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THE EQUILIBRIUM OF DISTORTED POLYTROPES

(I). The Rotational Problem.

S. Chandrasekhar.

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§ 1. Emden's well-known researches on the equilibrium of polytropic gas spheres has been of fundamental importance in its repercussions on the modern theories of stellar structure. But it is a matter of some surprise that scarcely any serious attempt has been made to extend Emden's researches to the case of rotating gas spheres which in their non-rotating states have polytropic distributions described by the so-called Emden functions.

The problem is exceedingly simple in its classical severity and can be formulated as follows :—

We have a gas sphere in gravitational equilibrium. It is given that the total pressure P is related to the density ρ by means of the relation

$$P=K\rho^{1+\frac{1}{n}}. \tag{1}$$

This defines for the non-rotating gas sphere a distribution of density and pressure governed by the differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^n \qquad (2)^*$$

—*Emden's differential equation of index n.* Now the gas sphere is set rotating at a constant small angular velocity ω. *The problem is to determine the shape and density distribution in such a gas sphere and explicitly to relate the structure of such rotating masses with the Emden functions which describe the polytropic non-rotating state.*

§ 2. In treating this problem we shall assume that the rotation is so slow that the configurations are only slightly oblate. In other words, the purpose of this paper is to specify *completely* those configurations for the polytropic model which correspond to the Maclaurin spheroids † in the case of the "incompressible rotating stellar masses."

It is thus clear that the point of view and the problem here is different from that considered by Jeans under the heading "Adiabatic-model" in his book on *The Problems of Cosmogony and Stellar-dynamics* (p. 165). The problem treated by Jeans is to enumerate the complete sequence of the geometry of the configurations for the whole range of ω. Further, the analysis of Jeans is to establish a general result that a gas sphere rotating as a rigid body can break up in two distinct ways—either by fission into two detached masses or by a process of equatorial break-up after assuming

* The significance of ξ and θ are explained later.

† For small ω.

a lenticular shape as in the Roche model—according as the central condensation is "weak" or "pronounced." We shall not enter into such questions—which of course are of fundamental significance in any problem of cosmogony—but confine ourselves to the comparatively more simple and elementary problem of specifying completely the "polytropic Maclaurin spheroids." Also the fundamental mathematical point of view will be different from that of Jeans. We will explicitly relate the geometry and the physical properties of these configurations with the Emden functions describing the polytropic non-rotating gas spheres.

Substantially the case $n=3$ has been treated already by Milne * and von Zeipel.† Indeed it was in this connection that von Zeipel discovered his fundamental theorem.‡ The method of solution adopted in the sequel will in main be that of Milne, to whose memoir the present paper owes a great deal.

§ 3. *Equations of the Problem.*—The equations of mechanical equilibrium are, taking the Z-axis as the axis of rotation,

$$\left.\begin{aligned} \frac{\partial P}{\partial x} &= \rho\frac{\partial V}{\partial x} + \rho\omega^2 x, \\ \frac{\partial P}{\partial y} &= \rho\frac{\partial V}{\partial y} + \rho\omega^2 y, \\ \frac{\partial P}{\partial z} &= \rho\frac{\partial V}{\partial z}, \end{aligned}\right\} \tag{3}$$

where V is the gravitational potential, which of course satisfies Poisson's equation

$$\sum_{x,y,z}\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = -4\pi G\rho. \tag{4}$$

Introducing polar co-ordinates r, θ, ϕ and neglecting ϕ on account of symmetry we find that

$$\left.\begin{aligned} \frac{\partial P}{\partial r} &= \rho\frac{\partial V}{\partial r} + \rho\omega^2 r(1-\mu^2), \\ \frac{\partial P}{\partial \mu} &= \rho\frac{\partial V}{\partial \mu} - \rho\omega^2 r^2\mu, \end{aligned}\right\} \tag{3'}$$

where $\mu = \cos\theta$. Also (4) in these variables reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left((1-\mu^2)\cdot\frac{\partial V}{\partial \mu}\right) = -4\pi G\rho. \tag{4'}$$

From (3′) and (4′) we deduce the fundamental equation of the problem:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left(\frac{1-\mu^2}{\rho}\frac{\partial P}{\partial \mu}\right) = -4\pi G\rho + 2\omega^2, \tag{5}$$

* E. A. Milne, *M.N.*, **83**, 118, 1923. † H. von Zeipel, *M.N.*, **84**, 665, 684, 1924.

‡ For a general discussion of the whole problem we refer the reader to Milne's article in the *Handbuch der Astrophysik*, Band III/1, p. 235.

where

$$P = K\rho^{1+\frac{1}{n}}. \tag{6}$$

§ 4. If we introduce the new variables

$$\rho = \lambda\Theta^n ; \qquad P = \lambda^{1+\frac{1}{n}} . K . \Theta^{n+1}, \tag{7}$$

(5) reduces to

$$\frac{\lambda^{\frac{1}{n}-1}K(n+1)}{4\pi G}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right)\right\} = -\Theta^n + \frac{\omega^2}{2\pi G\lambda}. \tag{8}$$

Put

$$r = \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{\frac{1}{2}}\xi \tag{9}$$

and

$$v = \frac{\omega^2}{2\pi G\lambda}. \tag{10}$$

We get

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right) = -\Theta^n + v. \tag{11}$$

§ 5. *Non-rotating Configuration.*—If the gas sphere is non-rotating then ω and therefore v is zero. If, further, now

$$\rho = \lambda\theta^n, \tag{7'}$$

we see that θ satisfies the differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n, \tag{12}$$

—Emden's differential equation of index n.

