THE EQUIVARIANT COHOMOLOGY OF WEIGHTED PROJECTIVE SPACES

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Abstract. Our main purpose is to compute the equivariant cohomology of weighted projective space, with integer coefficients and pairwise coprime weights; we work in the context of piecewise polynomials. We also outline aspects of the problem that place it within the realms of toric topology, on which we shall elaborate in a forthcoming article. These include relationships with weighted lens spaces, homotopy colimits, the Bousfield-Kan spectral sequence, and weighted face rings. By way of application, we describe the multiplicative structure in the integral cohomology ring of the weighted projectivisation of certain complex vector bundles; these have been determined additively by Al-Amrani.

1. Introduction

Let $\chi = (\chi_0, \ldots, \chi_n)$ be a vector of positive natural numbers. The associated *weighted* (sometimes known as *twisted*) *projective space* is the quotient

$$\mathbb{P}(\chi) = S^{2n+1}/S^1\langle \chi_0, \ldots, \chi_n \rangle, \tag{1.1}$$

where the χ_i indicate the weights with which S^1 acts on $S^{n+1} \subset \mathbb{C}^{n+1}$ via

$$g \cdot (x_0, \ldots, x_n) = (g^{\chi_0} x_0, \ldots, g^{\chi_n} x_n). \tag{1.2}$$

Since this quotient does not change if all weights are multiplied by a constant, we may always assume that the greatest common divisor of the weights is equal to 1. In this note we will make the stronger assumption that the weights are pairwise coprime; for $n \leq 2$, this is not a restriction.

Note that $\mathbb{P}(\chi)$ is equipped with an action of the n-dimensional torus

$$T = (S^1)^{n+1}/S^1\langle \chi_0, \ldots, \chi_n \rangle, \tag{1.3}$$

where the quotient is defined as before. Our aim is to describe $H^*_T(\mathbb{P}(\chi))$, the T-equivariant cohomology of $\mathbb{P}(\chi)$ with integer coefficients.

Kawasaki [K] has computed the ordinary cohomology ring of $\mathbb{P}(\chi)$ with integer coefficients. Additively, it is isomorphic to that of ordinary complex projective space, but multiplicatively, it is distinct. More precisely, if $c_1 \in H^2(\mathbb{P}(\chi))$ is a generator, then $H^*(\mathbb{P}(\chi))$ is generated by $c_0 = 1 \in H^0(\mathbb{P}(\chi))$ and the elements

$$c_m = \frac{c_1}{(\chi_0 \cdots \chi_n)^{m-1}} \in H^{2m}(\mathbb{P}(\chi)) \tag{1.4}$$

for $1 \leq m \leq n$, with the obvious multiplication. This result already uses our assumption that the weights are pairwise coprime.

Of course c_1 is the first Chern class of a complex line bundle ψ over $\mathbb{P}(\chi)$. The circle bundle $S(\psi)$ is the *weighted lens space*

$$\mathbb{L}(\chi) = S^{2n+1}/\mathbb{Z}\langle \chi_0, \ldots, \chi_n \rangle, \tag{1.5}$$

whose quotient by the free action of the circle $S^1\langle\chi\rangle$ is $\mathbb{P}(\chi)$. Kawasaki also proves that the integral cohomology group $H^{2m}(\mathbb{L}(\chi))$ is isomorphic to the cyclic group $\mathbb{Z}/?$ for $m \geq 2$, and is zero otherwise. Moreover, a simple geometric argument confirms that $\mathbb{L}(\chi)$ is homeomorphic to the double suspension $\Sigma^2\mathbb{L}(\chi')$, where ...

Our main result is Theorem 3.7, which presents $H^*_T(\mathbb{P}(\chi))$ in terms of generators and relations. The generators are 2–dimensional elements ...

2. From equivariant cohomology to piecewise polynomials

Let $\iota\colon \mathbb{P}(\chi) \to \mathbb{P}(\chi)_T$ be an inclusion of the fibre into the Borel construction.

Lemma 2.1. *$H_T^*(\mathbb{P}(\chi))$ is a free $H^*(BT)$-module. Moreover, as a ring it is generated by the image of $H^2(BT)$ in $H_T^2(\mathbb{P}(\chi))$ together with any subring $A \subset H_T^*(\mathbb{P}(\chi))$ which surjects onto $H^*(\mathbb{P}(\chi))$ under ι^*.*

Proof. According to Kawasaki, $H^*(\mathbb{P}(\chi))$ is free over $\mathbb{Z}$ and concentrated in even degrees. Hence, the Serre spectral sequence of the fibration $\mathbb{P}(\chi) \hookrightarrow \mathbb{P}(\chi)_T \to BT$ degenerates at the E_2 level, and $H_T^*(\mathbb{P}(\chi)) \cong H^*(\mathbb{P}(\chi)) \otimes H^*(BT)$ as $H^*(BT)$-modules by the Leray–Hirsch theorem. This isomorphism is induced by any (additive) section to ι^*. Since we can assume this section to take values in A, our claim is proven. □

The equivariant cohomology of ordinary complex projective space, which corresponds to the case of all weights being equal to 1, is well-known. A convenient description of it comes from the theory of toric varieties.

