(\alpha_{w^{(k)}}, \beta_{w^{(k)}}, \gamma_{w^{(k)}})). \end{aligned} \quad (6.6)$$

By using Lemma 4.4(i) and induction with respect to k it is shown that

$$\begin{aligned} |\alpha_{v^{(k)}}| &\leq |\beta| |\gamma|^{k+1} \quad \text{and} \quad |\beta_{v^{(k)}}| \leq |\beta| |\gamma|^k \quad \text{if } k \geq 0 \text{ and } k \text{ even;} \\ |\alpha_{v^{(k)}}| &\leq |\beta| |\gamma|^k \quad \text{and} \quad |\beta_{v^{(k)}}| \leq |\beta| |\gamma|^{k+1} \quad \text{if } k \geq 0 \text{ and } k \text{ odd,} \end{aligned}$$

by which one can easily see that the first series on the right-hand side of (6.6) converges for all $(m, n, r) \in M_C$ (so, $m+n \geq 1$; further, note that $\gamma_{v^{(k)}} = \gamma$ for all k):

$$\begin{aligned} &\sum_{k=0}^{\infty} |(1-\alpha_{v^{(k)}})\alpha_{v^{(k)}}^m (1-\beta_{v^{(k)}})\beta_{v^{(k)}}^n (1-\gamma_{v^{(k)}})\gamma_{v^{(k)}}^r| \\ &= |1-\gamma| |\gamma|^r \sum_{k=0}^{\infty} |(1-\alpha_{v^{(k)}})\alpha_{v^{(k)}}^m (1-\beta_{v^{(k)}})\beta_{v^{(k)}}^n| \\ &\leq 8 \sum_{k=0}^{\infty} |\alpha_{v^{(k)}}^m \beta_{v^{(k)}}^n| \leq 8 \sum_{k=0}^{\infty} (|\beta| |\gamma|^k)^{m+n} \leq 8 \sum_{k=0}^{\infty} |\beta| |\gamma|^k = \frac{8|\beta|}{1-|\gamma|}. \end{aligned}$$

Since $\alpha_{w^{(k)}} = \alpha_{v^{(k-1)}}$ and $\beta_{w^{(k)}} = \beta_{v^{(k-1)}}$ for all $k \geq 1$, we have

$$\max[|\alpha_{w^{(k)}}|, |\beta_{w^{(k)}}|, |\gamma_{w^{(k)}}|] = \max[|\alpha_{v^{(k-1)}}|, |\beta_{v^{(k-1)}}|] \leq |\beta| |\gamma|^{k-1}$$

for all $k \geq 1$. Combining this result with Lemma 6.1 shows that also the second series on the right-hand side of (6.6) converges for all $(m, n, r) \in M_C$:

$$\begin{aligned} \sum_{k=1}^{\infty} \text{abs}(x_{m,n,r}^{(3)}(\alpha_{w^{(k)}}, \beta_{w^{(k)}}, \gamma_{w^{(k)}})) &\leq \sum_{k=1}^{\infty} \frac{8|\beta| |\gamma|^{k-1}}{1-2|\beta| |\gamma|^{k-1}} \\ &\leq \sum_{k=1}^{\infty} \frac{8|\beta| |\gamma|^{k-1}}{1-2|\beta|} = \frac{8|\beta|}{(1-2|\beta|)(1-|\gamma|)}. \end{aligned}$$

As a result, for all $(m, n, r) \in M_C$ the sum $\text{abs}(x_{m,n,r}^{(1)}(\alpha, \beta, \gamma))$ is finite, i.e. $x_{m,n,r}^{(1)}(\alpha, \beta, \gamma)$ converges absolutely. \square

Proof of Theorem 6.1.

Now we are able to prove part (i) of Theorem 6.1. Let $i \in I$ and $(\alpha, \beta, \gamma) \in P'_i$. Define the constant C by $C = \max[|\alpha|, |\beta|, |\gamma|]$. Since $C < 1$, there is an integer $k \geq 0$ such that $C^{1/2k+1} < 1/2$. By repeated application of (6.4), we get

$$\begin{aligned} \text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)) &= \sum_{\substack{v \in V_i \\ l(v) < k}} |(1-\alpha_v)\alpha_v^m (1-\beta_v)\beta_v^n (1-\gamma_v)\gamma_v^r| \\ &\quad + \sum_{\substack{v \in V_i \\ l(v)=k}} \text{abs}(x_{m,n,r}^{(v_{l(v)})}(\alpha, \beta, \gamma)). \end{aligned} \tag{6.7}$$

By (6.5), for all $v \in V_i$ with $l(v)=k$, we have $|\beta_v\gamma_v| \leq C^{k+2}$ and therefore

$$\min[|\beta_v|, |\gamma_v|] \leq \sqrt{C^{k+2}} = C^{1/2k+1} < 1/2;$$

and similarly for $\min[|\alpha_v|, |\gamma_v|]$ and $\min[|\alpha_v|, |\beta_v|]$. So, by Lemma 6.2, for all $(m,n,r) \in M_c$ all terms of the second sum in (6.7) converge. Since this sum, and also the first sum in (6.7), consists of only a finite number of terms, we can conclude that for all $(m,n,r) \in M_c$ the sum $\text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma))$ is finite, i.e. $x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)$ converges absolutely. This completes the proof of Theorem 6.1(i).

Let us now prove the second part of Theorem 6.1. The validity of the first inequality is trivial and the third inequality immediately follows from part (i). The second inequality is proved as follows. From the definition of M_c , it follows that

$$\begin{aligned} \sum_{(m,n,r) \in M_c} \text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)) &= \sum_{(0,n,r) \in M_{(1)}} \text{abs}(x_{0,n,r}^{(i)}(\alpha, \beta, \gamma)) + \sum_{(m,0,r) \in M_{(2)}} \text{abs}(x_{m,0,r}^{(i)}(\alpha, \beta, \gamma)) \\ &\quad + \sum_{(m,n,0) \in M_{(3)}} \text{abs}(x_{m,n,0}^{(i)}(\alpha, \beta, \gamma)) + \sum_{(m,n,r) \in M_o} \text{abs}(x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)). \end{aligned} \tag{6.8}$$

By using Lemma 4.4(ii), one easily derives the following bound for the first sum on the right-hand side of (6.8):

$$\begin{aligned} \sum_{(0,n,r) \in M_{(1)}} \text{abs}(x_{0,n,r}^{(i)}(\alpha, \beta, \gamma)) &= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{v \in V_i} |(1-\alpha_v)(1-\beta_v)\beta_v^n (1-\gamma_v)\gamma_v^r| \\ &= \sum_{v \in V_i} |1-\alpha_v| |1-\beta_v| \frac{|\beta_v|}{1-|\beta_v|} |1-\gamma_v| \frac{|\gamma_v|}{1-|\gamma_v|} \leq \frac{\text{abs}(x_{0,1,1}^{(i)}(\alpha, \beta, \gamma))}{(1-|\beta|)(1-|\gamma|)}. \end{aligned} \tag{6.9}$$

In a similar way one derives bounds for the other sums on the right-hand side of (6.8), after which substitution of all these bounds in (6.8) completes the proof. \square

7. The equilibrium distribution

Due to the projection property, we have been able to prove that each formal solution $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$ converges absolutely in all states of the convergence region M_c , which consists of the interior and the three boundary planes. According to the reasoning given in the

last paragraph of Section 3, this implies that each formal solution $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$ satisfies all equilibrium equations for the states in M_c having no incoming transitions from states outside M_c . As one can easily verify, the only states in M_c which have incoming transitions from states outside M_c , i.e. from states on the axes or from the origin, are the states $(m, 0, 1)$, $(m, 1, 0)$, $(0, n, 1)$, $(1, n, 0)$, $(0, 1, r)$ and $(1, 0, r)$. Let M_c' be the set of all states for which the equilibrium equations are satisfied by a formal solution. Then

$$M_c' = \{ (m_1, m_2, m_3) \in M \mid m_i + m_j \geq 2 \text{ for all } i, j \in I, i \neq j \}.$$

Since each formal solution satisfies the equilibrium equations for all states in M_c' , also each linear combination of formal solutions satisfies the equations for this set M_c' . This gives us some freedom in finding a solution which also satisfies the equilibrium equations for the states in $M \setminus M_c'$, i.e. in finding the equilibrium distribution $\{p_{m,n,r}\}$. Now the question is which formal solutions should be linearly combined, or better, which starting solutions have to be selected. First, due to the projection property, we are able to derive a nice characterization for the starting solutions, from which we learn that each set P_i of starting solutions has an uncountable number of elements. It appears that out of these uncountable sets of candidates only a countable number of starting solutions is needed to obtain a linear combination of formal solutions which also satisfies the equilibrium equations for the countable set $M \setminus M_c'$. For the selection of the appropriate candidates, and also for the choice of the coefficients of the linear combination, we shall use the explicit expressions for the two-dimensional marginal distributions $\{p_{m_1, m_2}^{(i,j)}\}$ (note that we only were able to derive these expressions after the introduction of the projection property). In fact, it is at this point that induction has to be used to extend the main result of this paper to general N . Finally, we remark that the problem of selecting the appropriate starting solutions did not appear in the two-dimensional case, where one gets only a finite number of starting solutions, which all have to be used for the construction of the equilibrium distribution (see [8] and [13]).