§ 6. *Solution for the Rotating Gas Sphere.*—We will now seek a solution of (11) in terms of those of (12), and indeed we will assume the following form for our solution:—

$$\Theta = \theta + v\Psi + v^2\Phi + \ldots \tag{13}$$

We shall work consistently only up to the first order in v, i.e. we consider only such slow rotations that the effects arising from ω^4 can be neglected. Ψ then should satisfy the differential equation, remembering that θ is a spherically symmetrical function and therefore independent of μ,

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Psi}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Psi}{\partial\mu}\right) = -n\theta^{n-1}\Psi + 1. \tag{14}$$

Now we shall assume for Ψ the following form: *

$$\Psi = \psi_0(\xi) + \sum_{j=1}^{\infty} A_j\psi_j(\xi)P_j(\mu), \tag{15}$$

* For a formal justification of this assumption see von Zeipel (*loc. cit.*), p. 691.

where the $P_j(\mu)$'s are the Legendre functions of the various indices, the function with index j satisfying the differential equation

$$\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial P_j}{\partial\mu}\right)+j(j+1)P_j=0. \tag{16}$$

Further, the ψ_j's are functions of ξ only.

Substituting (15) in (14), and using (16) and equating coefficients of P_j, we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)=-n\theta^{n-1}\psi_0+1, \tag{17}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)=\left(\frac{j(j+1)}{\xi^2}-n\theta^{n-1}\right)\psi_j, \tag{18}$$

$$(j=1, 2, \ldots).$$

So far the A_j's are arbitrary. To determine them we must evaluate the potential V. For equation (5) (from which (14) is deduced) contains no explicit reference to the potential and is the same whatever the external gravitational field. To remove this indeterminateness we must use the solution found (with the arbitrary A_j's) to calculate the potential arising from the matter and then determine the A_j's such that the equations of equilibrium (3′) are satisfied.

Poisson's equation in the ξ, μ variables takes the form

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial V}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial V}{\partial\mu}\right)$$
$$=-(n+1)K\lambda^{\frac{1}{n}}\left[\theta^n+n\theta^{n-1}v\left\{\psi_0+\sum_j A_j\psi_jP_j\right\}\right]. \tag{19}$$

We develop V in the form (to the first order in v)

$$V=U+v\left\{V_0(\xi)+\sum_j V_j(\xi)P_j(\mu)\right\}, \tag{20}$$

where U is the potential of the non-rotating configuration. Substitution of (20) in (19) yields on equating the coefficients of $P_j(\mu)$,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dU}{d\xi}\right)=-R\theta^n, \tag{21}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_0}{d\xi}\right)=-Rn\theta^{n-1}\psi_0, \tag{22}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_j}{d\xi}\right)=\frac{j(j+1)}{\xi^2}V_j-Rn\theta^{n-1}A_j\psi_j, \tag{23}$$

where we have used the abbreviation

$$R=(n+1)K\lambda^{\frac{1}{n}}.$$

Remembering that θ satisfies the Emden equation with index n we deduce from (21) that

$$U=R\theta+\text{constant}. \tag{24}$$

Using the differential equation (17) defining ψ_0, we derive from (22) that

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_0}{d\xi}\right)=R\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)-1\right]$$
$$=R\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}(\psi_0-\tfrac{1}{6}\xi^2)\right)\right].$$

Hence

$$V_0=R(\psi_0-\tfrac{1}{6}\xi^2)+\text{constant}. \tag{25}$$

Similarly from (23) and (18) we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_j}{d\xi}\right)-\frac{j(j+1)}{\xi^2}V_j=A_jR\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)-\frac{j(j+1)}{\xi^2}\psi_j\right]. \tag{26}$$

A particular solution of (26) is

$$V_j=RA_j\psi_j+\text{constant}. \tag{27}$$

The general solution is obtained by adding any *regular* solution of the equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_j}{d\xi}\right)-\frac{j(j+1)}{\xi^2}V_j=0, \tag{28}$$

which is $RB_j\xi^j$ where B_j is arbitrary. Hence

$$V_j=R(A_j\psi_j+B_j\xi^j)+\text{constant}. \tag{29}$$

Combining the results (24), (25) and (29) we get after some minor rearrangement of the terms (to the first order in v)

$$V=R\left[\Theta+v\left\{\sum_{j=1}^{\infty}B_j\xi^jP_j(\mu)-\tfrac{1}{6}\xi^2\right\}\right]. \tag{30}$$

We have to substitute now (30) in (3′), which in the ξ, μ variables takes the form

$$\frac{dP}{d\xi}=\rho\frac{\partial V}{\partial\xi}+\frac{2}{3}\rho\omega^2\xi\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right](1-P_2(\mu)).$$

Substituting (30) in the above equation and remembering that

$$P=\lambda^{1+\frac{1}{n}}\,.\,K\,.\,\Theta^{n+1},$$

and equating coefficients of $P_j(\mu)$ we find that

$$B_j=0, \qquad j\neq 2$$

and

$$B_2=\tfrac{1}{6}. \tag{31}$$

Finally, we obtain

$$V=R(\Theta-\tfrac{1}{6}v(\xi^2-P_2(\mu)\xi^2))+\text{constant}. \tag{32}$$

We have still to ensure that V is the *actual* potential arising from the mass. This will determine the A_j's.

