Research Article

Bing He, Yong Hong*, and Qiang Chen

The equivalent parameter conditions for constructing multiple integral half-discrete Hilbert-type inequalities with a class of nonhomogeneous kernels and their applications

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Abstract: In this paper, we establish equivalent parameter conditions for the validity of multiple integral half-discrete Hilbert-type inequalities with the nonhomogeneous kernel $G(n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})$ $(\lambda_1\lambda_2 > 0)$ and obtain best constant factors of the inequalities in specific cases. In addition, we also discuss their applications in operator theory.

Keywords: multiple integral half-discrete Hilbert-type inequality, nonhomogeneous kernel, equivalent parameter condition, best constant factor, operator norm, bounded operator

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1 Introduction and preliminary knowledge

Suppose that $\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $\widetilde{a}=\{a_m\}\in l_p$, $\widetilde{b}=\{b_n\}\in L_q$, the classical Hilbert series inequality was obtained in 1925 [1]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}\leq\frac{\pi}{\sin(\pi/p)}\|\widetilde{a}\|_p\|\widetilde{b}\|_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best.

If $f(x)\in L_p(0,+\infty)$, $g(y)\in L_q(0,+\infty)$, the corresponding Hilbert integral inequality was obtained in 1934 [2]:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y\leq\frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best.

Since (1) is of great significance for the study of boundedness and norm of series operator in l_p and (2) is of great significance for the study of boundedness and norm of integral operator in $L_p(0,+\infty)$, Hilbert in-

* **Corresponding author: Yong Hong,** Department of Applied Mathematics, Guangzhou Huashang College, Guangzhou, China, e-mail: yonghonglbz@163.com
Bing He: Department of Mathematics, Guangdong University of Education, Guangzhou, China, e-mail: hzs314@163.com
Qiang Chen: Department of Computer Science, Guangdong University of Education, Guangzhou, China, e-mail: cq_c@gdei.edu.cn

equality has been widely concerned. To study the operator boundedness and operator norm from sequence space l to function space L or from function space L to sequence space l, the half-discrete Hilbert inequality has been paid more attention. In 2011, the following results were obtained [3]: If $\tilde{a} = \{a_n\} \in l_p$, $f(x) \in L_q(0, +\infty)$, then

$$\int_0^{+\infty} \sum_{n=1}^{\infty} \frac{a_n f(x)}{n + x} \mathrm{d}x \le \frac{\pi}{\sin(\pi/p)} \|\tilde{a}\|_p \|f\|_q, \tag{3}$$

the constant factor is also the best. Later on the equivalent conditions for validity of multiple integral half-discrete Hilbert-type inequality with generalized homogeneous kernel were discussed [4]. In [5], the parameter conditions for the optimal constant factor of half-discrete Hilbert-type inequality with homogeneous kernel in one dimension were established. Good results were obtained.

To further discuss the multiple integral half-discrete Hilbert-type inequality, we need to introduce the following notations: Suppose that $m \in \mathbb{N}_+$, $x = (x_1, x_2, \ldots, x_m)$, $\mathbb{R}_+^m = \{x = (x_1, x_2, \ldots, x_m) : x_i > 0, i = 1, 2, \ldots, m\}$. For $\rho > 0$, the norm of x is defined by

$$\|x\|_{m,\rho} = (x_1^\rho + x_2^\rho + \cdots + x_m^\rho)^{1/\rho}.$$

Spaces l and L are defined by, respectively,

$$l_p^\alpha = \left\{ \tilde{a} = \{a_n\} : \|\tilde{a}\|_{p,\alpha} = \left(\sum_{n=1}^{\infty} n^\alpha |a_n|^p \right)^{1/p} < +\infty \right\},$$

$$L_q^\beta(\mathbb{R}_+^m) = \left\{ f(x) : \|f\|_{q,\beta} = \left(\int_{\mathbb{R}_+^m} \|x\|_{m,\rho}^\beta |f(x)|^q \mathrm{d}x \right)^{1/q} < +\infty \right\}.$$

If $K(n, \|x\|_{m,\rho}) = G(n^{\lambda_1} \|x\|_{m,\rho}^{\lambda_2}) \ge 0$, then $K(n, \|x\|_{m,\rho})$ is a nonhomogeneous nonnegative function. In this paper, we will discuss the equivalent parameter conditions under which the multiple integral half-discrete Hilbert-type inequality

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}_+^m} G(n^{\lambda_1} \|x\|_{m,\rho}^{\lambda_2}) a_n f(x) \mathrm{d}x \le M \|\tilde{a}\|_{p,\alpha} \|f\|_{q,\beta} \tag{4}$$

can be established when $\lambda_1 \lambda_2 > 0$. That is, what conditions do the parameters $\alpha, \beta, \lambda_1, \lambda_2, p, q$ meet if there is a constant $M > 0$ such that (4) holds? On the contrary, if there exists a constant $M > 0$ such that (4) holds, then what conditions do the parameters $\alpha, \beta, \lambda_1, \lambda_2, p, q$ satisfy? Such problems are undoubtedly very important theoretical problems, which have not been well solved at present. At the same time, we also discuss the best constant factor of (4) and its application in operator theory. More related literature can be found in [6–21].