Let us start with the derivation of the nice characterization for the starting solutions $(\alpha, \beta, \gamma) \in P_i$. Consider a solution $(\alpha, \beta, \gamma) \in P_1$, i.e. a starting solution on the boundary $m=0$. Such a solution has to satisfy the equilibrium equations (2.1) and (2.2), i.e. (α, β, γ) has to satisfy the quadratic equation (3.1) and the equation

$$\beta\gamma = \sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} \alpha \beta^{1-t_2} \gamma^{1-t_3} + \sum_{(0, t_2, t_3) \in T} q_{0, t_2, t_3}^{(1)} \beta^{1-t_2} \gamma^{1-t_3}, \quad (7.1)$$

which is obtained by substituting the product form $\alpha^m \beta^n \gamma^r$ in (2.2) (see also the definition of $K(\alpha, \beta, \gamma)$ at the beginning of Section 3). Because of the projection property, the rates $q_{0, t_2, t_3}^{(1)}$ in (7.1) may be replaced by $q_{0, t_2, t_3} + q_{-1, t_2, t_3}$. Subsequently, multiplying both sides of (7.1) by α and subtracting (7.1) from both sides of (3.1) leads to

$$0 = \sum_{(1, t_2, t_3) \in T} q_{1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3} - \alpha \sum_{(-1, t_2, t_3) \in T} q_{-1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3},$$

which shows that α has to be equal to $f_1(\beta, \gamma)$. Rewriting (3.1) to a quadratic equation in α (see (3.3)), dividing all terms by α and next substituting $\alpha = f_1(\beta, \gamma)$ shows that β and γ have to satisfy the equation

$$\beta\gamma = \sum_{(t_1, t_2, t_3) \in T} q_{t_1, t_2, t_3} \beta^{1-t_2} \gamma^{1-t_3}, \quad (7.2)$$

which is equivalent to (3.1) for fixed $\alpha=1$. Finally, we have to evaluate the condition $0 < |\alpha| < |\beta\gamma|$. Let (β, γ) be a solution of (7.2) with $0 < |\beta| < 1$ and $0 < |\gamma| < 1$, then for these fixed β and γ the quadratic equation (3.1) has two solutions: $\alpha=1$ and $\alpha=f_1(\beta, \gamma)$. Since $1 > |\beta\gamma|$, according to Lemma 4.1(i), the second root $\alpha=f_1(\beta, \gamma)$ satisfies $0 < |\alpha| < |\beta\gamma|$. This proves part (i) of the following lemma; the other two parts may be proved along the same lines.

Lemma 7.1

- (i) (α, β, γ) is a starting solution on the boundary plane $m=0$, i.e. $(\alpha, \beta, \gamma) \in P_1$, if and only if $\beta^n \gamma^r$ is a solution of the quadratic equation (3.1) for fixed $\alpha=1$ and α is equal to $\alpha=f_1(\beta, \gamma)$;
- (ii) (α, β, γ) is a starting solution on the boundary plane $n=0$, i.e. $(\alpha, \beta, \gamma) \in P_2$, if and only if $\alpha^m \gamma^r$ is a solution of the quadratic equation (3.1) for fixed $\beta=1$ and β is equal to $\beta=f_2(\alpha, \gamma)$;
- (iii) (α, β, γ) is a starting solution on the boundary plane $r=0$, i.e. $(\alpha, \beta, \gamma) \in P_3$, if and only if $\alpha^m \beta^n$ is a solution of the quadratic equation (3.1) for fixed $\gamma=1$ and γ is equal to $\gamma=f_3(\alpha, \beta)$.

When reading part (i) of Lemma 7.1, we realize the following. Since the quadratic equation (3.1) for fixed $\alpha=1$ is equivalent to the quadratic equation for the two-dimensional random walk describing the behavior for the components n and r (see (5.6) and Figure 4), all product forms present in formula (5.4) for $\{p_{n,r}^{(2,3)}\}$ may be extended to starting solutions on the boundary $m=0$. So, what comes up in our mind now is that we have to do something with the product forms of the marginal distributions $\{p_{m_1, m_2}^{(i,j)}\}$. This thought is strengthened by the remarkable property of the formal solutions $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$ described in the next paragraph.

Because of the projection property we were able to derive the explicit formula (6.2) for a_ν, b_ν, c_ν , which has led to the simplified formula (6.3) for $\{x_{m,n,r}^{(i)}(\alpha, \beta, \gamma)\}$. Formula (6.3) shows that each formal solution is a kind of alternating sum of product-form distributions. As a consequence, for each formal solution two terms with the same values for two of the three factors α_ν, β_ν and γ_ν vanish when taking the summation over the coordinate belonging to the third factor. For example, for a formal solution $\{x_{m,n,r}^{(1)}(\alpha, \beta, \gamma)\}$ two terms with the same β - and γ -factor vanish if we take the summation of $x_{m,n,r}^{(1)}(\alpha, \beta, \gamma)$ over $m=0$ to ∞ , by which

$$\begin{aligned} \sum_{m=0}^{\infty} x_{m,n,r}^{(1)}(\alpha, \beta, \gamma) &= \sum_{m=0}^{\infty} \left[(1-\alpha_\emptyset) \alpha_\emptyset^m (1-\beta_\emptyset) \beta_\emptyset^n (1-\gamma_\emptyset) \gamma_\emptyset^r \right. \\ &\quad \left. + \sum_{\substack{\nu \in V_1 \setminus \{\emptyset\} \\ \nu_{l(\nu)}=1}} (-1)^{l(\nu)} [(1-\alpha_{p(\nu)}) \alpha_{p(\nu)}^m - (1-\alpha_\nu) \alpha_\nu^m] (1-\beta_{p(\nu)}) \beta_{p(\nu)}^n (1-\gamma_{p(\nu)}) \gamma_{p(\nu)}^r \right] \\ &= (1-\beta) \beta^n (1-\gamma) \gamma^r \quad \text{for all } n, r \geq 1. \end{aligned}$$

Here, the last equality is found after changing summations, which is allowed by Theorem 6.1(ii). The first term of $x_{m,n,r}^{(1)}(\alpha, \beta, \gamma)$ does not vanish when summing over m , since this term does not have a companion term with the same β - and γ -factor. When summing over n , all

terms have a companion term with the same α - and γ -factor, by which

$$\begin{aligned} & \sum_{n=0}^{\infty} x_{m,n,r}^{(1)}(\alpha, \beta, \gamma) \\ &= \sum_{n=0}^{\infty} \sum_{\substack{v \in V_1 \setminus \{\emptyset\} \\ v_{l(v)}=2}} (-1)^{l(p(v))} [(1-\beta_{p(v)})\beta_{p(v)}^n - (1-\beta_v)\beta_v^n] (1-\alpha_{p(v)})\alpha_{p(v)}^n (1-\gamma_{p(v)})\gamma_{p(v)}^n \\ &= 0 \quad \text{for all } m, r \geq 1; \end{aligned}$$

and similarly when summing over r . This proves Lemma 7.2 for $i=1$; the cases with $i=2$ or 3 are treated in a similar way.

Lemma 7.2

Let $i, j \in I$ and $(\alpha_1, \alpha_2, \alpha_3) \in P_i$. Then

$$\sum_{m_j=0}^{\infty} x_{m_1, m_2, m_3}^{(i)}(\alpha_1, \alpha_2, \alpha_3) = \begin{cases} \prod_{l \in \Lambda\{i\}} (1-\alpha_l)\alpha_l^{m_l} & \text{if } j=i; \\ 0 & \text{if } j \neq i, \end{cases}$$

for all $m_l \geq 1, l \in \Lambda\{j\}$.

Together with the expressions for the two-dimensional marginal distributions $\{p_{m_1, m_2}^{(i, j)}\}$, the results stated in the Lemmas 7.1 and 7.2 suggest which starting solutions have to be selected and how the coefficients of the linear combination of the corresponding formal solutions should be chosen to obtain the equilibrium distribution $\{p_{m,n,r}\}$. We shall now first define the suggested linear combination, whereafter it will be shown that the suggested linear combination indeed satisfies all equilibrium equations.