A little consideration shows that (compare Milne (*loc. cit.*), p. 134) to the order of accuracy we are working, the potential (and its derivative) given by (32) should be continuous with an expression of the type

$$V_{\text{external}} = R\left[\frac{C_0}{\xi} + v\sum_{j=1}^{\infty} \frac{C_j}{\xi^{j+1}} P_j(\mu)\right] + \text{constant} \tag{33}$$

on a sphere of radius ξ_1 the first zero of the Emden's function with index n. Comparing the "inner" and the "external" potentials at $\xi = \xi_1$ and also their derivatives we obtain

$$A_j = C_j = 0, \qquad j \neq 2.$$

But, if $j = 2$, we get

$$\left.\begin{aligned} \frac{C_2}{\xi_1^3} &= A_2\psi_2(\xi_1) + \tfrac{1}{6}\xi_1^2, \\ -\frac{3C_2}{\xi_1^4} &= A_2\psi_2'(\xi_1) + \tfrac{1}{3}\xi_1. \end{aligned}\right\} \tag{34}$$

This gives

$$A_2 = -\frac{5}{6}\,\frac{\xi_1^2}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}. \tag{35}$$

Hence the solution to the problem is given by

$$\Theta = \theta + v\left[\psi_0(\xi) - \frac{5}{6}\,\frac{\xi_1^2}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}\psi_2(\xi)P_2(\mu)\right], \tag{36}$$

where ψ_0 and ψ_2 satisfy the differential equations

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) = -n\theta^{n-1}\psi_0 + 1, \tag{37$_1$}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right) = \left(-n\theta^{n-1} + \frac{6}{\xi^2}\right)\psi_2, \tag{37$_2$}$$

which are for purposes of numerical integration more conveniently written as

$$\frac{d^2\eta_0}{d\xi^2} = -n\theta^{n-1}\eta_0 + \xi\,; \qquad \frac{d^2\eta_2}{d\xi^2} = \left(-n\theta^{n-1} + \frac{6}{\xi^2}\right)\eta_2, \tag{37$'$}$$

where

$$\eta_{0,\,2} = \xi\psi_{0,\,2}. \tag{37$''$}$$

§ 7. *Expansion, Ellipticity and Oblateness.*—The boundary ξ_0 is given by $\Theta = 0$, and hence by (36)

$$\xi_0 = \xi_1 + \frac{v}{|\,\theta_1'\,|}\left[\psi_0(\xi_1) - \frac{5}{6}\,\frac{\xi_1^2\psi_2(\xi_1)P_2(\mu)}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}\right]. \tag{38}$$

Thus there is an expansion of the star as a whole of amount $v\psi_0(\xi_1)/|\,\theta_1'\,|$ and superposed on this an ellipticity. At the equator $P_2(\mu) = -\frac{1}{2}$ and at the poles $P_2(\mu) = +1$. Hence we get for the *oblateness* of the boundary the general expression

$$\sigma = \frac{5}{4}\,\frac{v}{|\,\theta_1'\,|}\cdot\frac{\xi_1\psi_2(\xi_1)}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}. \tag{39}$$

§ 8. *Mass Relation.*—The mass is given by

$$M=2\pi\iint\rho r^2 dr d\mu.$$

The ellipticity term does not clearly contribute to the mass on the average. Introducing the ξ, θ variables in the above integral we have

$$M=4\pi\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{3/2}\lambda\int_0^{\xi_1+d\xi_1}(\theta^n+n\theta^{n-1}v\psi_0)\xi^2 d\xi,$$

which to the first order in v can be replaced by

$$M=4\pi\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-\frac{1}{3}}\right]^{3/2}\left\{\int_0^{\xi_1}\theta^n\xi^2 d\xi+v\int_0^{\xi_1}n\theta^{n-1}\psi_0\xi^2 d\xi\right\}.$$

Now

$$\int_0^{\xi_1}\theta^n\xi^2 d\xi=-\int_0^{\xi_1}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\xi_1^2\left(\frac{d\theta}{d\xi}\right)_1,$$

$$\int_0^{\xi_1}n\theta^{n-1}\psi_0\xi^2 d\xi=-\int_0^{\xi_1}\left\{\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)-\xi^2\right\}d\xi$$

$$=-\xi_1^2\psi_0'(\xi_1)+\tfrac{1}{3}\xi_1^3.$$

Hence

$$M=-4\pi\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{3-n}{3n}}\right]^{3/2}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_1\cdot\left[1+v\frac{\frac{1}{3}\xi_1-\psi_0'(\xi_1)}{|\theta_1'|}\right].\qquad(40)$$

If $v=0$, we have for the case of non-rotating stars

$$M_0=-4\pi\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{3-n}{3n}}\right]^{3/2}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_1.\qquad(41)$$

Hence the "mass relation" for two gas spheres with *equal central densities*—one rotating with an angular velocity ω and the other non-rotating—is

$$M_\omega=M_0\cdot\left[1+v\frac{\frac{1}{3}\xi_1-\psi_0'(\xi_1)}{|\theta_1'|}\right].\qquad(42)$$

Hence the rotating configuration has a *greater mass*, as indeed we should expect on general grounds.*

§ 9. *Volume and Relation between Mean and Central Density.*—The volume N of the configuration is clearly given by

$$N=\frac{4\pi}{3}\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{3/2}\xi_1^3+\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{3/2}\cdot\int_0^{2\pi}\int_{-1}^{+1}(\xi_0-\xi_1)\xi_1^2 d\mu d\phi,$$

which by (38) yields

$$N=\frac{4}{3}\pi\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{3/2}\xi_1^3\cdot\left[1+v\cdot\frac{3\psi_0(\xi_1)}{\xi_1|\theta_1'|}\right].\qquad(43)$$

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* Compare Milne's article in the *Handbuch* ("General Effects of Rotation," p. 236).