2 Some lemmas

By using the Hölder's inequality of integral and series, we can easily get the following lemma.

Lemma 2.1. *Assume that* $\frac{1}{p} + \frac{1}{q} = 1 \, (p > 1)$, $a_n(x) \ge 0$, $b_n(x) \ge 0$, Ω *is measurable. Then, the mixed Hölder's inequality can be obtained*

$$\int_\Omega \sum_{n=1}^{\infty} a_n(x) b_n(x) \mathrm{d}x = \sum_{n=1}^{\infty} \int_\Omega a_n(x) b_n(x) \mathrm{d}x \le \left(\int_\Omega \sum_{n=1}^{\infty} a_n^p(x) \mathrm{d}x \right)^{1/p} \left(\int_\Omega \sum_{n=1}^{\infty} b_n^q(x) \mathrm{d}x \right)^{1/q}.$$

Lemma 2.2. [22] *If $m \in \mathbb{N}_+$, $\rho > 0$, $r > 0$, $\psi(u)$ is measurable, then*

$$\int_{\|x\|_{m,\rho} \le r} \psi(\|x\|_{m,\rho})\mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^r \psi(u)u^{m-1}\mathrm{d}u,$$

$$\int_{\|x\|_{m,\rho} \ge r} \psi(\|x\|_{m,\rho})\mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_r^{+\infty} \psi(u)u^{m-1}\mathrm{d}u,$$

where $\Gamma(t)$ is the Gamma function, and $\|x\|_{m,\rho} \le r$ represents the region $\Omega_r = \{x = (x_1, x_2, \dots, x_m) : x_i \ge 0, \|x\|_{m,\rho} \le r\}$.

According to Lemma 2.2, one gets

$$\int_{\mathbb{R}_+^m} \psi(\|x\|_{m,\rho})\mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^{+\infty} \psi(u)u^{m-1}\mathrm{d}u.$$

Lemma 2.3. *Assume that $m \in \mathbb{N}_+$, $\rho > 0$, $\frac{1}{p} + \frac{1}{q} = 1\,(p > 1)$, $\lambda_1\lambda_2 > 0$, $K(n, \|x\|_{m,\rho}) = G(n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})$ is nonnegative measurable, $\frac{1}{\lambda_2}\left(\frac{m\lambda_1 + \alpha\lambda_2}{p} - \frac{\lambda_2 + \beta\lambda_1}{q}\right) = c$, and $K(t, 1)t^{-\frac{\alpha+1}{p}+c}$ is monotonically decreasing in $(0, +\infty)$. Denote that*

$$W_1 = \int_0^{+\infty} K(1, t)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t, \quad W_2 = \int_0^{+\infty} K(t, 1)t^{-\frac{\alpha+1}{p}+c}\mathrm{d}t.$$

Then,

$$\lambda_1 W_2 = \lambda_2 W_1$$

and

$$\omega_1(n) = \int_{\mathbb{R}_+^m} K(n, \|x\|_{m,\rho})\|x\|_{m,\rho}^{-\frac{\beta+m}{q}}\mathrm{d}x = n^{\frac{\lambda_1}{\lambda_2}\left(\frac{\beta+m}{q}-m\right)} \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} W_1,$$

$$\omega_2(x) = \sum_{n=1}^{\infty} K(n, \|x\|_{m,\rho})n^{-\frac{\alpha+1}{p}+c} \le \|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-1-c\right)} W_2.$$

Proof. It follows from $\frac{m\lambda_1 + \alpha\lambda_2}{p} - \frac{\lambda_2 + \beta\lambda_1}{q} = \lambda_2 c$ that $\frac{1}{\lambda_1}\left(-\frac{\alpha\lambda_2}{p} + \frac{\lambda_2}{q} + \lambda_2 c\right) - 1 = -\frac{\beta + m}{q} + m - 1$, then

$$\begin{aligned} W_2 &= \int_0^{+\infty} K\left(1, t^{\frac{\lambda_1}{\lambda_2}}\right)t^{-\frac{\alpha+1}{p}+c}\mathrm{d}t \\ &= \frac{\lambda_2}{\lambda_1}\int_0^{+\infty} K(1, u)u^{\frac{\lambda_2}{\lambda_1}\left(-\frac{\alpha+1}{p}+c\right)+\frac{\lambda_2}{\lambda_1}-1}\mathrm{d}u \\ &= \frac{\lambda_2}{\lambda_1}\int_0^{+\infty} K(1, u)u^{\frac{1}{\lambda_1}\left(-\frac{\alpha\lambda_2}{p}+\frac{\lambda_2}{q}+\lambda_2 c\right)-1}\mathrm{d}u \\ &= \frac{\lambda_2}{\lambda_1}\int_0^{+\infty} K(1, u)u^{-\frac{\beta+m}{q}+m-1}\mathrm{d}u = \frac{\lambda_2}{\lambda_1}W_1. \end{aligned}$$

Hence, $\lambda_1 W_2 = \lambda_2 W_1$.