Combining the results of the Lemmas 7.1 and 7.2 gives us the idea to define a linear combination of all formal solutions with starting solutions coming from the product forms in formula (5.4) for the marginal distribution $\{p_{n,r}^{(2,3)}\}$. Define $\alpha_k^{(1)} = f_1(\beta_k^{(1)}, \gamma_k^{(1)})$ and $\hat{\alpha}_k^{(1)} = f_1(\hat{\beta}_k^{(1)}, \hat{\gamma}_k^{(1)})$ for all $k \geq 0$, then, by Lemma 7.1(i), all solutions $(\alpha_k^{(1)}, \beta_k^{(1)}, \gamma_k^{(1)})$ and $(\hat{\alpha}_k^{(1)}, \hat{\beta}_k^{(1)}, \hat{\gamma}_k^{(1)})$ are starting solutions on the boundary plane $m=0$. Next, defining

$$y_{m,n,r}^{(1)} = \sum_{k=0}^{\infty} (-1)^k x_{m,n,r}^{(1)}(\alpha_k^{(1)}, \beta_k^{(1)}, \gamma_k^{(1)}) + \sum_{k=0}^{\infty} (-1)^k x_{m,n,r}^{(1)}(\hat{\alpha}_k^{(1)}, \hat{\beta}_k^{(1)}, \hat{\gamma}_k^{(1)}), \quad (m, n, r) \in M_C,$$

gives us a solution $\{y_{m,n,r}^{(1)}\}$, for which, by Lemma 7.2,

$$\begin{aligned} \sum_{m=0}^{\infty} y_{m,n,r}^{(1)} &= \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty} x_{m,n,r}^{(1)}(\alpha_k^{(1)}, \beta_k^{(1)}, \gamma_k^{(1)}) + \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty} x_{m,n,r}^{(1)}(\hat{\alpha}_k^{(1)}, \hat{\beta}_k^{(1)}, \hat{\gamma}_k^{(1)}) \\ &= p_{n,r}^{(2,3)} \quad \text{for all } n, r \geq 1. \end{aligned}$$

So, when summing $\{y_{m,n,r}^{(1)}\}$ over the m -component, one gets the marginal distribution for the other two components. This indicates that we are on the right track with our search for $\{p_{m,n,r}\}$, since this property is satisfied by the equilibrium distribution $\{p_{m,n,r}\}$ by definition (see the definition of the marginal distributions $\{p_{m_1, m_2}^{(i, j)}\}$ in Section 5). Summing $\{y_{m,n,r}^{(1)}\}$

over n or r leads to:

$$\sum_{n=0}^{\infty} y_{m,n,r}^{(1)} = 0 \text{ for all } m, r \geq 1, \quad \sum_{r=0}^{\infty} y_{m,n,r}^{(1)} = 0 \text{ for all } m, n \geq 1.$$

As we see, in this case the result is zero instead of a marginal probability. Fortunately, this is corrected by adding linear combinations of formal solutions $\{x_{m,n,r}^{(2)}(\alpha, \beta, \gamma)\}$ and $\{x_{m,n,r}^{(3)}(\alpha, \beta, \gamma)\}$.

The definition and properties for $\{y_{m,n,r}^{(1)}\}$ are easily extended to solutions $\{y_{m,n,r}^{(i)}\}$, $i \in I$.

Let

$$\begin{aligned} \alpha_k^{(1)} &= f_1(\beta_k^{(1)}, \gamma_k^{(1)}), & \hat{\alpha}_k^{(1)} &= f_1(\hat{\beta}_k^{(1)}, \hat{\gamma}_k^{(1)}); \\ \beta_k^{(2)} &= f_2(\alpha_k^{(2)}, \gamma_k^{(2)}), & \hat{\beta}_k^{(2)} &= f_2(\hat{\alpha}_k^{(2)}, \hat{\gamma}_k^{(2)}); \\ \gamma_k^{(3)} &= f_3(\alpha_k^{(3)}, \beta_k^{(3)}), & \hat{\gamma}_k^{(3)} &= f_3(\hat{\alpha}_k^{(3)}, \hat{\beta}_k^{(3)}), \end{aligned}$$

for all $k \geq 0$ and let the solutions $\{y_{m,n,r}^{(i)}\}$, $i \in I$, be defined by

$$y_{m,n,r}^{(i)} = \sum_{k=0}^{\infty} (-1)^k x_{m,n,r}^{(i)}(\alpha_k^{(i)}, \beta_k^{(i)}, \gamma_k^{(i)}) + \sum_{k=0}^{\infty} (-1)^k x_{m,n,r}^{(i)}(\hat{\alpha}_k^{(i)}, \hat{\beta}_k^{(i)}, \hat{\gamma}_k^{(i)}), \quad (m, n, r) \in M_c.$$

Then for all $i, j \in I$, $k, l \in I \setminus \{i\}$, $k < l$, and all $m_k, m_l \geq 1$, it holds that

$$\sum_{m_j=0}^{\infty} y_{m_1, m_2, m_3}^{(i)} = \begin{cases} p_{m_k, m_l}^{(k, l)} & \text{if } j = i; \\ 0 & \text{if } j \neq i. \end{cases} \quad (7.3)$$

Obviously, the solution $\{y_{m,n,r}\}$ defined as the sum of the solutions $\{y_{m,n,r}^{(i)}\}$, i.e.

$$y_{m,n,r} = \sum_{i \in I} y_{m,n,r}^{(i)}, \quad (m, n, r) \in M_c,$$

satisfies the desired property: for all $i \in I$, $k, l \in I \setminus \{i\}$, $k < l$, it holds that

$$\sum_{m_i=0}^{\infty} y_{m_1, m_2, m_3} = p_{m_k, m_l}^{(k, l)} \quad \text{for all } m_k, m_l \geq 1. \quad (7.4)$$

Before continuing, we remark that the solution $\{y_{m,n,r}\}$, so far being defined for all states (m, n, r) in the convergence region M_c , is well-defined, since all six series constituting $\{y_{m,n,r}\}$ are absolutely convergent for all states in M_c :

$$\begin{aligned} \sum_{k=0}^{\infty} |x_{m,n,r}^{(i)}(\alpha_k^{(i)}, \beta_k^{(i)}, \gamma_k^{(i)})| &< \infty \quad \text{and} \\ \sum_{k=0}^{\infty} |x_{m,n,r}^{(i)}(\hat{\alpha}_k^{(i)}, \hat{\beta}_k^{(i)}, \hat{\gamma}_k^{(i)})| &< \infty, \quad (m, n, r) \in M_c, \quad i \in I. \end{aligned} \quad (7.5)$$

For $\sum_{k=0}^{\infty} x_{m,n,r}^{(1)}(\alpha_k^{(1)}, \beta_k^{(1)}, \gamma_k^{(1)})$, the absolute convergence is proved by using the bound given in Lemma 6.1 and the property that the factors $\beta_k^{(1)}$ and $\gamma_k^{(1)}$ decrease monotonically and exponentially fast (see (5.7)); and similarly for the other series. Further, we have to remark that the properties stated in (7.3) and (7.4) have been derived after changing summations; this was allowed, since

$$\sum_{(m,n,r) \in M_c} \sum_{k=0}^{\infty} |x_{m,n,r}^{(i)}(\alpha_k^{(i)}, \beta_k^{(i)}, \gamma_k^{(i)})| < \infty \quad \text{and}$$

$$\sum_{(m,n,r) \in M_c} \sum_{k=0}^{\infty} |x_{m,n,r}^{(i)}(\hat{\alpha}_k^{(i)}, \hat{\beta}_k^{(i)}, \hat{\gamma}_k^{(i)})| < \infty, \quad i \in I, \quad (7.6)$$

which is proved by using (7.5), Theorem 6.1(ii) and the property that all factors of the starting solutions $(\alpha_k^{(i)}, \beta_k^{(i)}, \gamma_k^{(i)})$ and $(\hat{\alpha}_k^{(i)}, \hat{\beta}_k^{(i)}, \hat{\gamma}_k^{(i)})$ decrease monotonically and exponentially fast.

The solution $\{y_{m,n,r}\}$ defined for all states $(m,n,r) \in M_c$ up to now, satisfies two properties. In the first place, since $\{y_{m,n,r}\}$ is a linear combination of formal solutions, $\{y_{m,n,r}\}$ satisfies the equilibrium equations for all states $(m,n,r) \in M_c'$. Secondly, $\{y_{m,n,r}\}$ satisfies (7.4). Now, define $y_{m,n,r}$ on the m -axis by

$$y_{m,0,0} = p_m^{(1)} - \sum_{\substack{n,r \geq 0 \\ n+r \geq 1}} y_{m,n,r} \quad \text{for all } m \geq 1,$$

and similarly for the n -axis and r -axis. Finally, define $y_{0,0,0}$ by

$$y_{0,0,0} = 1 - \sum_{(m_1, m_2, m_3) \in M \setminus \{(0,0,0)\}} y_{m_1, m_2, m_3}$$

(use (7.6) to show the correctness of these definitions, i.e. to show that the series at the right-hand sides are absolutely convergent). Then $\{y_{m,n,r}\}$ may be shown to satisfy (7.4) also for $m_k = 0$ and/or $m_l = 0$ (see Lemma 7.3), whereafter we are able to finish the proof that $\{y_{m,n,r}\}$ also satisfies the equilibrium equations for the states outside M_c' . From this, we may conclude that $\{y_{m,n,r}\}$ equals the equilibrium distribution $\{p_{m,n,r}\}$.

Lemma 7.3

Let $i \in I$ and $k, l \in \Lambda\{i\}$, $k < l$. Then

$$\sum_{m_i=0}^{\infty} y_{m_1, m_2, m_3} = p_{m_k, m_l}^{(k,l)} \quad \text{for all } m_k, m_l \geq 0.$$

Proof.