Using the mass relation (40) we get for the mean density ρ_m the formula

$$\rho_m = -3\lambda\frac{1}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1 \cdot \frac{1+v\,.\,\dfrac{\frac{1}{3}\xi_1-\psi_0'(\xi_1)}{|\,\theta_1'\,|}}{1+v\,.\,\dfrac{3\psi_0(\xi_1)}{\xi_1|\,\theta_1'\,|}},$$

or to the order of accuracy we are working we have, remembering that $\lambda=\rho_c$ (θ and Θ are both unity at the centre),

$$\rho_m = -\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1\rho_c\,.\left[1+v\frac{\frac{1}{3}\xi_1^2-\psi_0'(\xi_1)\xi_1-3\psi_0(\xi_1)}{\xi_1|\,\theta_1'\,|}\right]. \tag{44}$$

If $v=0$, we get the well-known formula for the non-rotating polytropes

$$\rho_m = -\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1\rho_c. \tag{45}$$

From (44) we deduce (to the first order) that

$$v=\frac{\omega^2}{2\pi G\rho_c}=-\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1\cdot\frac{\omega^2}{2\pi G\rho_m}. \tag{46}$$

Formulæ (38, (39), (40), (42), (43) can all now be expressed in terms of ρ_m (instead of ρ_c) by means of (46).

§ 10. *Numerical Integration of the Differential Equations.*—Thus we see that the structure of a slowly rotating polytropic gas configuration is completely specified when the pair of differential equations ($37_{1,\,2}$) are solved. There is just one case where integration can be effected at once. The equation for ψ_0 for the index $n=1$ can be written as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d(\psi_0-1)}{d\xi}\right)=-(\psi_0-1), \tag{47}$$

which is just Emden's equation with index 1. Remembering the boundary conditions that ψ_0 and ψ_0' are to be zero at the origin, we see that

$$\psi_0=1-\frac{\sin\xi}{\xi}, \tag{48}$$

is the required solution. [By (48) we see that

$$\psi_0(\xi_1)=1\,;\qquad \psi_0'(\xi_1)=\frac{1}{\pi}\,;\qquad \xi_1=\pi. \tag{48'}$$

The mass relation (42) and the volume relation (43) therefore take for this case the neat forms

$$M_\omega=M_0[1+v(\tfrac{1}{3}\pi^2-1)], \tag{48''}$$

$$N_\omega=N_0[1+3v]. \tag{48'''}$$

In all other cases numerical integration must be adopted. To do this we must have a power series in ξ for ψ_0 and ψ_2 at the origin, and with a start thus made the integration has to be continued by any of the known standard methods.

Now it is seen that near the origin Emden's equation with index n has the expansion *

$$\theta = 1 - \frac{1}{3!}\xi^2 + \frac{n}{5!}\xi^4 - \frac{8n^2 - 5n}{3 \cdot 7!}\xi^6 + \ldots \tag{49}$$

which yields for θ^{n-1} the expansion

$$\theta^{n-1} = 1 - \frac{(n-1)}{6}\xi^2 + \frac{(n-1)(4n-5)}{180}\xi^4 - \ldots \tag{50}$$

Substituting this in equations ($37_{1,\,2}$), and assuming for ψ_0 and ψ_2 power series of the type

$$\psi_{0,\,2} = a\xi^2 + b\xi^3 + c\xi^4 + \ldots, \tag{51}$$

and solving, we obtain the following expansions for ψ_0 and ψ_2 near the origin :—

$$\psi_0 = \frac{1}{6}\xi^2 - \frac{n}{120}\xi^4 + \frac{n(13n-10)}{42 \cdot 360}\xi^6 - \frac{n(90n^2 - 157n + 70)}{72 \cdot 42 \cdot 360}\xi^8 + \ldots \tag{52}$$

$$\psi_2 = \xi^2 - \frac{n}{14}\xi^4 + \frac{n(10n-7)}{42 \cdot 36}\xi^6 - \frac{n(308n^2 - 503n + 210)}{42 \cdot 36 \cdot 330}\xi^8 + \ldots \tag{53}$$

The numerical integration was carried out for the cases $n = 1$, 1·5, 2, 3, 4, and the functions ψ_0 and ψ_2 are tabulated in Tables I–V appended to the end of this paper. At the bottom of each table the value of ψ_0 and ψ_2 and their derivatives at ξ_1—the first zero of Emden's equation—are given. The method of integration adopted is the one attributed to Adams and sketched at the end of the second of von Zeipel's papers referred to at the outset. In the following Table VI the values of $\psi_0(\xi_1)$, $\psi_0'(\xi_1)$, $\psi_2(\xi_1)$, $\psi_2'(\xi_1)$, ξ_1 and θ_1' are given.