By Lemma 2.2, one gets

$$\begin{aligned}
\omega_1(n) &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_0^{+\infty} K(n,t)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t\\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_0^{+\infty} K\left(1, t\cdot n^{\frac{\lambda_1}{\lambda_2}}\right)t^{-\frac{\beta+m}{q}+m-1}\mathrm{d}t\\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{-\frac{\lambda_1}{\lambda_2}\left(-\frac{\beta+m}{q}+m-1\right)-\frac{\lambda_1}{\lambda_2}}\int_0^{+\infty} K(1,u)u^{-\frac{\beta+m}{q}+m-1}\mathrm{d}u\\
&= n^{\frac{\lambda_1}{\lambda_2}\left(\frac{\beta+m}{q}-m\right)}\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}W_1.
\end{aligned}$$

Note that $K(t,1)t^{-\frac{\alpha+1}{p}+c}$ is monotonically decreasing in $(0,+\infty)$, thus

$$\begin{aligned}
\omega_2(x) &= \sum_{n=1}^{\infty} K\left(n\cdot\|x\|_{m,\rho}^{\lambda_2/\lambda_1},1\right)n^{-\frac{\alpha+1}{p}+c}\\
&= \|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-c\right)}\sum_{n=1}^{\infty} K\left(n\cdot\|x\|_{m,\rho}^{\lambda_2/\lambda_1},1\right)\left(n\cdot\|x\|_{m,\rho}^{\lambda_2/\lambda_1}\right)^{-\frac{\alpha+1}{p}+c}\\
&\le \|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-c\right)}\int_0^{+\infty} K\left(t\cdot\|x\|_{m,\rho}^{\lambda_2/\lambda_1},1\right)\left(t\cdot\|x\|_{m,\rho}^{\lambda_2/\lambda_1}\right)^{-\frac{\alpha+1}{p}+c}\mathrm{d}t\\
&= \|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-1-c\right)}\int_0^{+\infty} K(u,1)u^{-\frac{\alpha+1}{p}+c}\mathrm{d}u\\
&= \|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-1-c\right)}W_2.
\end{aligned}$$

□

3 Main results

Theorem 3.1. *Suppose that* $m\in\mathbb{N}_+$, $\rho>0$, $\lambda_1\lambda_2>0$, $\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $K(n,\|x\|_{m,\rho})=G\left(n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2}\right)$ *is non-negative measurable,* $\frac{1}{\lambda_2}\left(\frac{m\lambda_1+\alpha\lambda_2}{p}-\frac{\lambda_2+\beta\lambda_1}{q}\right)=c$, $K(t,1)t^{-\frac{\alpha+1}{p}}$ *and* $K(t,1)t^{-\frac{\alpha+1}{p}+c}$ *are monotonically decreasing in* $(0,+\infty)$, *and*

$$W_0=|\lambda_1|\int_0^{+\infty}K(t,1)t^{-\frac{\alpha+1}{p}}\mathrm{d}t<+\infty.$$

Then,

(*i*) *The necessary and sufficient conditions for the validity of inequality*

$$\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K(n,\|x\|_{m,\rho})a_n f(x)\mathrm{d}x\le M\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} \tag{5}$$

with some constant $M>0$ *is* $c\ge 0$, *where* $\tilde{a}=\{a_n\}\in l_p^{\alpha}$, $f(x)\in L_q^{\beta}(\mathbb{R}_+^m)$.

(*ii*) *For* $c=0$, *the best constant factor of* (5) *is*

$$\inf M=\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}.$$

Proof. (i) Suppose that (5) holds. If $c < 0$, for $\varepsilon = \frac{-c}{2|\lambda_1|} > 0$, taking

$$a_n = n^{(-\alpha-1-|\lambda_1|\varepsilon)/p}, \quad n = 1, 2, \ldots$$

and

$$f(x) = \begin{cases} \|x\|_{m,\rho}^{(-\beta-m+|\lambda_2|\varepsilon)/q}, & 0 < \|x\|_{m,\rho} \le 1, \\ 0, & \|x\|_{m,\rho} > 1. \end{cases}$$

It follows from Lemma 2.2 that

$$\begin{aligned} M\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} &= M\left(\sum_{n=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\right)^{1/p}\left(\int\limits_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{-m+|\lambda_2|\varepsilon}\mathrm{d}x\right)^{1/q} \\ &\le M\left(1 + \int\limits_{1}^{+\infty} t^{-1-|\lambda_1|\varepsilon}\mathrm{d}t\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int\limits_0^1 t^{-1+|\lambda_2|\varepsilon}\mathrm{d}t\right)^{1/q} \\ &= \frac{M}{\varepsilon|\lambda_1|^{1/p}|\lambda_1|^{1/q}}(1 + |\lambda_1|\varepsilon)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \\ &= \frac{2M}{-c}\left(\frac{\lambda_1}{\lambda_2}\right)^{1/q}\left(1 - \frac{c}{2}\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} < +\infty. \end{aligned}$$