The result stated in (7.4) is extended in two steps. In the first step (7.4) is extended to

$$\sum_{m_i=0}^{\infty} y_{m_1, m_2, m_3} = p_{m_k, m_l}^{(k,l)} \quad \text{for all } m_k, m_l \geq 0, \quad m_k + m_l \geq 1, \quad (7.7)$$

where $i \in I$ and $k, l \in \Lambda\{i\}$, $k < l$. This extension is proved by rewriting the expressions for $y_{m,n,r}$ on the axes. For example, for the case $i = 1$, so $k = 2$ and $l = 3$, we may rewrite $y_{0,n,0}$ for all $n \geq 1$ as (use (7.4))

$$\begin{aligned} y_{0,n,0} &= p_n^{(2)} - \sum_{\substack{m,r \geq 0 \\ m+r \geq 1}} y_{m,n,r} = p_n^{(2)} - \sum_{m=1}^{\infty} y_{m,n,0} - \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} y_{m,n,r} \\ &= p_n^{(2)} - \sum_{m=1}^{\infty} y_{m,n,0} - \sum_{r=1}^{\infty} p_{n,r}^{(2,3)} = p_{n,0}^{(2,3)} - \sum_{m=1}^{\infty} y_{m,n,0}, \end{aligned}$$

which proves that $\sum_{m=0}^{\infty} y_{m,n,0} = p_{n,0}^{(2,3)}$ for all $n \geq 1$; rewriting $y_{0,0,r}$ for all $r \geq 1$ proves the extension of (7.4) for the case $n=0$ and $r \geq 1$. In the second step (7.7) is extended to the result stated in Lemma 7.3; this extension is proved by rewriting $y_{0,0,0}$. \square

In the final part of the proof of the Main Theorem we have to show that $\{y_{m,n,r}\}$ also satisfies the equilibrium equations for the states outside M'_c . For this we shall use the *balance principle*:

$$\text{the rate out of a set } M' = \text{the rate into this set } M', \quad M' \subset M. \quad (7.8)$$

Obviously, for a subset M' consisting of a single state the balance principle is equivalent to the equilibrium equation for that state. Therefore $\{y_{m,n,r}\}$ satisfies (7.8) for all states of M'_c . Further, by Lemma 7.3, $\{y_{m,n,r}\}$ satisfies (7.8) for all subsets of the form

$$M' = \{ (n_1, n_2, n_3) \in M \mid n_i \geq 0 \text{ and } n_j = m_j \text{ for all } j \in \Lambda\{i\} \}, \quad (7.9)$$

where $i \in I$ and $m_j \geq 0, j \in \Lambda\{i\}$, since for such a subset the balance principle is equivalent to the equilibrium equation in the state (m_k, m_l) of one of the two-dimensional marginal random walk describing the behavior for the components m_k and $m_l, k, l \in \Lambda\{i\}, k < l$. For example, for the subset $M' = \{(m, n, r) \mid m \geq 0\}$ with fixed $n, r \geq 1$ the balance principle is equivalent to (take the sum of (2.1) over $m \geq 1$ and add (2.2), after having replaced $q_{0,t_2,t_3}^{(1)}$ by $q_{0,t_2,t_3} + q_{-1,t_2,t_3}$)

$$p_{n,r}^{(2,3)} = \sum_{t_1, t_2 \in \{-1, 0, 1\}} q_{t_2, t_3}^{(2,3)} p_{n-t_2, r-t_3}^{(2,3)},$$

which is the equilibrium equation for the state (n, r) of the random walk describing the behavior for the last two components (see Figure 4). Now, by considering differences $M_1 \setminus M_2$ with $M_2 \subset M_1$ of the sets given in (7.9) and sets consisting of states of M'_c , $\{y_{m,n,r}\}$ may be shown to satisfy also the equilibrium equations for the states outside M'_c . For example, for all $m \geq 1$, (7.8) is satisfied for the set $M_1 = \{(m, n, 1) \mid n \geq 0\}$ (see (7.9)) and (7.8) is satisfied for the set $M_2 = \{(m, n, 1) \mid n \geq 1\}$, since $\{y_{m,n,r}\}$ satisfies the balance principle for every state of this set. Therefore, $\{y_{m,n,r}\}$ also satisfies the balance principle (7.8) for $M_1 \setminus M_2 = \{(m, 0, 1)\}$ (since (7.8) for $M_1 \setminus M_2$ is obtained by subtracting (7.8) for M_2 from (7.8) for M_1). This proves that $\{y_{m,n,r}\}$ also satisfies the equilibrium equations for the states $(m, 0, 1), m \geq 1$. One can easily check that all other states outside M'_c may be treated in a similar way. Hence we may conclude that $\{y_{m,n,r}\}$ satisfies all equilibrium equations. Since, by the definition of $y_{0,0,0}$, the solution $\{y_{m,n,r}\}$ already adds up to one, this completes the proof of the Main Theorem (cf. the main result stated in [32] for the symmetric case).

Theorem 7.1 (Main Theorem)

$$p_{m,n,r} = y_{m,n,r} \quad \text{for all } (m,n,r) \in M.$$

The Main Theorem states that the equilibrium distribution $\{p_{m,n,r}\}$, restricted to the interior and the boundary planes, may be written as the sum of six alternating series of alternating binary trees of product-form distributions. Looking in less detail, we can say that $\{p_{m,n,r}\}$ may be written as one (alternating) sum of product-form distributions. As we know, all these

product forms, and also all product forms appearing in the formulae for the marginal distributions, are solutions of the equilibrium equation for the interior, i.e. of the quadratic equation (3.1). By considering the definitions of all factors, we may conclude that all these product forms are obtained by taking the trivial solution (1, 1, 1) of (3.1) and generating new solutions of (3.1) by letting one factor free each time. The tree of solutions which we get in this way is depicted in Figure 5.

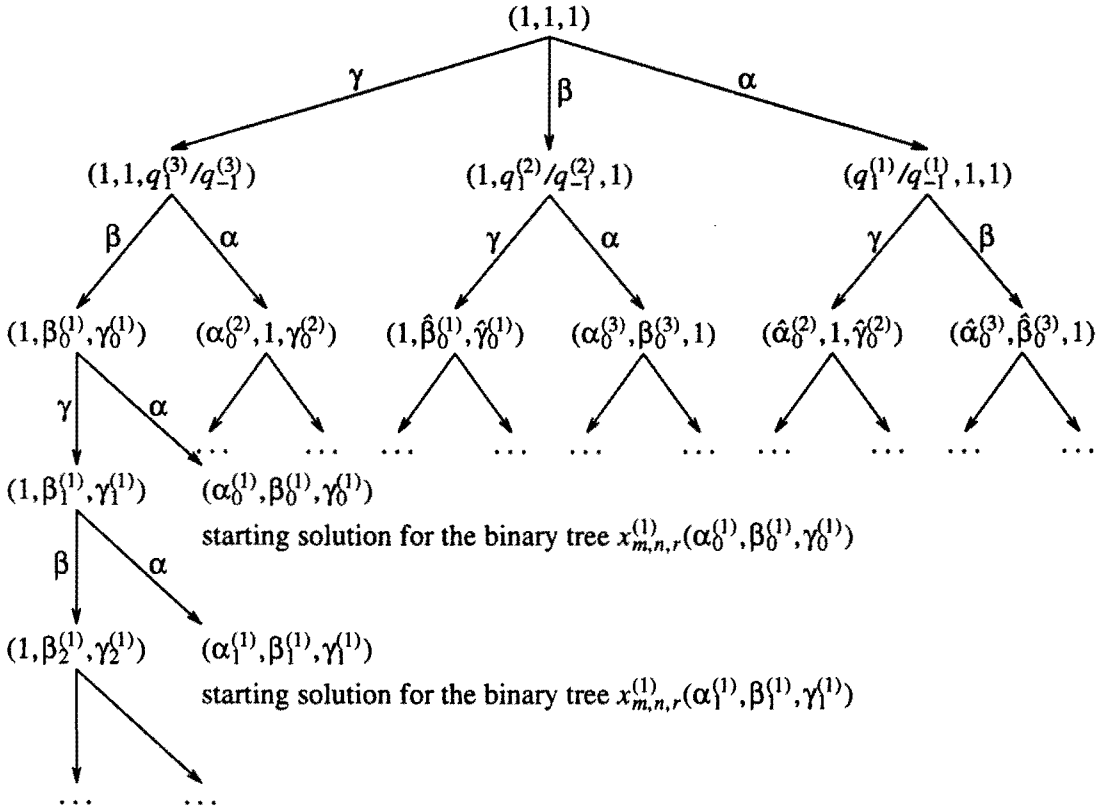


Figure 5. All relevant solutions of (3.1) needed for the equilibrium distribution $\{p_{m,n,r}\}$ and all its marginal distributions.

