The Equivalence of Hamiltonians of Holstein-Primakoff and Dyson in Spin-Wave Theory in Ferromagnetism

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The Heisenberg model of a ferromagnet is described by the following Hamiltonian,

$$\mathcal{H}=g\mu H\sum_{l} S_l^z-2J\sum_{\langle l,m\rangle}\boldsymbol{S}_l\cdot\boldsymbol{S}_m. \quad (1)$$

Here the notations are those that are commonly used. Following Holstein and Primakoff,[1] we express (1) by the creation and annihilation operators of spin deviation as follows,

$$\mathcal{H}=g\mu H\sum_{l}(S-a_l^* a_l)$$
$$-2J\sum_{\langle l,m\rangle}[(S-a_l^* a_l)(S-a_m^* a_m)$$
$$+S(f_l a_l a_m^* f_m+a_l^* f_l f_m a_m)], \quad (2)$$

where

$$f_l=(1-a_l^* a_l/2S)^{1/2}. \quad (3)$$

We have calculated[2] the spontaneous magnetization from (2) by the spin-wave method taking account of the spin-wave interactions as the perturbation and obtained the same result to the first order in 1/S as Dyson's[3] which is regarded as rigorous at low temperatures. Now we shall show that (2) can be reduced to Dyson's Hamiltonian $\mathcal{H}_D$ by a simple transformation so that (2) is completely equivalent to $\mathcal{H}_D$.

Let us transform $\mathcal{H}$ by a non-unitary matrix T as follows,

$$\mathcal{H}'=T^{-1}\mathcal{H}T, \quad (4)$$

where $\mathcal{H}'$ is expressed by a_l', $a_l'^*$, etc. instead of a_l, a_l^*, etc.; a_l' and $a_l'^*$ are defined by

$$a_l'=T^{-1}a_lT,\ a_l'^*=T^{-1}a_l^*T, \quad (5)$$

and they satisfy the following commutation rule,

$$[a_l',\ a_m'^*]=\delta_{l,m}. \quad (6)$$

We note that a_l' and $a_l'^*$ are not Hermitian conjugate to each other, therefore $\mathcal{H}'$ is not Hermitian. The eigenvalues of $\mathcal{H}'$, however, are the same as those of $\mathcal{H}$. We introduce the operators α_l and α_l^* in the forms

$$a_l'=(1-\alpha_l^*\alpha_l/2S)^{1/2}\alpha_l,$$
$$a_l'^*=\alpha_l^*(1-\alpha_l^*\alpha_l/2S)^{-1/2}, \quad (7)$$

where α_l and α_l^* satisfy the following relations,

$$\alpha_l^*\alpha_l=a_l'^*a_l', \quad (8)$$

$$[\alpha_l,\ \alpha_m^*]=\delta_{l,m}. \quad (9)$$

Using (7) to (9), $\mathcal{H}'$ can be written as follows,

$$\mathcal{H}'=Ng\mu SH-NzJS^2-g\mu H\sum_l\alpha_l^*\alpha_l$$
$$+2JS\sum_{\langle l,m\rangle}(\alpha_l^*-\alpha_m^*)(\alpha_l-\alpha_m)$$
$$+J\sum_{\langle l,m\rangle}\alpha_l^*\alpha_m^*(\alpha_l-\alpha_m)^2. \quad (10)$$

This is perfectly in agreement with $\mathcal{H}_D$.

A similar method can be applied to antiferromagnetism. We use the operators α_l, α_l^* defined in (7) for + spin sublattice and β_m, β_m^* which are derived from the annihilation and creation operators b_m', $b_m'^*$ of the spin deviation for $-$spin sublattice as follows,

$$b_m'=\left(1-\frac{\beta_m^*\beta_m}{2S}\right)^{-1/2}\beta_m,$$

$$b_m'^*=\beta_m^*\left(1-\frac{\beta_m^*\beta_m}{2S}\right)^{1/2} \quad (11)$$

$$[\beta_m,\ \beta_{m'}^*]=\delta_{m,m'}. \quad (12)$$

Thus the Hamiltonian can be written as

$$\mathcal{H}'=-z|J|NS^2-\kappa NS^2$$
$$+2|J|S\sum_{\langle l,m\rangle}(\alpha_l^*\alpha_l+\beta_m^*\beta_m$$
$$+\alpha_l\beta_m+\alpha_l^*\beta_m^*)$$
$$+g\mu H(-\sum_l\alpha_l^*\alpha_l+\sum_m\beta_m^*\beta_m)$$
$$+2\kappa s(1-1/2S)$$
$$\times(\sum_l\alpha_l^*\alpha_l+\sum_m\beta_m^*\beta_m)$$
$$-|J|\sum_{\langle l,m\rangle}\alpha_l^*(\alpha_l+\beta_m^*)^2\beta_m$$
$$-\kappa(\sum_l\alpha_l^{*2}\alpha_l^2+\sum_m\beta_m^{*2}\beta_m^2), \quad (13)$$

where H is directed along the antiferromagnetic axis, and κ is a constant proportional to uniaxial anisotropy energy. It is a very interesting problem to obtain a ground-state energy from (13).

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1) T. Holstein and H. Primakoff, Phys. Rev. **58** (1940), 1908.
2) T. Oguchi, Phys. Rev. **117** (1960), 117.
3) F. J. Dyson, Phys. Rev. **102** (1956), 1217, 1230.