TABLE VI

n	1	1·5	2	3	4
ξ_1	3·14159	3·6538	4·3529	6·8968	14·9715
$-\theta_1'$	0·31831	0·20330	0·12725	0·04243	0·0080181
$\psi_0(\xi_1)$	1·00000	1·2942	1·9153	5·8380	33·5327
$\psi_0'(\xi_1)$	0·31831	0·6364	0·9961	2·0391	4·8812
$\psi_2(\xi_1)$	4·55940	4·7820	5·6431	11·2780	46·5444
$\psi_2'(\xi_1)$	0·42060	1·1495	1·7559	3·0409	6·1766

§ 11. With the use of the values given in the above table we can now numerically evaluate many of the "coefficients" occurring in the formulæ

* See the introduction (by D. H. Sadler) to the *Mathematical Tables*, vol. ii, on "Emden Functions," issued by the British Association for the Advancement of Science.

of §§ 7, 8 and 9. We shall now give the precise forms for the cases where the numerical integration has been effected.

Firstly, equation (36), expressing Θ, takes the following forms :—

$$\left.\begin{aligned} \Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}5483\psi_2(\xi)P_2(\mu)], & n&=1 \\ \Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}5999\psi_2(\xi)P_2(\mu)], & n&=1{\cdot}5 \\ \Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}6426\psi_2(\xi)P_2(\mu)], & n&=2 \\ \Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}72325\psi_2(\xi)P_2(\mu)], & n&=3 \\ \Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}8048\psi_2(\xi)P_2(\mu)], & n&=4 \end{aligned}\right\}. \qquad (54)$$

The equations of the boundary ξ_0 *are* :

$$\left.\begin{aligned} \xi_0 &= 3{\cdot}1416 + 3{\cdot}1416v[1 - 2{\cdot}5000P_2(\mu)], & n&=1 \\ \xi_0 &= 3{\cdot}6538 + 4{\cdot}9188v[1{\cdot}2942 - 2{\cdot}8687P_2(\mu)], & n&=1{\cdot}5 \\ \xi_0 &= 4{\cdot}3529 + 7{\cdot}8585v[1{\cdot}9153 - 3{\cdot}6260P_2(\mu)], & n&=2 \\ \xi_0 &= 6{\cdot}8968 + 23{\cdot}568v\,[5{\cdot}8380 - 8{\cdot}1568P_2(\mu)], & n&=3 \\ \xi_0 &= 14{\cdot}9715 + 124{\cdot}718v\,[33{\cdot}533 - 37{\cdot}460P_2(\mu)], & n&=4 \end{aligned}\right\}. \qquad (55)$$

Remembering that $P_2(\mu) = -\frac{1}{2}$ at the equator and $+1$ at the poles we obtain from (55) the following table giving *the fractional elongation at the equator, the fractional contraction at the poles and the oblateness* as expressed by equation (39):—

Table VII

Geometry of the Boundary

n	Fractional Elongation at the Equator	Fractional Contraction at the Poles	Oblateness σ
1	$2{\cdot}2500v$	$1{\cdot}5000v$	$3{\cdot}7500v$
1·5	$3{\cdot}6730v$	$2{\cdot}1195v$	$5{\cdot}7926v$
2	$6{\cdot}7309v$	$3{\cdot}0884v$	$9{\cdot}8194v$
3	$33{\cdot}888v$	$7{\cdot}9240v$	$41{\cdot}811v$
4	$435{\cdot}35v$	$32{\cdot}714v$	$468{\cdot}07v$

The mass relations are :

$$\left.\begin{aligned} M_\omega &= M_0\,.\,(1 + 2{\cdot}290v), & n&=1 \\ M_\omega &= M_0\,.\,(1 + 2{\cdot}860v), & n&=1{\cdot}5 \\ M_\omega &= M_0\,.\,(1 + 3{\cdot}575v), & n&=2 \\ M_\omega &= M_0\,.\,(1 + 6{\cdot}123v), & n&=3 \\ M_\omega &= M_0\,.\,(1 + 13{\cdot}632v), & n&=4 \end{aligned}\right\}. \qquad (56)$$

The volume relations are :

$$\left.\begin{aligned} N_\omega &= N_0\,.\,(1 + 3v), & n&=1 \\ N_\omega &= N_0\,.\,(1 + 5{\cdot}227v), & n&=1{\cdot}5 \\ N_\omega &= N_0\,.\,(1 + 10{\cdot}373v), & n&=2 \\ N_\omega &= N_0\,.\,(1 + 59{\cdot}849v), & n&=3 \\ N_\omega &= N_0\,.\,(1 + 838{\cdot}00v), & n&=4 \end{aligned}\right\}. \qquad (57)$$

The relations between the mean and the central densities are :

$$\left.\begin{aligned} \rho_c &= \rho_m \times 3\cdot2899[1+0\cdot710v], & n&=1 \\ \rho_c &= \rho_m \times 5\cdot9907[1+2\cdot367v], & n&=1\cdot5 \\ \rho_c &= \rho_m \times 11\cdot4025[1+6\cdot797v], & n&=2 \\ \rho_c &= \rho_m \times 54\cdot1825[1+53\cdot724v], & n&=3 \\ \rho_c &= \rho_m \times 622\cdot408\ [1+814\cdot37v], & n&=4 \end{aligned}\right\}. \tag{58}$$

§ 12. *Comparison of Configurations with equal Mass.*—The above sections (in particular § 11) give a complete answer to the problem formulated at the outset in § 1. But it is of interest to compare two configurations, one "stationary" and the other rotating with a slow angular velocity ω, both having the *same mass*. Now for this problem to have a meaning, the radius of the non-rotating polytrope must be determined when the mass is given. As is well known (and as is clear from equation (41)) this is not the case when $n=3$. Hence we should expect (and indeed as it turns out subsequently) that $n=3$ must be of the nature of a singularity with respect to this problem.