Using this tree of product forms enables us to give more compact formulae for $\{p_{m,n,r}\}$ and its marginal distributions. Let V be the set of vectors given in Section 3. Define $(\alpha_\emptyset, \beta_\emptyset, \gamma_\emptyset) = (1, 1, 1)$ and let for all other vectors $v \in V$ the factors of $(\alpha_v, \beta_v, \gamma_v)$ be defined by

$$\begin{aligned} \alpha_v &= f_1(\beta_{p(v)}, \gamma_{p(v)})/\alpha_{p(v)}, & \beta_v &= \beta_{p(v)}, & \gamma_v &= \gamma_{p(v)} & \text{if } v_{l(v)} = 1; \\ \alpha_v &= \alpha_{p(v)}, & \beta_v &= f_2(\alpha_{p(v)}, \gamma_{p(v)})/\beta_{p(v)}, & \gamma_v &= \gamma_{p(v)} & \text{if } v_{l(v)} = 2; \\ \alpha_v &= \alpha_{p(v)}, & \beta_v &= \beta_{p(v)}, & \gamma_v &= f_3(\alpha_{p(v)}, \beta_{p(v)})/\gamma_{p(v)} & \text{if } v_{l(v)} = 3. \end{aligned}$$

Then the set of all solutions depicted in Figure 5 is given by

$$P^* = \{ (\alpha_v, \beta_v, \gamma_v) \mid v \in V \}.$$

For each solution $(\alpha_v, \beta_v, \gamma_v)$ in this set, all factors are real numbers in the interval (0, 1] and

therefore P^* may be partitioned into the subsets

$$P_J^* = \{ (\alpha_1, \alpha_2, \alpha_3) \in P^* \mid \alpha_i < 1 \text{ for all } i \in J \text{ and } \alpha_i = 1 \text{ for all } i \notin J \}, \quad J \subset I.$$

As one can easily check, for the marginal distribution $\{p_m^{(1)}\}$ only the unique solution $(\alpha_v, \beta_v, \gamma_v) \in P^*$ with $\alpha_v < 1$ and $\beta_v = \gamma_v = 1$ is needed:

$$p_m^{(1)} = \sum_{(\alpha_v, \beta_v, \gamma_v) \in P_{\{1\}}^*} (-1)^{l(v)-1} (1-\alpha_v) \alpha_v^m = (1-\alpha_{(1)}) \alpha_{(1)}^m, \quad m \geq 0;$$

and similarly for $\{p_n^{(2)}\}$ and $\{p_r^{(3)}\}$. Considering the formula for $\{p_{m,n}^{(1,2)}\}$ shows that $\{p_{m,n}^{(1,2)}\}$ consists of the product forms $\pm (1-\alpha_v) \alpha_v^m (1-\beta_v) \beta_v^n$ with $(\alpha_v, \beta_v, \gamma_v) \in P^*$ and $\alpha_v < 1$, $\beta_v < 1$ and $\gamma_v = 1$, where the sign depends on the distance between the node v and the node \emptyset :

$$p_{m,n}^{(1,2)} = \sum_{(\alpha_v, \beta_v, \gamma_v) \in P_{\{1,2\}}^*} (-1)^{l(v)-2} (1-\alpha_v) \alpha_v^m (1-\beta_v) \beta_v^n, \quad m, n \geq 0, m+n \geq 1;$$

and similarly for $\{p_{m,r}^{(1,3)}\}$ and $\{p_{m,n}^{(1,2)}\}$. Finally, for the equilibrium distribution $\{p_{m,n,r}\}$ all solutions $(\alpha_v, \beta_v, \gamma_v) \in P^*$ with $\alpha_v < 1$, $\beta_v < 1$ and $\gamma_v < 1$ are needed:

$$p_{m,n,r} = \sum_{(\alpha_v, \beta_v, \gamma_v) \in P^*} (-1)^{l(v)-3} (1-\alpha_v) \alpha_v^m (1-\beta_v) \beta_v^n (1-\gamma_v) \gamma_v^r, \quad (m, n, r) \in M_c. \quad (7.10)$$

Together with the expressions for the equilibrium probabilities for the axes and the origin, this formula represents an alternative formulation of the Main Theorem. In Section 9 it is shown that the above formulae may easily be extended to random walks with dimension four or more.

8. Numerical analysis and some results

After having found the explicit expressions for the equilibrium probabilities in the previous section, the question arises how the probabilities and functions of them may be efficiently computed. In this section we shortly describe three procedures: a simple procedure, a simple procedure with bounds and a sophisticated procedure. For a more extensive description the reader is referred to [33].

Suppose that we want to compute the equilibrium probabilities $p_{m,n,r}$ for all states (m, n, r) lying sufficiently close to the origin, for example for all states with m, n and r smaller than or equal to some threshold. Then one may do this by first computing the equilibrium probabilities for the states in the convergent region, whereafter the probabilities for the axes and the origin may be computed with the help of appropriately chosen equilibrium equations.

Let us focus on the computation of an equilibrium probability $p_{m,n,r}$ with $(m, n, r) \in M_c$. An expression for this probability is given by formula (7.10). An equivalent, but more appropriate formula for computational purposes is given by

$$p_{m,n,r} = \sum_{v \in V} 1_{\{\alpha_v, \beta_v, \gamma_v < 1\}} (-1)^{l(v)-3} (1-\alpha_v) \alpha_v^m (1-\beta_v) \beta_v^n (1-\gamma_v) \gamma_v^r, \quad (m, n, r) \in M_c, \quad (8.1)$$

where $1_{\{\alpha_v, \beta_v, \gamma_v < 1\}}$ is equal to 1 if all three factors α_v , β_v and γ_v are smaller than 1 and equal to 0 else. Both (7.10) as well as (8.1) show that $p_{m,n,r}$ is equal to a sum consisting of an

infinite number of terms, which, for numerical purposes, has to be truncated in some way. By Lemma 4.4 (and the behavior of the factors $\alpha_k^{(i)}$, $\beta_k^{(i)}$, $\gamma_k^{(i)}$ and $\hat{\alpha}_k^{(i)}$, $\hat{\beta}_k^{(i)}$, $\hat{\gamma}_k^{(i)}$), for each $v \in V \setminus \{\emptyset\}$ the factors α_v , β_v , γ_v are smaller than or equal to the factors $\alpha_{p(v)}$, $\beta_{p(v)}$, $\gamma_{p(v)}$ of the parent $p(v)$. So, the sum is dominated by the terms corresponding to short vectors $v \in V$ and therefore $p_{m,n,r}$ may be easily computed within a given relative or absolute accuracy by approximating the infinite sum by finite sums consisting of dominant terms. The main distinction between the simple procedure (with or without bounds) and the sophisticated one consists of the way in which dominant terms are selected.

In the *simple procedure*, one computes finite (partial) sums $p_{m,n,r}(d)$ consisting of all terms up to and including depth d , i.e. all terms corresponding to vectors v with length $l(v) \leq d$:

$$p_{m,n,r}(d) = \sum_{v \in V(d)} 1_{\{\alpha_v, \beta_v, \gamma_v < 1\}} (-1)^{l(v)-3} (1-\alpha_v) \alpha_v^m (1-\beta_v) \beta_v^n (1-\gamma_v) \gamma_v^r, \quad (8.2)$$

where $V(d) = \{v \in V \mid l(v) \leq d\}$, $d \in \mathbb{N}_0$. Since $\{p_{m,n,r}(d)\}_{d \in \mathbb{N}_0}$ constitutes an (alternating) sequence with limit $p_{m,n,r}$, the probability $p_{m,n,r}$ may be determined by successively computing $p_{m,n,r}(d)$ for $d=0, 1, \dots$. This computing process should be stopped as soon as for some d the relative or absolute error of $p_{m,n,r}(d)$ with respect to $p_{m,n,r}$ is sufficiently small. An indication of the magnitude of this error is given by the difference between $p_{m,n,r}(d)$ and $p_{m,n,r}(d-1)$.

In the *simple procedure with bounds*, one uses a bound for $|p_{m,n,r} - p_{m,n,r}(d)|$ to determine when the computing process has to be stopped. Such a bound is given by

$$\begin{aligned} |p_{m,n,r} - p_{m,n,r}(d)| &\leq \sum_{v \in V \setminus V(d)} |1_{\{\alpha_v, \beta_v, \gamma_v < 1\}} (-1)^{l(v)-3} (1-\alpha_v) \alpha_v^m (1-\beta_v) \beta_v^n (1-\gamma_v) \gamma_v^r| \\ &\leq \sum_{v \in V \setminus V(d)} \alpha_v^m \beta_v^n \gamma_v^r \\ &= \sum_{v \in V, l(v)=d} b_{m,n,r}(v), \end{aligned} \quad (8.3)$$

where $b_{m,n,r}(v)$ is the sum of the terms $\alpha_w^m \beta_w^n \gamma_w^r$ over all vectors w corresponding to the nodes, except v itself, of the subtree starting at node v :

$$b_{m,n,r}(v) = \sum_{w \in S(v) \setminus \{v\}} \alpha_w^m \beta_w^n \gamma_w^r$$

with

$$S(v) = \{w \in V \mid l(w) \geq l(v) \text{ and } w_k = v_k \text{ for all } k = 1, \dots, l(v)\}, \quad v \in V.$$

In fact, $b_{m,n,r}(v)$ gives an upper bound for the absolute error when approximating the sum over all terms of a subtree $S(v)$ by its dominant term corresponding to v .