Since $K(t, 1)t^{-\frac{\alpha+1}{p}}$ is monotonically decreasing in $(0, +\infty)$, then

$$\begin{aligned} \int\limits_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} K(n, \|x\|_{m,\rho})a_n f(x)\mathrm{d}x &= \int\limits_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{(-\beta-m+|\lambda_2|\varepsilon)/q}\left(\sum_{n=1}^{\infty} K(n, \|x\|_{m,\rho})n^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\right)\mathrm{d}x \\ &= \int\limits_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{\frac{-\beta-m+|\lambda_2|\varepsilon}{q}-\frac{\lambda_2}{\lambda_1}\frac{-\alpha-1-|\lambda_1|\varepsilon}{p}}\left[\sum_{n=1}^{\infty} K\left(n\cdot\|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}}, 1\right)\left(n\cdot\|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}}\right)^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\right]\mathrm{d}x \\ &\ge \int\limits_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+\frac{\lambda_2}{\lambda_1}\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\left[\int\limits_1^{\infty} K\left(u\cdot\|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}}, 1\right)\left(u\cdot\|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}}\right)^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}u\right]\mathrm{d}x \\ &= \int\limits_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+\frac{\lambda_2}{\lambda_1}\frac{\alpha+1+|\lambda_1|\varepsilon}{p}-\frac{\lambda_2}{\lambda_1}}\left(\int\limits_{\|x\|_{m,\rho}^{\frac{\lambda_2}{\lambda_1}}}^{\infty} K(t, 1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t\right)\mathrm{d}x \\ &\ge \int\limits_{0<\|x\|_{m,\rho}\le 1} \|x\|_{m,\rho}^{-m+\frac{\lambda_2}{\lambda_1}c+|\lambda_2|\varepsilon}\left(\int\limits_1^{+\infty} K(t, 1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t\right)\mathrm{d}x \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int\limits_0^1 t^{-1+\frac{\lambda_2}{\lambda_1}c+|\lambda_2|\varepsilon}\mathrm{d}t\int\limits_1^{+\infty} K(t, 1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int\limits_0^1 t^{-1+\frac{\lambda_2}{2\lambda_1}c}\mathrm{d}t\int\limits_1^{+\infty} K(t, 1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t. \end{aligned}$$

Consequently,

$$\int\limits_0^1 t^{-1+\frac{\lambda_2}{2\lambda_1}c}\mathrm{d}t\int\limits_1^{+\infty} K(t, 1)t^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}}\mathrm{d}t \le \frac{2M}{-c}\left(\frac{\lambda_1}{\lambda_2}\right)^{1/q}\left(1 - \frac{c}{2}\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{-1/q} < +\infty. \tag{6}$$

Note that $\frac{\lambda_2}{2\lambda_1}c < 0$, hence $\int_0^1 t^{-1+\frac{\lambda_2}{2\lambda_1}c}\mathrm{d}t = +\infty$, which contradicts (6). Therefore, $c \geq 0$.

On the contrary, suppose that $c \geq 0$. It follows from the mixed Hölder's inequality and Lemma 2.3 that

$$\begin{aligned}
\int\limits_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} K(n, \|x\|_{m,\rho}) a_n f(x)\mathrm{d}x &= \int\limits_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}\left(\frac{n^{(\alpha+1-pc)/(pq)}}{\|x\|_{m,\rho}^{(\beta+m)/(pq)}}\cdot a_n\right)\left(\frac{\|x\|_{m,\rho}^{(\beta+m)/(pq)}}{n^{(\alpha+1-pc)/(pq)}}\cdot f(x)\right)K(n, \|x\|_{m,\rho})\mathrm{d}x \\
&\leq \left(\int\limits_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}\frac{n^{(\alpha+1-pc)/q}}{\|x\|_{m,\rho}^{(\beta+m)/q}}\cdot|a_n|^p\cdot K(n, \|x\|_{m,\rho})\mathrm{d}x\right)^{1/p} \\
&\quad\times\left(\int\limits_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}\frac{\|x\|_{m,\rho}^{(\beta+m)/p}}{n^{(\alpha+1-pc)/p}}\cdot|f(x)|^q\cdot K(n, \|x\|_{m,\rho})\mathrm{d}x\right)^{1/q} \\
&= \left(\sum_{n=1}^{\infty} n^{(\alpha+1-pc)/q}|a_n|^p\omega_1(n)\right)^{1/p}\left(\int\limits_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{(\beta+m)/p}|f(x)|^q\omega_2(x)\mathrm{d}x\right)^{1/q} \\
&\leq W_1^{1/p}W_2^{1/q}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{\frac{\alpha+1-pc}{q}+\frac{\lambda_1}{\lambda_2}\left(\frac{\beta+m}{q}-m\right)}|a_n|^p\right)^{1/p} \\
&\quad\times\left(\int\limits_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\frac{\beta+m}{p}+\frac{\lambda_2}{\lambda_1}\left(\frac{\alpha+1}{p}-1-c\right)}|f(x)|^q\mathrm{d}x\right)^{1/q} \\
&= \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{\alpha-pc}|a_n|^p\right)^{1/p}\left(\int\limits_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\beta}|f(x)|^q\mathrm{d}x\right)^{1/q} \\
&\leq \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{\alpha}|a_n|^p\right)^{1/p}\left(\int\limits_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\beta}|f(x)|^q\mathrm{d}x\right)^{1/q} \\
&= \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}.
\end{aligned}$$

Taking $M \geq \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}$ arbitrarily, one can get (5).