We have already seen that if a rotating and a non-rotating configuration are to have the same central density, then the non-rotating configuration has a smaller mass. To secure the same mass we must alter the central density of one of them—say the rotating one. The problem then is to find the fractional increase (or decrease) in λ (*i.e.* the central density) such that the masses of the two configurations are the same, *i.e.* we have to find $\delta\lambda$ such that

$$M(\lambda+\delta\lambda,\ \omega)=M(\lambda,\ 0). \tag{59}$$

From (41) we easily find by differentiation with respect to λ that

$$\frac{\delta\lambda}{\lambda}=-\frac{2n}{\dfrac{(3-n)}{v}\dfrac{|\theta_1'|}{\frac{1}{3}\xi_1-\psi_0'(\xi_1)}-3(n-1)}. \tag{60}$$

If $n \neq 3$ we can rewrite the above relation approximately to the first order as

$$\frac{\delta\lambda}{\lambda}=-v\cdot\frac{2n}{3-n}\frac{\frac{1}{3}\xi_1-\psi_0'(\xi_1)}{|\theta_1'|}. \tag{61}$$

It is at once clear from the above formula that *if $n>3$ the central density of the rotating configuration is greater than that of the non-rotating configuration, while if $n<3$ the converse is true.*

Of course the above result is precisely what we should expect. For if the central densities of the two configurations are equal the non-rotating configuration has a *smaller* mass, and as we have

$$M_0 \propto \lambda^{\frac{3-n}{2n}}, \tag{62}$$

to *increase* the mass of the non-rotating sphere we should increase or decrease λ according as n is less than or greater than 3. This is just what (61) shows. When $n=3$ the radius of the non-rotating configuration is indeterminate and the question "the fractional change in the central density" has no meaning, for the radius of the polytrope $n=3$ is first determined only when the mean density (and therefore also the central density) is given (in addition to M).

If $n \neq 3$ the change in the central density is proportional to v and hence *formulæ* (54), (55), (57), (58) *and the results of* Table VI *continue to be true* (*to the first order in* v) *when we are comparing two configurations with equal mass.*

Finally, the following short table gives precise values for $\frac{\delta\lambda}{\lambda}$:—

n	1	1·5	2	3	4
$-\delta\lambda/\lambda$	2·290v	5·720v	14·299v	...	− 109·06v

§ 13. *General Discussion.*—There are a number of points which can now be discussed in connection with the above calculations and their bearing on actual physical facts, but I shall make only some brief comments and reserve more detailed references to possible applications for a separate communication after treating the closely related problem of the double stars.

(1) Though the problem has been formulated in an abstract form and the whole calculation proceeds on the assumption that there exists a relation $P=K\rho^{1+\frac{1}{n}}$ between the total pressure and density, it can actually be shown that the whole investigation applies to the more general "standard model."

(2) A glance at Table VII shows that as the central condensation of the configuration increases the fractional elongation at the equator and the fractional contraction at the poles increase rapidly for a given angular velocity ω. But what is noteworthy is that the fractional elongation at the equator increases much more rapidly than the fractional contraction at the poles. Thus the ratio of the fractional elongation at the equator to the fractional contraction at the poles increases from 1·5 for $n=1$ to 13·3 for $n=4$. This is precisely what one would expect if for *large values* of ω the configurations with large central condensations should tend to assume lenticular forms for equatorial break-up.

(3) In § 12 we considered the fractional change in the central density ($\delta\lambda/\lambda$) which takes place when a gas sphere is set rotating with a slow angular velocity ω. It was found that if $n<3$, $\delta\lambda/\lambda$ is *negative*, while if $n>3$, $\delta\lambda/\lambda$ is *positive.* Thus from this standpoint configurations with "large" central condensations behave quite differently from configurations with comparatively weak central condensations. One is tempted to conjecture that these two distinct different *initial* behaviours of rotating masses correspond to the two distinct types of break-up—namely, the fissional break-up and the equatorial break-up—which occur when ω gets large. But this conjecture, though fascinating, is by no means well founded.

In conclusion, I wish to record my thanks to Professor N. Bohr for allowing me the very valuable privileges of his Institute, where the above work was carried out. My thanks are also due to Professor E. A. Milne for his interest and advice.

APPENDIX

Solutions of the differential equations

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) = -n\theta^{n-1}\psi_0 + 1,$$

and

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right) = \left(-n\theta^{n-1} + \frac{6}{\xi^2}\right)\psi_2,$$

for n = 1, 1·5, 2, 3 *and* 4.

In the following tables, in addition to ψ_0 and ψ_2, $n\theta^{n-1}$ is also tabulated. The values of the Emden functions were taken from *Mathematical Tables*, vol. ii, on "Emden Functions," issued by L. J. Comrie on behalf of the British Association for the Advancement of Science. Only as many decimal figures are retained as are regarded to be reliable.