Instead of being able to compute $b_{m,n,r}(v)$, we are able to derive a computable upper bound which also suffices for our purpose. To find such an upper bound we consider ratios of connected terms of a subtree $S(v)$. For a vector $w \in S(v)$, $w \neq v$, the ratio of the term corresponding to w and the term corresponding to its parent $p(w)$ is given by

$$\frac{\alpha_w^m \beta_w^n \gamma_w^r}{\alpha_{p(w)}^m \beta_{p(w)}^n \gamma_{p(w)}^r} = \begin{cases} (\alpha_w / \alpha_{p(w)})^m = (h_1(\beta_w, \gamma_w))^m & \text{if } w_{l(w)} = 1; \\ (\beta_w / \beta_{p(w)})^n = (h_2(\alpha_w, \gamma_w))^n & \text{if } w_{l(w)} = 2; \\ (\gamma_w / \gamma_{p(w)})^r = (h_3(\alpha_w, \beta_w))^r & \text{if } w_{l(w)} = 3, \end{cases}$$

where the function $h_1(\beta, \gamma)$ is defined as the ratio of the smallest root α and the largest root α of the quadratic equation (3.1) for fixed β and γ with $\beta, \gamma \in [0, 1]$ (since both roots may be shown to be real, $h_1(\beta, \gamma)$ is a real valued function); and the functions $h_2(\alpha, \gamma)$ and $h_3(\alpha, \beta)$ are defined similarly. Since the functions $h_i(\cdot, \cdot)$ may be shown to be increasing in both arguments and since, by Lemma 4(ii), all factors α_w, β_w and γ_w are smaller than α_v, β_v and γ_v , we find that

$$\frac{\alpha_w^m \beta_w^n \gamma_w^r}{\alpha_{p(w)}^m \beta_{p(w)}^n \gamma_{p(w)}^r} \leq \begin{cases} (h_1(\beta_v, \gamma_v))^m & \text{if } w_{l(w)} = 1; \\ (h_2(\alpha_v, \gamma_v))^n & \text{if } w_{l(w)} = 2; \\ (h_3(\alpha_v, \beta_v))^r & \text{if } w_{l(w)} = 3. \end{cases}$$

Define $x_1 = (h_1(\beta_v, \gamma_v))^m$, $x_2 = (h_2(\alpha_v, \gamma_v))^n$ and $x_3 = (h_3(\alpha_v, \beta_v))^r$, then by induction one may show that

$$\frac{\alpha_w^m \beta_w^n \gamma_w^r}{\alpha_v^m \beta_v^n \gamma_v^r} \leq y_w \quad \text{for all } w \in S(v),$$

where $y_w := 1$ for $w = v$ and for all $w \in S(v) \setminus \{v\}$ the terms y_w are recursively defined by

$$y_w := \begin{cases} x_1 y_{p(w)} & \text{if } w_{l(w)} = 1; \\ x_2 y_{p(w)} & \text{if } w_{l(w)} = 2; \\ x_3 y_{p(w)} & \text{if } w_{l(w)} = 3. \end{cases}$$

As a consequence,

$$b_{m,n,r}(v) \leq \hat{b}_{m,n,r}(v), \tag{8.4}$$

where

$$\hat{b}_{m,n,r}(v) := \alpha_v^m \beta_v^n \gamma_v^r \sum_{w \in S(v) \setminus \{v\}} y_w.$$

As one can easily see, the values of the terms y_w only depend on x_1, x_2 and x_3 . The sum of all y_w also depends on the kind of subtree $S(v)$ we have. In case $v \neq \emptyset$, the subtree $S(v)$ is a binary tree (if $v = \emptyset$, then v itself has three successors) and the sum of all y_w further only depends on $v_{l(v)}$; the value of $v_{l(v)}$ determines which factor x_i is not used when computing the terms y_w for the successors $w \in O(v)$ of v . Let $G^{(i)}(x_1, x_2, x_3)$ denote the sum of all terms y_w of a subtree $S(v)$ with $v \neq \emptyset$ and $v_{l(v)} = i$. Then, by using elementary algebra, one may prove that the geometric tree $G^{(i)}(x_1, x_2, x_3)$ converges absolutely if and only if

$$d(x_1, x_2, x_3) := 1 - (x_1 x_2 + x_1 x_3 + x_2 x_3 + 2x_1 x_2 x_3) > 0, \tag{8.5}$$

and if (8.5) is satisfied then

$$G^{(i)}(x_1, x_2, x_3) = \sum_{w \in S(v)} y_w = \frac{1}{d(x_1, x_2, x_3)} \prod_{k \in \Lambda\{i\}} (1 + x_k). \tag{8.6}$$

Combining (8.4) and (8.6) shows that $b_{m,n,r}(v)$ is bounded by

$$\hat{b}_{m,n,r}(v) = \alpha_v^m \beta_v^n \gamma_v^r (G^{(v_{l(v)})}(x_1, x_2, x_3) - 1), \quad v \in V \setminus \{\emptyset\}, \tag{8.7}$$

provided that $d(x_1, x_2, x_3) > 0$. Together with (8.3) this bound is used in the simple

procedure with bounds to get an upper bound for the relative or absolute error of $p_{m,n,r}(d)$, $d \geq 1$.

In the simple procedure each time the selection of new terms of the sum in (8.1) is based on the distance of the indices ν of the terms up to the root \emptyset . In the *sophisticated procedure* it is tried to select the dominant terms of (8.1) in a more sophisticated way. This procedure computes the sum in (8.1) within a given absolute accuracy by truncating subtrees at different depths. A set V' stores the nodes where the computing process still has to be continued. This set is initiated by the set of nodes at depth 1. For each node $\nu \in V'$ one stores the solution $(\alpha_\nu, \beta_\nu, \gamma_\nu)$ and the bound $\hat{b}_{m,n,r}(\nu)$, i.e. a bound for the absolute error when approximating the sum of all terms, except ν itself, of the binary subtree starting at ν . Further the absolute accuracy is stored with which the subtree starting at ν has to be computed. In the initialization step the initial allowed inaccuracy is spread over the three nodes at depth 1 proportionally to the values of their bounds $\hat{b}_{m,n,r}(\nu)$ (provided that each node gets at least 5%). In each next step, one selects a node ν of the set V' and computes the contribution of the corresponding term. Subsequently, it is checked whether the subtree starting at ν may be truncated below ν , i.e. whether the bound $\hat{b}_{m,n,r}(\nu)$ is smaller than or equal to the inaccuracy allocated to ν . If so, then one can continue with another element of V' , otherwise one first has to add two successors of ν to the set V' . Here, again the inaccuracy allocated to ν is spread over its successors proportionally to the values of their bounds $\hat{b}_{m,n,r}(\nu)$ (provided that each successor gets at least 5%). Finally, we remark that an equilibrium probability $p_{m,n,r}$ for a state in the convergent region M_c may be computed within a given *relative* accuracy by applying the sophisticated procedure for decreasing values of the allowed absolute inaccuracy.

In Table 1, the performance of the three procedures is compared on the hand of the computation of the equilibrium probability $p_{0,1,1}$ for the symmetric 2×3 switch, i.e. the 2×3 switch with equal arrival rates $r_1 = r_2 = \hat{r}$, where $\hat{r} \in (0, 1]$, and $\hat{r}_{k,l} = 1/3$ for all k and l (see Example 2.2). For all cases an absolute accuracy of 10^{-6} has been required. In the first two columns the range of chosen values of \hat{r} and the corresponding values of $p_{0,1,1}$ are depicted. In the fourth column the number of computed terms of the sum in (8.1) is given, while the fifth column denotes the number of computed relevant terms, i.e. the number of computed terms for which $\alpha_\nu, \beta_\nu, \gamma_\nu < 1$. The third column denotes the maximal depth reached during the computing process, i.e. the maximal length of the indices ν of the computed terms. For the simple procedure with or without bounds this value is equal to the smallest d for which $p_{0,1,1}(d)$ approximates $p_{0,1,1}$ within the required accuracy. The sixth column gives the upper bound for the absolute accuracy with which $p_{0,1,1}$ has been computed. Of course, for the simple procedure with bounds this value is equal to the sum of the bound $\hat{b}_{m,n,r}(\nu)$ over all ν at depth d (see (8.3)), i.e. the depth depicted in the third column. For the sophisticated procedure this value is equal to the bound $\hat{b}_{m,n,r}(\nu)$ summed up over all ν where subtrees have been truncated. In the simple procedure without bounds we have used $|p_{m,n,r}(d) - p_{m,n,r}(d-1)|$ as an indication for the absolute accuracy of $p_{m,n,r}(d)$, where d is assumed to be larger than or equal to 3 since $p_{m,n,r}(d) = 0$ for $d = 0, 1, 2$. Finally, in the last column the absolute accuracy itself has been depicted. These values have been computed after having determined $p_{0,1,1}$ with a smaller absolute accuracy.