(ii) For $c = 0$, assuming the best constant factor of (5) is M_0, then we can see from the previous proof that

$$M_0 \leq \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p},$$

$$\int\limits_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K(n, \|x\|_{m,\rho})a_n f(x)\mathrm{d}x \leq M_0\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}.$$

For $\varepsilon > 0$, sufficiently small and N sufficiently large, taking

$$a_n = n^{(-\alpha-1-|\lambda_1|\varepsilon)/p}, \quad n = 1, 2, \ldots$$

$$f(x) = \begin{cases}\|x\|_{m,\rho}^{(-\beta-m+|\lambda_2|\varepsilon)/q}, & 0 < \|x\|_{m,\rho} \leq N,\\ 0, & \|x\|_{m,\rho} > N.\end{cases}$$

Then,

$$
\begin{aligned}
M_0\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} &= M_0\left(\sum_{k=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\right)^{1/p}\left(\int\limits_{0<\|x\|_{m,\rho}\le N} \|x\|_{m,\rho}^{-m+|\lambda_2|\varepsilon}\mathrm{d}x\right)^{1/q} \\
&\le M_0\left(1+\int_1^{+\infty} t^{-1-|\lambda_1|\varepsilon}\mathrm{d}t\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_0^N t^{-1+|\lambda_2|\varepsilon}\mathrm{d}t\right)^{1/q} \\
&= \frac{M_0}{\varepsilon|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q} N^{|\lambda_2|\varepsilon/q}(1+|\lambda_1|\varepsilon)^{1/p},
\end{aligned}
$$

$$
\begin{aligned}
\int\limits_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} K(n,\|x\|_{m,\rho})a_n f(x)\mathrm{d}x &= \sum_{n=1}^{\infty} n^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\left(\int\limits_{0<\|x\|_{m,\rho}\le N} K(n,\|x\|_{m,\rho})\|x\|_{m,\rho}^{(-\beta-m+|\lambda_2|\varepsilon)/q}\mathrm{d}x\right) \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\left(\int_0^N K(n,t)t^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t\right) \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\left(\int_0^N K\left(1,t\cdot n^{\lambda_1/\lambda_2}\right)t^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t\right) \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-\frac{\alpha+1+|\lambda_1|\varepsilon}{p}+\frac{\lambda_1}{\lambda_2}\left(\frac{\beta+m-|\lambda_2|\varepsilon}{q}-m+1\right)-\frac{\lambda_1}{\lambda_2}}\int_0^{N\cdot n^{\lambda_1/\lambda_2}} K(1,u)u^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\int_0^{N\cdot n^{\lambda_1/\lambda_2}} K(1,u)u^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u \\
&\ge \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\int_0^{N} K(1,u)u^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u \\
&\ge \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_1^{+\infty} t^{-1-|\lambda_1|\varepsilon}\mathrm{d}t\int_0^{N} K(1,u)u^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u \\
&= \frac{\Gamma^m(1/\rho)}{|\lambda_1|\varepsilon\rho^{m-1}\Gamma(m/\rho)}\int_N^{+\infty} K(1,u)u^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u.
\end{aligned}
$$

Consequently,

$$
\frac{\Gamma^m(1/\rho)}{|\lambda_1|\varepsilon\rho^{m-1}\Gamma(m/\rho)}\int_0^N K(1,u)u^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u \le \frac{M_0}{\varepsilon|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q} N^{|\lambda_2|\varepsilon/q}(1+|\lambda_1|\varepsilon)^{1/p},
$$

$$
\frac{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}{|\lambda_1|}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\int_0^N K(1,u)u^{-\frac{\beta+m-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u \le M_0\cdot N^{|\lambda_2|\varepsilon/q}(1+|\lambda_1|\varepsilon)^{1/p}.
$$

Let $\varepsilon \to 0^+$, then

$$
\frac{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}{|\lambda_1|}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\int_0^N K(1,u)u^{-\frac{\beta+m}{q}+m-1}\mathrm{d}u \le M_0.
$$

In addition, let $N \to +\infty$, we have

$$\frac{1}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}|\lambda_2|\int_0^{+\infty}K(1,u)u^{-\frac{\beta+m}{q}+m-1}\mathrm{d}u \le M_0.$$