TABLE I

Index $n = 1$

ξ	$\frac{6}{\xi^2} - 1$	ψ_0	ψ_2
0·	∞	0·	0·
0·2	149·0000	0·00665	0·039886
0·4	36·5000	0·02645	0·158180
0·6	15·6667	0·05893	0·350835
0·8	8·3750	0·10330	0·611258
1·0	5·0000	0·15853	0·93053
1·2	3·16666	0·22330	1·2977
1·4	2·06122	0·28611	1·7001
1·6	1·34375	0·37527	2·1239
1·8	0·85185	0·45897	2·5544
2·0	0·50000	0·54535	2·9767
2·2	0·23967	0·63250	3·3759
2·4	0·04167	0·71856	3·7380
2·6	−0·11243	0·80173	4·0499
2·8	−0·23469	0·88036	4·3002
3·0	−0·33333	0·95296	4·4795
3·2	−0·41406	...	4·5804
3·4	−0·48097	...	4·5982

$\xi_1 = 3·14159,$

$\psi_0(\xi_1) = 1·00000, \quad \psi_2(\xi_1) = 4·55940,$

$\psi_0'(\xi_1) = 0·31831, \quad \psi_2'(\xi_1) = 0·42060.$

TABLE II

Index $n = 1.5$

ξ	$1.5\theta^{\frac{1}{2}}$	ψ_0	ψ_2
0.0	1.5000	0.	0.
0.2	1.4950	0.0066467	0.0398291
0.4	1.4801	0.026351	0.157289
0.6	1.4556	0.058423	0.346477
0.8	1.4217	0.101785	0.59813
1.0	1.3790	0.155061	0.90040
1.2	1.3281	0.21669	1.23978
1.4	1.2698	0.28506	1.6022
1.6	1.2046	0.35862	1.9741
1.8	1.1332	0.43601	2.3430
2.0	1.0564	0.51612	2.6985
2.2	0.97449	0.59822	3.0324
2.4	0.88802	0.68194	3.3401
2.6	0.79699	0.76739	3.6197
2.8	0.70067	0.85511	3.8714
3.0	0.59787	0.94618	4.0992
3.2	0.48403	1.04226	4.3095
3.4	0.35286	1.14581	4.5121
3.6	0.15797	1.26074	4.7218

$\xi_1 = 3.6538,$

$\psi_0(\xi_1) = 1.2942, \qquad \psi_2(\xi_1) = 4.7820,$

$\psi_0'(\xi_1) = 0.6364, \qquad \psi_2'(\xi_1) = 1.1495.$

TABLE III

Index $n = 2$

ξ	2θ	ψ_0	ψ_2
0.	2.00000	0.	0.
0.2	1.98672	0.0066401	0.039773
0.4	1.94751	0.026249	0.15641
0.6	1.88419	0.057935	0.34226
0.8	1.79959	0.100359	0.58572
1.0	1.69731	0.152110	0.87277
1.2	1.58134	0.21126	1.1886
1.4	1.45582	0.27629	1.5190
1.6	1.32472	0.34605	1.8519
1.8	1.19165	0.41957	2.1778
2.0	1.05967	0.49641	2.4904
2.2	0.93130	0.57652	2.7862
2.4	0.80842	0.66030	3.0643
2.6	0.69237	0.74845	3.3263
2.8	0.58398	0.84195	3.5751
3.0	0.48365	0.94200	3.8151
3.2	0.39145	1.0500	4.0515
3.4	0.30720	1.1675	4.2898
3.6	0.23051	1.2963	4.5361
3.8	0.16086	1.4381	4.7966
4.0	0.09768	1.5949	5.0776
4.2	0.04032	1.7691	5.3855
4.4	...	1.9629	5.7274

$\xi_1 = 4.3529,$

$\psi_0(\xi_1) = 1.9153, \qquad \psi_2(\xi_1) = 5.6431,$

$\psi_0'(\xi_1) = 0.9961, \qquad \psi_2'(\xi_1) = 1.7559.$

TABLE IV

Index $n=3$

ξ	$3\theta^2$	ψ_0	ψ_2
0·	3·000	0·	0·
0·2	2·960	0·0066271	0·039657
0·4	2·846	0·026050	0·15470
0·6	2·668	0·057010	0·33425
0·8	2·444	0·09768	0·56286
1·0	2·194	0·14637	0·8237
1·2	1·932	0·20142	1·1015
1·4	1·676	0·26168	1·3841
1·6	1·434	0·32655	1·6632
1·8	1·215	0·39596	1·9341
2·0	1·019	0·47021	2·195
2·2	0·8486	0·5499	2·448
2·4	0·7023	0·6360	2·693
2·6	0·5785	0·7292	2·935
2·8	0·4743	0·8306	3·175
3·0	0·3870	0·9412	3·419
3·2	0·3147	1·0618	3·668
3·4	0·2549	1·1932	3·924
3·6	0·2055	1·3364	4·192
3·8	0·1648	1·4920	4·472
4·0	0·1314	1·6607	4·767
4·2	0·1040	1·8431	5·078
4·4	0·08168	2·0398	5·407
4·6	0·06352	2·2513	5·753
4·8	0·04876	2·4780	6·121
5·0	0·03682	2·7203	6·506
5·2	0·02730	2·9787	6·916
5·4	0·01971	3·2532	7·346
5·6	0·01376	3·5443	7·798
5·8	0·00853	3·8520	8·272
6·0	0·00574	4·1765	8·768
6·2	0·00325	4·5180	9·288
6·4	0·00155	4·8760	9·829
6·6	0·00052	5·2510	10·39
6·8	0·00005	5·6429	10·98
7·0	...	6·0510	11·59