procedure	\hat{r}	$p_{0,1,1}$	depth	terms	relevant terms	error bound	absolute error
simple procedure without bounds	0.01	0.000000	3	22	6	$2.8 \cdot 10^{-15}$	$6.9 \cdot 10^{-25}$
	0.2	0.000000	3	22	6	$2.3 \cdot 10^{-7}$	$1.0 \cdot 10^{-11}$
	0.4	0.000020	4	46	24	$1.6 \cdot 10^{-8}$	$6.1 \cdot 10^{-12}$
	0.6	0.000323	5	94	66	$3.4 \cdot 10^{-9}$	$7.6 \cdot 10^{-12}$
	0.8	0.002593	5	94	66	$4.2 \cdot 10^{-7}$	$3.4 \cdot 10^{-9}$
	1.0	0.013901	6	190	156	$6.4 \cdot 10^{-7}$	$1.6 \cdot 10^{-8}$
simple procedure with bounds	0.01	0.000000	3	22	6	$2.8 \cdot 10^{-15}$	$6.9 \cdot 10^{-25}$
	0.2	0.000000	3	22	6	$2.1 \cdot 10^{-7}$	$1.0 \cdot 10^{-11}$
	0.4	0.000020	4	46	24	$3.1 \cdot 10^{-7}$	$6.1 \cdot 10^{-12}$
	0.6	0.000323	5	94	66	$5.0 \cdot 10^{-7}$	$7.6 \cdot 10^{-12}$
	0.8	0.002593	7	382	342	$1.1 \cdot 10^{-7}$	$2.7 \cdot 10^{-13}$
	1.0	0.013901	9	1534	1482	$1.5 \cdot 10^{-7}$	$3.3 \cdot 10^{-13}$
sophisticated procedure	0.01	0.000000	2	10	0	$2.5 \cdot 10^{-10}$	$2.8 \cdot 10^{-15}$
	0.2	0.000000	3	14	2	$6.5 \cdot 10^{-7}$	$2.3 \cdot 10^{-7}$
	0.4	0.000020	5	34	12	$2.7 \cdot 10^{-8}$	$1.6 \cdot 10^{-8}$
	0.6	0.000323	6	54	26	$4.8 \cdot 10^{-8}$	$1.3 \cdot 10^{-9}$
	0.8	0.002593	7	78	46	$3.7 \cdot 10^{-7}$	$1.6 \cdot 10^{-7}$
	1.0	0.013901	9	134	92	$5.5 \cdot 10^{-7}$	$2.2 \cdot 10^{-7}$

Table 1. Performance characteristics for the computation of $p_{0,1,1}$ with absolute accuracy 10^{-6} for the symmetric 2×3 switch.

Table 1 shows that for the simple procedure with bounds a lot more (relevant) terms have to be computed than for the sophisticated procedure, especially for high values of \hat{r} (i.e. for high workloads). This seems to be caused by the bad quality of the upper bound used for the absolute accuracy (compare the values in the last two columns). Probably, this bad quality is mainly due to the use of the bound $\hat{b}_{m,n,r}(v)$, or better $b_{m,n,r}(v)$, at nodes v for which at least one of the factors of $(\alpha_v, \beta_v, \gamma_v)$ is equal to one. The subtree starting at such a node v has a whole series of nodes w with α_w, β_w or γ_w equal to one. Since the contribution of these terms is equal to zero for $p_{m,n,r}$, but equal to $\alpha_w^m \beta_w^n \gamma_w^r$ for $b_{m,n,r}(v)$, a large gap arises between the upper bound for the absolute accuracy and the absolute accuracy itself, especially when we are at a node v near the root of the tree. The sophisticated procedure overcomes the problem caused by the bad quality of $b_{m,n,r}(v)$, and of $\hat{b}_{m,n,r}(v)$, for nodes v with at least one of the factors α_v, β_v and γ_v equal to 1, by going deeper in the tree at such nodes than at other nodes. As a consequence, the sophisticated procedure performs much better than the simple procedure with bounds. The results in Table 1 also show that in the simple procedure without bounds the indication $|p_{m,n,r}(d) - p_{m,n,r}(d-1)|$ appears to serve as a good upper bound for the absolute accuracy of $p_{m,n,r}(d)$. Therefore this procedure may serve as a good alternative

for the sophisticated procedure (in case one wants to minimize the programming work, for example).

Except for the equilibrium distribution, the above procedures may also be used for quantities such as moments of queue lengths (however, note that for a moment $\mathbb{E}Q_1^{k_1} Q_2^{k_2} Q_3^{k_3}$, where Q_i denotes the length of the queue at server i , it suffices to analyze a 2×2 switch in case one or more of the powers k_i equal to 0) and the distribution of the number N of non-empty queues at the beginning of a time unit. Let the probability that N equals i be denoted by $p(i)$. By using (8.1), we find

$$\begin{aligned}
 p(3) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} p_{m,n,r} \\
 &= \sum_{v \in V} 1_{\{\alpha_v, \beta_v, \gamma_v < 1\}} (-1)^{l(v)-3} \alpha_v^m \beta_v^n \gamma_v^r \tag{8.8}
 \end{aligned}$$

and a similar expression may be found for $p(2)$; $p(0) = p_{0,0,0}$ and $p(1)$ follows from the property that the probabilities $p(i)$ add up to 1. The sum in (8.8) may be computed in the same way as the sums for the equilibrium probabilities given in (8.1) and one may use the same bounds.

system	\hat{r}	$\hat{p}(0)$	$\hat{p}(1)$	$\hat{p}(2)$	$\hat{p}(3)$	$\mu(\hat{N})$	$\sigma(\hat{N})$	$cv(\hat{N})$
2x3 switch	0.01	0.9801	0.0198	0.0001	0.0000	0.02	0.140	7.024
	0.2	0.6302	0.3397	0.0299	0.0002	0.4	0.548	1.371
	0.4	0.3345	0.5336	0.1294	0.0025	0.8	0.659	0.823
	0.6	0.1302	0.5549	0.2997	0.0152	1.2	0.671	0.559
	0.8	0.0245	0.4091	0.5084	0.0580	1.6	0.636	0.398
	1.0	0.0000	0.1732	0.6536	0.1732	2.0	0.589	0.294
independent servers	0.01	0.9801	0.0198	0.0001	0.0000	0.02	0.141	7.047
	0.2	0.6510	0.3004	0.0462	0.0024	0.4	0.589	1.472
	0.4	0.3944	0.4302	0.1564	0.0190	0.8	0.766	0.957
	0.6	0.2160	0.4320	0.2880	0.0640	1.2	0.849	0.707
	0.8	0.1016	0.3485	0.3982	0.1517	1.6	0.864	0.540
	1.0	0.0370	0.2222	0.4445	0.2963	2.0	0.816	0.408

Table 2. The distribution of the number of working servers during a time unit for the symmetric 2×3 switch; the second part gives the distribution for independent servers.

Out of the distribution of N , one can easily compute the distribution of the number \hat{N} of working servers during a time unit. In Table 2 this distribution ($\hat{p}(i)$ denotes the probability that \hat{N} equals i), and also its first moment $\mu(\hat{N})$, deviation $\sigma(\hat{N})$ and coefficient of variation $cv(\hat{N})$, are given for the symmetric 2×3 switch. In the second part of the table the same

quantities are depicted for the corresponding system with independent servers, i.e. the system consisting of three, parallel servers where each server has two Bernoulli streams of arriving jobs with rate $\hat{r}/3$. The results in Table 2 show that for all values of \hat{r} the 2×3 switch has a smaller variability in the number of working servers than the system with independent servers, which, of course, is due to the (negative) coupling between the streams of arriving jobs. For large values of \hat{r} this coupling has a considerable impact, while for small values of \hat{r} the impact is almost negligible.

9. N-dimensional random walks

All results derived in the previous sections for three-dimensional random walks may easily be extended to N -dimensional random walks with general $N \geq 2$. In this section the main results are gathered. As the reader can easily check, all these results also appear to hold for the case $N = 1$.

Consider an N -dimensional, irreducible, positive recurrent, strongly homogeneous, nearest-neighboring random walk with state space

$$M = \{ (m_1, \dots, m_N) \mid m_i \in \mathbb{N}_0 \text{ for all } i \in I \},$$

where $N \geq 2$ and $I := \{1, \dots, N\}$. For such a random walk, the set of feasible transitions for the interior points is given by

$$T = \{ (t_1, \dots, t_N) \mid t_i \in \{-1, 0, 1\} \text{ for all } i \in I \}$$

and the corresponding transition rates are denoted by q_{t_1, \dots, t_N} (assume that $\sum_{(t_1, \dots, t_N) \in T} q_{t_1, \dots, t_N} = 1$). The equilibrium equation for the interior points is given by:

$$p_{m_1, \dots, m_N} = \sum_{(t_1, \dots, t_N) \in T} q_{t_1, \dots, t_N} p_{m_1 - t_1, \dots, m_N - t_N}, \quad m_i > 0 \text{ for all } i \in I. \quad (9.1)$$

To determine the equilibrium distribution $\{p_{m_1, \dots, m_N}\}$ one can use the compensation approach, which tries to construct a solution of all equilibrium equations by linearly combining product-form solutions which satisfy the equilibrium equation (9.1) for the interior. By substituting the product form $\prod_{i=1}^N \alpha_i^{m_i}$ into equation (9.1), it follows that this equation is satisfied if and only if $(\alpha_1, \dots, \alpha_N)$ satisfies the quadratic equation

$$\prod_{i=1}^N \alpha_i = \sum_{(t_1, \dots, t_N) \in T} q_{t_1, \dots, t_N} \prod_{i=1}^N \alpha_i^{1-t_i}. \quad (9.2)$$

Without explicitly defining the formal solutions constructed by the compensation approach, we can say the following. To avoid special cases one has to assume that

$$\sum_{\substack{(t_1, \dots, t_N) \in T \\ t_i=1}} q_{t_1, \dots, t_N} > 0 \quad \text{and} \quad \sum_{\substack{(t_1, \dots, t_N) \in T \\ t_i=-1}} q_{t_1, \dots, t_N} > 0 \quad \text{for all } i \in I, \quad (9.3)$$

i.e. for all interior points and each coordinate direction, both the total rate in the positive direction and the total rate in the negative direction have to be positive; if this assumption is not satisfied then one may use an alternative method (see Remark 4.1). If (9.3) is satisfied,

then, for the sake of the absolute convergence of the formal solutions, (in general) one has to require that the transition rates q_{t_1, \dots, t_N} for the interior points satisfy the necessary condition (cf. (1.1) and (4.3) and cf. condition (12) stated in [32])

$$q_{t_1, \dots, t_N} = 0 \quad \text{if } t_i + t_j > 0 \text{ for some } i, j \in I, i \neq j. \quad (9.4)$$

Further, for each formal solution the starting solution also has to satisfy an equilibrium equation for one of the boundary planes, by which each formal solution reduces to an $(N-1)$ -ary tree of product forms. This second condition does not restrict the applicability of the compensation approach, but the condition stated in (9.4) does, especially for $N \geq 3$. However, for $N \geq 3$ there still is a queueing system satisfying (9.4): the $2 \times N$ switch. The questions whether condition (9.4) is also sufficient for the absolute convergence of the formal solutions and which formal solutions should be linearly combined, i.e. which starting solutions have to be taken, to get the equilibrium distribution are still open. We are able to answer these questions for random walks which also satisfy the projection property.