To wit, the space $\mathbb{P}(\chi)$ is an n-dimensional projective toric variety. It is defined by any complete simplicial fan Σ spanned by vectors $v_0, \ldots, v_n \in N = \mathbb{Z}^n$ with the following properties:

(1) The vectors $v_0, \ldots, v_n$ span N.
(2) They satisfy the relation

$$\chi_0 v_0 + \cdots + \chi_n v_n = 0. \tag{2.1}$$

[mf: insert reference. maybe Fulton?]

The equivariant cohomology of ordinary projective n-space can be described as the integral Stanley–Reisner algebra of the fan Σ,

$$\mathbb{Z}[\Sigma] = \mathbb{Z}[a_0, \ldots, a_n]/(a_0 \cdots a_n), \tag{2.2}$$

where each generator a_i has cohomological degree 2. For the general case, we will give a similar description of $H_T^*(\mathbb{P}(\chi))$ as some kind of "weighted Stanley–Reisner algebra". Our main tool will be piecewise polynomials, to which we turn now.

A function $f\colon N_{\mathbb{Q}} = \mathbb{Q}^n \to \mathbb{Q}$ is called *piecewise polynomial* if on each cone $\sigma \in \Sigma$ it coincides with some (global) polynomial $g \in \mathbb{Z}[N]$. [Is it necessary to switch from N to $N_{\mathbb{Q}}$ here?]

Proposition 2.2. *$H_T^*(\mathbb{P}(\chi))$ is isomorphic as $H^*(BT)$-algebra to the algebra of piecewise polynomials on Σ.*

Under this isomorphism, the cup product corresponds to the usual pointwise multiplication of functions, and the canonical map $H^*(BT) \to H_T^*(\mathbb{P}(\chi))$ to the inclusion of (global) polynomials.

Proof. Since $H_T^*(\mathbb{P}(\chi))$ is free over $H^*(BT)$ and moreover all isotropy groups of $\mathbb{P}(\chi)$ are connected (as for any toric variety), the so-called Chang–Skjelbred sequence

$$0 \longrightarrow H_T^*(\mathbb{P}(\chi)) \xrightarrow{j^*} H_T^*(\mathbb{P}(\chi)^T) \xrightarrow{\delta} H_T^{*+1}(\mathbb{P}(\chi)_1, \mathbb{P}(\chi)^T) \tag{2.3}$$

is exact (Franz–Puppe [FP]). Here $\mathbb{P}(\chi)^T$ denotes the T-fixed points, $\mathbb{P}(\chi)_1$ the union of $\mathbb{P}(\chi)^T$ and all 1-dimensional orbits, j the inclusion $\mathbb{P}(\chi)^T \to \mathbb{P}(\chi)$ and δ the differential of the long exact cohomology sequence for the pair $(\mathbb{P}(\chi)_1, \mathbb{P}(\chi)^T)$.

The piecewise polynomials are a way to represent the kernel of the map δ. Write $\mathcal{O}_\sigma$ for the orbit under the complexification $T_{\mathbb{C}}$ of T corresponding to $\sigma \in \Sigma$, and $\mathbb{Z}[\sigma]$ for the polynomials with integer coefficients on the linear hull of σ. Note that a polynomial on the linear hull of σ is uniquely defined by its restriction to σ.

We have for full-dimensional $\sigma \in \Sigma^n$

$$H_T^*(\mathcal{O}_\sigma) = H^*(BT) = \mathbb{Z}[\sigma], \tag{2.4}$$

and in one dimension lower for $\tau \in \Sigma^{n-1}$

$$H_T^{*+1}(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \cong H_T^{*+1}(\mathbb{CP}^1, \{0, \infty\}), \tag{2.5}$$

which after a choice of orientation of the interval $\mathcal{O}_\tau/T$ gives an isomorphism

$$H_T^{*+1}(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \cong \mathbb{Z}[\tau]. \tag{2.6}$$

Moreover, it turns out that for a facet τ of σ the differential

$$H_T^*(\mathcal{O}_\sigma) \to H_T^{*+1}(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \tag{2.7}$$

is the canonical restriction $\mathbb{Z}[\sigma] \to \mathbb{Z}[\tau]$, multiplied by ± 1 depending on the chosen orientation of $\mathcal{O}_\tau/T$.