It follows from Lemma 2.3 that

$$\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \le M_0.$$

Hence, the best constant factor of (5) is

$$M_0 = \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}. \qquad \square$$

4 Applications in operator theory

The series operator T_1 and singular integral operator T_2 are defined by, respectively,

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty}K(n,\|x\|_{m,\rho})a_n, \quad T_2(f)_n = \int_{\mathbb{R}_+^m}K(n,\|x\|_{m,\rho})f(x)\mathrm{d}x. \tag{7}$$

According to the basic theory of Hilbert-type inequality, (5) can be equivalently written as the following two expressions:

$$\|T_1(\tilde{a})\|_{p,\beta(1-p)} \le M\|\tilde{a}\|_{p,\alpha} \quad \text{and} \quad \|T_2(f)\|_{q,\alpha(1-q)} \le M\|f\|_{q,\beta}.$$

Thus, by Theorem 3.1, one has

Theorem 4.1. *Assume that* $m \in \mathbb{N}_+$, $\rho > 0$, $\lambda_1\lambda_2 > 0$, $\frac{1}{p}+\frac{1}{q} = 1\,(p > 1)$, $K(n,\|x\|_{m,\rho}) = G(n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})$ *is non-negative measurable,* $\frac{1}{\lambda_2}\left(\frac{m\lambda_1+\alpha\lambda_2}{p} - \frac{\lambda_2+\beta\lambda_1}{q}\right) = c$, $K(t,1)t^{-\frac{\alpha+1}{p}}$ *and* $K(t,1)t^{-\frac{\alpha+1}{p}+c}$ *are monotonically decreasing in* $(0,+\infty)$, *and*

$$W_0 = |\lambda_1|\int_0^{+\infty}K(t,1)t^{-\frac{\alpha+1}{p}}\mathrm{d}t$$

is convergent. Then,

(*i*) $T_1 : l_p^{\alpha} \to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ *and* $T_2 : L_q^{\beta}(\mathbb{R}_+^m) \to l_q^{\alpha(1-q)}$ *are bounded operators if and only if* $c \ge 0$;

(*ii*) *For* $c = 0$, *i.e.,* $\frac{m\lambda_1+\alpha\lambda_2}{p} = \frac{\lambda_2+\beta\lambda_1}{q}$, *the operator norms of* T_1 *and* T_2 *are as follows*:

$$\|T_1\| = \|T_2\| = \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}.$$

Corollary 4.1. *Suppose that* $m \in \mathbb{N}_+$, $\rho > 0$, $\frac{1}{p}+\frac{1}{q} = 1\,(p > 1)$, $a > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\frac{1}{\lambda_2}\left(\frac{m\lambda_1+\alpha\lambda_2}{p} - \frac{\lambda_2+\beta\lambda_1}{q}\right) = l$, $\frac{p}{q} + \max\{p\lambda_1(b-a), p(\lambda_1 b-1), p(\lambda_1 c-1), p(\lambda_1 b+l-1), p(\lambda_1 c+l-1)\} < \alpha < \frac{p}{q} + p\lambda_1 c$, *and*

$$W_0 = \int_0^1 \frac{1}{(1+t)^a}\left[t^{c+\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})-1} + t^{a-b-\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})-1}\right]\mathrm{d}t.$$

The operators T_1 and T_2 *are defined by, respectively,*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} \frac{(\max\{1, n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2}\})^b \ (\min\{1, n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2}\})^c}{(1 + n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})^a} \cdot a_n,$$

$$T_2(f)_n = \int_{\mathbb{R}_+^m} \frac{(\max\{1, n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2}\})^b \ (\min\{1, n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2}\})^c}{(1 + n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})^a} \cdot f(x)\mathrm{d}x.$$

Then,

(*i*) *T_1 is a bounded operator from l_p^{α} to $L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ and T_2 is a bounded operator from $L_q^{\beta}(\mathbb{R}_+^m)$ to $l_q^{\alpha(1-q)}$ if and only if $l \geq 0$;*

(*ii*) *For $l = 0$, i.e., $\frac{m\lambda_1 + \alpha\lambda_2}{p} = \frac{\lambda_2 + \beta\lambda_1}{q}$, the operator norms of T_1 and T_2 are as follows:*

$$\|T_1\| = \|T_2\| = \frac{W_0}{\lambda_1^{1/q}\lambda_2^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}.$$

Proof. It follows from $\frac{p}{q} + p\lambda_1(b - a) < \alpha < \frac{p}{q} + p\lambda_1 c$ that $c + \frac{1}{\lambda_1}\left(1 - \frac{\alpha+1}{p}\right) > 0$, $a - b - \frac{1}{\lambda_1}\left(1 - \frac{\alpha+1}{p}\right) > 0$, and the integral in W_0 is convergent. Denote that

$$K(n, \|x\|_{m,\rho}) = \frac{(\max\{1, n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2}\})^b \ (\min\{1, n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2}\})^c}{(1 + n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})^a}.$$