Consider an N -dimensional, irreducible, positive recurrent, homogeneous, nearest-neighboring random walk with the projection property (remark that these random walks form a subclass of the class described in the previous paragraph, since the homogeneity and the projection property imply strong homogeneity). For a random walk of this class, (9.3) is always satisfied, since

$$\sum_{\substack{(t_1, \dots, t_N) \in T \\ t_i = -1}} q_{t_1, \dots, t_N} > \sum_{\substack{(t_1, \dots, t_N) \in T \\ t_i = 1}} q_{t_1, \dots, t_N} > 0 \quad \text{for all } i \in I, \quad (9.5)$$

which is necessary and sufficient for the random walk to be positive recurrent; together with (9.4), this condition also guarantees the irreducibility. Further, for such a random walk condition (9.4) may be shown to be also sufficient for the absolute convergence of the formal solutions, at least for all states (m_1, \dots, m_N) with $m_i = 0$ for at most one $i \in I$.

By using induction with respect to N , we now can prove that the equilibrium distribution and all its marginal distributions are equal to alternating sums of pure product-form distributions, where all product forms are obtained by taking the trivial solution $(1, \dots, 1)$ of the quadratic equation (9.2) and generating new solutions of (9.2) by letting one factor free each time. Let the set of vectors V be defined by

$$V = \{ (v_1, \dots, v_l) \mid l \in \mathbb{N}_0, v_1 \in I \text{ and } v_k \in I \setminus \{v_{k-1}\} \text{ for all } k \geq 2 \}.$$

Next, define $(\alpha_{1, \emptyset}, \dots, \alpha_{N, \emptyset}) = (1, \dots, 1)$ and let for all other vectors $v \in V$ the factors of $(\alpha_{1, v}, \dots, \alpha_{N, v})$ be defined by

$$\alpha_{i, v} = \begin{cases} f_i(\alpha_{1, p(v)}, \dots, \alpha_{i-1, p(v)}, \alpha_{i+1, p(v)}, \dots, \alpha_{N, p(v)}) / \alpha_{i, p(v)} & \text{if } v_{l(v)} = i; \\ \alpha_{i, p(v)} & \text{if } v_{l(v)} \neq i, \end{cases}$$

where $p(v)$ and $l(v)$ are the parent and the length of a vector v and the function $f_i(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_N)$ denotes the product of the two roots of the quadratic equation (9.2) for fixed $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_N$. Then the set of all relevant product-form solutions is given by

$$P^* = \{ (\alpha_{1, v}, \dots, \alpha_{N, v}) \mid v \in V \}.$$

For each solution $(\alpha_{1, v}, \dots, \alpha_{N, v})$ of this set, all factors are real numbers in the interval $(0, 1]$,

so we can partition P^* into the subsets

$$P_J^* = \{ (\alpha_1, \dots, \alpha_N) \in P^* \mid \alpha_i < 1 \text{ for all } i \in J \text{ and } \alpha_i = 1 \text{ for all } i \notin J \}, \quad J \subset I.$$

The solutions in a set P_J^* are just the ones needed to describe the equilibrium behavior of the components which belong to J .

Theorem 9.1 (*extension of Theorem 7.1*)

Let $\{p_{m_1, \dots, m_L}^{(j_1, \dots, j_L)}\}$ be the equilibrium distribution for the components j_1, \dots, j_L , where $1 \leq L \leq N$ and $1 \leq j_1 < \dots < j_L \leq N$. Define $J = \{j_1, \dots, j_L\}$, then

$$p_{m_1, \dots, m_L}^{(j_1, \dots, j_L)} = \sum_{(\alpha_{1,v}, \dots, \alpha_{N,v}) \in P_J^*} (-1)^{l(v)-L} \prod_{i=1}^L (1 - \alpha_{j_i,v}) \alpha_{j_i,v}^{m_i} \quad (9.6)$$

for all (m_1, \dots, m_L) with $m_i \geq 0$ for all $i = 1, \dots, L$ and $m_i = 0$ for at most one i .

Remark that according to the notation used in this theorem the distribution $\{p_{m_1, \dots, m_N}\}$ for the full Markov chain/random walk is denoted by $\{p_{m_1, \dots, m_N}^{(1, \dots, N)}\}$. Further, note that all equilibrium probabilities $p_{m_1, \dots, m_L}^{(j_1, \dots, j_L)}$ for the states for which formula (9.6) does not hold, may be determined with the help of the marginal distributions of $\{p_{m_1, \dots, m_L}^{(j_1, \dots, j_L)}\}$ or with the help of the equilibrium equations of the random walk for the components j_1, \dots, j_L . Since the random walk describing the behavior of the $2 \times N$ switch has the projection property and satisfies condition (9.4), Theorem 9.1 may be applied to the $2 \times N$ switch to determine the equilibrium distribution and/or other interesting quantities such as the moments of the total number of jobs in the system. Here, one can use the three procedures described in Section 8; these procedures and the bounds used by them, which have been described in Section 8 for the three-dimensional case, may be easily extended to the N -dimensional case.

10. Final conclusions and suggestions for future research

The goal of this paper was to investigate for which multi-dimensional, irreducible, positive recurrent, homogeneous, nearest-neighboring random walks the compensation approach works, in particular for which random walks with dimension three or more. First, for the subclass of strongly homogeneous random walks, we have derived the necessary condition stated in (9.4), but this condition may be shown to hold also for the whole class. Next, we have shown that this condition is necessary and sufficient for the subclass of random walks with the projection property. For such a random walk, using the compensation approach shows that the equilibrium distribution may be written as a kind of alternating sum of pure product-form distributions. As we saw in Section 8 for the three-dimensional case, this last result may lead to an efficient algorithm for the computation of the equilibrium distribution and the quantities which are deducible from it.

From the research in [8] we know that for the two-dimensional case the condition stated in (9.4) is also sufficient for random walks without the projection property. Whether this also holds for the N -dimensional case with $N \geq 3$ is not known yet. We believe that for $N \geq 3$ the condition in (9.4) has to be extended a little bit. We conjecture that for a (strongly)

homogeneous, nearest-neighboring random walk the compensation approach works if and only if for all $J \subset I$

$$q_{t_1, \dots, t_N}^J = 0 \quad \text{if } t_i + t_j > 0 \text{ for some } i, j \in J, i \neq j. \quad (10.1)$$

As one can easily check, for random walks with the projection property this condition is equivalent to (9.4). Future research should confirm our conjecture. However, since for three- and higher-dimensional random walks the condition is that severe that almost no queueing problem satisfies this condition, other topics might be more interesting. Recently the compensation approach has been shown to be successful for the two-dimensional, symmetric shortest queue system with Erlang distributed service-times, see [4]. This problem is in the class of two-dimensional, homogeneous random walks where for each state transitions are allowed to all states within a given distance. Probably for this class the necessary and sufficient condition (1.1) for the application of the compensation approach is still valid. Future research should confirm this. The problem studied in [4] appears to satisfy this condition. A more important topic for future research, but also a more difficult one, is finding a new method similar to the compensation approach for two-dimensional random walks which violate condition (1.1). Such a new method maybe can be extended to a method for a less restrictive class of three- and higher-dimensional random walks.

If it is not possible to develop a method for the three- and higher-dimensional random walks which violate (9.4), then one can only use numerical methods, such as truncation, for the computation of the equilibrium distribution and the relevant performance measures. In general these methods are rather rough, by which they consume large amounts of computing time. Therefore, finding more sophisticated and less expensive numerical methods could also considerably improve the present situation. For some particular problems such methods are already available. For example, for the symmetric shortest queue system with $N \geq 2$ servers one can use the power series method, see [12], and the methods described in [29] and [1]. In the last paper, an upper and a lower bound model have been derived to approximate the mean waiting time as accurate as wished. Both models describe a slightly modified system of which the waiting time provides an upper or lower bound for the waiting time of the original system. Parameters introduced to describe the modifications, determine the impact of the modifications and may be varied to determine the waiting time within a given accuracy. Both modifications are such that the N -dimensional state space of the Markov chain which describes the symmetric shortest queue system is made finite in exactly $N-1$ directions, by which the waiting times of the modified systems may be determined with the help of the matrix-geometric approach (see [30]). Future research has to make clear whether this technique is also appropriate for the whole class of random walks studied in this paper.

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