$$\xi_1 = 6{\cdot}8969,$$

$$\psi_0(\xi_1) = 5{\cdot}8380, \qquad \psi_2(\xi_1) = 11{\cdot}2780,$$

$$\psi_0'(\xi_1) = 2{\cdot}0391, \qquad \psi_2'(\xi_1) = 3{\cdot}0409.$$

TABLE V

Index $n=4$

ξ	$4\theta^3$	ψ_0	ψ_2
0·	4·0000	0·	0·
0·2	3·9209	0·0066140	0·0395434
0·4	3·6976	0·025857	0·153027
0·6	3·3651	0·056144	0·32671
0·8	2·9682	0·095460	0·54239
1·0	2·5514	0·141909	0·78215
1·2	2·1490	0·19413	1·03149
1·4	1·7824	0·25144	1·2809
1·6	1·4625	0·31378	1·5255
1·8	1·1915	0·37044	1·7636
2·0	0·96690	0·45543	1·9962
2·2	0·78328	0·53624	2·2254
2·4	0·63448	0·62487	2·4536
2·6	0·51458	0·72217	2·6837
2·8	0·41824	0·82892	2·9181
3·0	0·34076	0·94583	3·1591
3·2	0·27853	1·07355	3·4087
3·4	0·22845	1·2126	3·6683
3·6	0·18807	1·3636	3·9394
3·8	0·15533	1·5268	4·2230
4·0	0·12869	1·7027	4·5200
4·2	0·10695	1·8916	4·8311
4·4	0·08916	2·0938	5·1569
4·6	0·07454	2·3096	5·4979
4·8	0·06251	2·5390	5·8546
5·0	0·05253	2·7825	6·2270
5·2	0·04424	3·0400	6·6161
5·6	0·03159	3·5980	7·4434
6·0	0·02271	4·2140	8·3380
6·4	0·01639	4·8886	9·3008
6·8	0·01190	5·6223	10·3326
7·2	0·008648	6·4156	11·4337
7·6	0·006285	7·2685	12·6045
8·0	0·004566	8·1811	13·8451
8·4	0·003306	9·1535	15·1558
8·8	0·002384	10·1856	16·5364
9·2	0·001707	11·2773	17·9870
9·6	0·001211	12·4301	19·5076
10·0	0·000850	13·6407	21·0980
10·4	0·000588	14·9118	22·7509
10·8	0·000399	16·2414	24·4756
11·2	0·000264	17·6294	26·2663
11·6	0·000170	19·0754	28·1281
12·0	0·000105	20·5792	30·0592
12·4	0·000062	22·1405	32·0591
12·8	0·000034	23·7589	34·1276
13·2	0·000017	25·4344	36·2645
13·6	0·000007	27·1664	38·4695
14·0	0·000002	28·9549	40·7424
14·4	0·0000004	30·7996	43·0830
14·8	0·0000000	32·7003	45·4911
15·2	...	34·6567	47·9667

$\xi_1 = 14{\cdot}9715$,

$\psi_0(\xi_1) = 33{\cdot}5327$, $\psi_2(\xi_1) = 46{\cdot}5444$,

$\psi_0'(\xi_1) = 4{\cdot}8812$, $\psi_2'(\xi_1) = 6{\cdot}1766$.

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PHOTOGRAPHIC OBSERVATIONS OF COMETS MADE AT THE ROYAL OBSERVATORY, GREENWICH, 1926–27.

G. Merton, M.A., Ph.D.

(Communicated by the Astronomer Royal)

Through the kindness of the Astronomer Royal I was enabled to obtain some photographic observations of comets with the 30-inch reflector, at times when it was available for such work between 1927 February and August, and with short focus cameras mounted on the 13-inch astrographic telescope.

The work with the short focus cameras was started with a Ross portrait lens of 17-inch focal length working at $F/5\frac{1}{2}$ to see what could be done with such an instrument. This work was continued after 1926 November 13 with an Aldis aeroplane lens and a Cooke Aviar lens, both of 20-inch focal length and working at $F/5{\cdot}6$. They gave excellent definition over nearly the whole of a 5×4-inch plate. The results show that the accuracy of the deduced positions is surprisingly good, especially when, as in this case, duplicate exposures are made; it is in fact of the same order as the best visual observations of comets, say $\pm 2''$. The results obtained with the 30-inch reflector are, of course, in general entitled to greater weight.

In Table I the last column indicates the instrument used, 30-inch reflector (R) or short focus cameras (S), and the number of plates or exposures on which each result depends. The cometary magnitudes given are based on careful estimates of the integrated light. One plate of Comet Pons-Winnecke taken by Mr. Witchell on June 23 with the 13-inch astrographic (A) was also reduced, and is included.

Table II gives the necessary information concerning the comparison stars used in the reductions.

Table I

Comet 1926 *f—Comas Sola*

	U.T. 1926	α 1926·0	δ 1926·0	$\log p_\alpha \Delta$	$\log p_\delta \Delta$	Comp. Stars	Mag.	Exp.
		h m s	° ′ ″					
1	Nov. 10·1394	2 51 35·43	+ 6 49 16·3	9·485	0·809	1 to 3	12	S3
2	,, 12·9296	2 48 46·63	7 0 6·7	9·014*n*	0·789	4 and 5	11½	S2
3	,, 14·9973	2 46 40·66	7 8 58·9	8·834	0·788	6 and 7	12	S2
4	Dec. 24·9133	2 21 41·59	12 24 58·9	9·218	0·750	8 and 9	11½	S2
	1927	α 1927·0	δ 1927·0					
5	Jan. 27·86425	2 44 26·29	+19 30 41·5	9·361	0·697	10 and 11	12·3	S2
6	Apr. 30·92346	6 26 59·63	33 10 27·5	9·640	0·772	12 and 13	...	R1
7	May 1·92211	6 30 1·34	33 9 56·2	9·640	0·772	14 to 17	...	R1
8	,, 8·88425	6 51 8·88	32 9 30·5	9·635	0·724	18 and 19	...	R1