Then,

$$\begin{aligned} W_0 &= \lambda_1 \int_0^{+\infty} K(t, 1)t^{-\frac{\alpha+1}{p}}\mathrm{d}t \\ &= \lambda_1 \int_0^{+\infty} \frac{(\max\{1, t^{\lambda_1}\})^b \ (\min\{1, t^{\lambda_1}\})^c}{(1 + t^{\lambda_1})^a} \cdot t^{-\frac{\alpha+1}{p}}\mathrm{d}t \\ &= \int_0^{+\infty} \frac{(\max\{1, u\})^b \ (\min\{1, u\})^c}{(1 + u)^a} \cdot u^{\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})-1}\mathrm{d}u \\ &= \int_0^1 \frac{u^{c+\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})-1}}{(1 + u)^a}\mathrm{d}u + \int_1^{+\infty} \frac{u^{b+\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})-1}}{(1 + u)^a}\mathrm{d}u \\ &= \int_0^1 \frac{1}{(1 + t)^a}\left[t^{c+\frac{1}{\lambda_1}\left(1-\frac{\alpha+1}{p}\right)-1} + t^{a-b-\frac{1}{\lambda_1}\left(1-\frac{\alpha+1}{p}\right)-1}\right]\mathrm{d}t. \end{aligned}$$

According to $\alpha > \frac{p}{q} + p(\lambda_1 b - 1)$ and $\alpha > \frac{p}{q} + p(\lambda_1 c - 1)$, one has $\lambda_1 b - \frac{\alpha+1}{p} < 0$, $\lambda_1 c - \frac{\alpha+1}{p} < 0$, and

$$K(t, 1)t^{-\frac{\alpha+1}{p}} = \frac{(\max\{1, t^{\lambda_1}\})^b \ (\min\{1, t^{\lambda_1}\})^c}{(1 + t^{\lambda_1})^a} \cdot t^{-\frac{\alpha+1}{p}} = \begin{cases} \dfrac{1}{(1 + t^{\lambda_1})^a} \cdot t^{\lambda_1 c-\frac{\alpha+1}{p}}, & 0 < t \leq 1, \\ \dfrac{1}{(1 + t^{\lambda_1})^a} \cdot t^{\lambda_1 b-\frac{\alpha+1}{p}}, & t > 1. \end{cases}$$

Thus, $K(t, 1)t^{-\frac{\alpha+1}{p}}$ is monotonically decreasing in $(0, +\infty)$.

Note that $\alpha > \frac{p}{q} + p(\lambda_1 b + l - 1)$ and $\alpha > \frac{p}{q} + p(\lambda_1 c + l - 1)$, we get $\lambda_1 b - \frac{\alpha+1}{p} + l < 0$, $\lambda_1 c - \frac{\alpha+1}{p} + l < 0$, and

$$K(t, 1)t^{-\frac{\alpha+1}{p}+l} = \begin{cases} \dfrac{1}{(1 + t^{\lambda_1})^a} \cdot t^{\lambda_1 c-\frac{\alpha+1}{p}+l}, & 0 < t \leq 1, \\ \dfrac{1}{(1 + t^{\lambda_1})^a} \cdot t^{\lambda_1 b-\frac{\alpha+1}{p}+l}, & t > 1. \end{cases}$$

Therefore, $K(t,1)t^{-\frac{\alpha+1}{p}+l}$ is monotonically decreasing in $(0,+\infty)$.

Finally, it follows from Theorem 4.1 that Corollary 4.1 holds. □

Taking $c = b$ in Corollary 4.1, according to the properties of Beta function, one gets

$$W_0 = \int_0^1 \frac{1}{(1+t)^a}\left[t^{b+\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})-1} + t^{a-\left(b+\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})\right)-1}\right]\mathrm{d}t = B\left(b+\frac{1}{\lambda_1}\left(1-\frac{\alpha+1}{p}\right), a-\left[b+\frac{1}{\lambda_1}\left(1-\frac{\alpha+1}{p}\right)\right]\right).$$

Hence, we have

Corollary 4.2. *Assume that* $m\in\mathbb{N}_+$, $\rho>0$, $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $a>0$, $\lambda_1>0$, $\lambda_2>0$, $\frac{1}{\lambda_2}\left(\frac{m\lambda_1+\alpha\lambda_2}{p}-\frac{\lambda_2+\beta\lambda_1}{q}\right)=l$, $\frac{p}{q}+\max\{p\lambda_1(b-a), p(\lambda_1 b-1), p(\lambda_1 b+l-1)\}<\alpha<\frac{p}{q}+p\lambda_1 b$, *operators* T_1 *and* T_2 *are defined by, respectively,*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty}\frac{(n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})^b}{(1+n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})^a}\cdot a_n,$$

$$T_2(f)_n = \int_{\mathbb{R}_+^m}\frac{(n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})^b}{(1+n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})^a}\cdot f(x)\mathrm{d}x.$$

Then,

(*i*) $T_1: l_p^{\alpha}\to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ *and* $T_2: L_q^{\beta}(\mathbb{R}_+^m)\to l_q^{\alpha(1-q)}$ *are bounded operators if and only if* $l\geq 0$.

(*ii*) *For* $l=0$, *i.e.,* $\frac{m\lambda_1+\alpha\lambda_2}{p}=\frac{\lambda_2+\beta\lambda_1}{q}$, *the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\| = \|T_2\| = \frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}}B\left(b+\frac{1}{\lambda_1}\left(1-\frac{\alpha+1}{p}\right), a-\left[b+\frac{1}{\lambda_1}\left(1-\frac{\alpha+1}{p}\right)\right]\right)\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}.$$

We can get the following results by taking $b=0$ in Corollary 4.2.

Corollary 4.3. *Assume that* $m\in\mathbb{N}_+$, $\rho>0$, $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $a>0$, $\lambda_1>0$, $\lambda_2>0$, $\frac{1}{\lambda_2}\left(\frac{m\lambda_1+\alpha\lambda_2}{p}-\frac{\lambda_2+\beta\lambda_1}{q}\right)=l$, $\max\{p(1-\lambda_1 a)-1, -1, pl-1\}<\alpha<p-1$, *operators* T_1 *and* T_2 *are defined by, respectively,*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty}\frac{a_n}{(1+n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})^a} \quad \text{and} \quad T_2(f)_n = \int_{\mathbb{R}_+^m}\frac{f(x)\mathrm{d}x}{(1+n^{\lambda_1}\|x\|_{m,\rho}^{\lambda_2})^a}.$$

Then,

(*i*) $T_1: l_p^{\alpha}\to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ *and* $T_2: L_q^{\beta}(\mathbb{R}_+^m)\to l_q^{\alpha(1-q)}$ *are bounded operators if and only if* $l\geq 0$.

(*ii*) *For* $l=0$, *i.e.,* $\frac{m\lambda_1+\alpha\lambda_2}{p}=\frac{\lambda_2+\beta\lambda_1}{q}$, *the operator norms of* T_1 *and* T_2 *are expressed as follows*:

$$\|T_1\| = \|T_2\| = \frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}}B\left(\frac{1}{\lambda_1}\left(1-\frac{\alpha+1}{p}\right), a-\frac{1}{\lambda_1}\left(1-\frac{\alpha+1}{p}\right)\right)\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}.$$

In Corollary 4.1, let $m=1$, $\alpha=\frac{1}{\lambda_2}(p\sigma-\lambda_1)$ and $\beta=\frac{1}{\lambda_1}(q\sigma-\lambda_2)$, then $\frac{m\lambda_1+\alpha\lambda_2}{p}=\sigma=\frac{\lambda_2+\beta\lambda_1}{q}$. The following results can be obtained.

Corollary 4.4. *Assume that* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $a>0$, $\lambda_1>0$, $\lambda_2>0$, $\frac{\lambda_1}{p}+\frac{\lambda_2}{q}+\max\{\lambda_1\lambda_2(b-a), \lambda_1\lambda_2 b-\lambda_2, \lambda_1\lambda_2 c-\lambda_2\}<\sigma<\frac{\lambda_1}{p}+\frac{\lambda_2}{q}+\lambda_1\lambda_2 c$, $\alpha=\frac{1}{\lambda_2}(p\sigma-\lambda_1)$, $\beta=\frac{1}{\lambda_1}(q\sigma-\lambda_2)$. *Denote that*

$$W_0 = \int_0^1\frac{1}{(1+t)^a}\left[t^{\frac{1}{\lambda_1\lambda_2}\left(\frac{\lambda_1}{p}+\frac{\lambda_2}{q}+\lambda_1\lambda_2 c-\sigma\right)-1} + t^{a-\frac{1}{\lambda_1\lambda_2}\left(\frac{\lambda_1}{p}+\frac{\lambda_2}{q}+\lambda_1\lambda_2 b-\sigma\right)-1}\right]\mathrm{d}t.$$

Operators T_1 and T_2 are defined by, respectively,

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} \frac{(\max\{1, n^{\lambda_1}x^{\lambda_2}\})^b \ (\min\{1, n^{\lambda_1}x^{\lambda_2}\})^c}{(1 + n^{\lambda_1}x^{\lambda_2})^a} \cdot a_n,$$

$$T_2(f)_n = \int_0^{+\infty} \frac{(\max\{1, n^{\lambda_1}x^{\lambda_2}\})^b \ (\min\{1, n^{\lambda_1}x^{\lambda_2}\})^c}{(1 + n^{\lambda_1}x^{\lambda_2})^a} \cdot f(x)\mathrm{d}x.$$

Then, $T_1 : l_p^{\alpha} \to L_p^{\beta(1-p)}(0, +\infty)$ and $T_2 : L_q^{\beta}(0, +\infty) \to l_q^{\alpha(1-q)}$ are bounded operators, and the operator norms of T_1 and T_2 are expressed as follows:

$$\|T_1\| = \|T_2\| = \frac{W_0}{\lambda_1^{1/q}\lambda_2^{1/p}}.$$

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