As result we get that the differential δ from (2.3) is a signed sum of restrictions of polynomials,

$$\delta\colon \bigoplus_{\sigma\in\Sigma^n} \mathbb{Z}[\sigma] \to \bigoplus_{\tau\in\Sigma^{n-1}} \mathbb{Z}[\tau], \tag{2.8}$$

where the component in $\mathbb{Z}[\tau]$ is the difference of the restrictions of the polynomials on the two full-dimensional cones having τ as their common facet. Hence, the kernel consists of those collections of polynomials on the full-dimensional cones which glue along their common facets. But this is the same as requiring that the polynomials collect along *any* intersection $\tau = \sigma \cap \sigma'$ of two cones σ, $\sigma' \in \Sigma$. The reason is that σ and σ are connected by a sequence of cones, each containing τ. (In other words, Σ is a hereditary fan, cf. [BR].) We therefore get that the kernel of δ are the piecewise polynomial functions on Σ, i.e., the functions which are polynomial on each $\sigma \in \Sigma$. □

Remark 2.3. The integral equivariant cohomology of any smooth, not necessarily compact toric variety X_Σ is given by the integral Stanley–Reisner algebra of Σ or, equivalently, by the piecewise polynomials on Σ, cf. [BDCP], [DJ], [Br]. A canonical isomorphism between the Stanley–Reisner algebra of Σ and the algebra of piecewise polynomials on Σ can be defined by assigning the Courant function a_ρ associated with the ray ρ to the Stanley–Reisner generator corresponding to ρ. This function a_ρ is the piecewise linear function on Σ that assumes the value 1 on the generator of ρ and 0 on all other rays. It is well-defined because the smoothness of X_Σ implies that the rays of any cone in Σ can be completed to a basis of the lattice N.

Similarly, for a simplicial fan Σ the rational equivariant cohomology $H_T(X_\Sigma; \mathbb{Q})$ is given by the rational Stanley–Reisner algebra $\mathbb{Q}[\Sigma]$. [reference needed?]

3. Generators of the ring of piecewise polynomials

For $i = 0, \ldots, n$ we will write $\sigma_i \in \Sigma$ for the full-dimensional cone spanned by all fan generators except v_i. Moreover, given a piecewise polynomial f, we will denote the unique polynomial which coincides with f on σ_i by $f^{(i)}$. We call a piecewise polynomial *reduced* if it is not divisible by any rational prime.

Let a_i be the *(integral) Courant function* corresponding to v_i. By this we mean the reduced piecewise linear function that assumes a positive value on v_i and vanishes on all v_j for $j \neq i$. Moreover, let b_{ij}, $i \neq j$, be the reduced linear function that assumes a positive value on v_i and vanishes on all v_k, $i \neq k \neq j$.

Lemma 3.1. *$b_{ij}(v_i) = \chi_j$ and $b_{ij}(v_j) = -\chi_i$ for $i \neq j$.*

Proof. Applying b_{ij} to the relation (2.1) yields

$$\chi_i\, b_{ij}(v_i) = -\chi_j\, b_{ij}(v_j). \tag{3.1}$$

This implies the claim because v_i and v_j span $N/\ker b_{ij} \cong \mathbb{Z}$ and χ_i and χ_j are coprime. □

Proposition 3.2. *The functions b_{ij}, $i \neq j$, generate the linear functions.*

Proof. For given i, let N_i be the span of the linear independent set $V_i = \{v_j : j \neq i\}$ and $N_i^\vee$ its dual. By Lemma 3.1, the restriction of each b_{ij}, $j \neq i$, to N_i is divisible by χ_i, and $-b_{ij}/\chi_i \in N_i^\vee$ is an element of the basis dual to V_i.

Denote by M_i the sublattice generated by the b_{ij}, $j \neq i$, inside the dual $N^\vee$ of N, and by M the one generated by all b_{ij}, $j \neq i$. We have

$$N_i^\vee/N^\vee = (N_i^\vee/M_i) \;/\; (N^\vee/M_i). \tag{3.2}$$

Hence, the order of $N^\vee/M_i$ divides that of $N_i^\vee/M_i$, which equals χ_i^n by what we have said so far. (In fact, $|N_i^\vee/N^\vee| = |N/N_i| = \chi_i$ and therefore $|N^\vee/M_i| = \chi_i^{n-1}$, but we won't need this.)

Since the order of $N^\vee/M_i$ divides χ_i^n, the same applies to that of $N^\vee/M$ because

$$N^\vee/M = (N^\vee/M_i) \;/\; (M/M_i). \tag{3.3}$$

This implies that the order of $N^\vee/M$ divides the greatest common divisor of all χ_i^n, which is 1. □

Lemma 3.3. *Together with the linear functions, each a_i generates the piecewise linear functions.*

Proof. Let f be piecewise linear. Then $f - f^{(i)}$ vanishes on σ_i, hence is a multiple of a_i. □

Lemma 3.4. $a_i(v_i) = \prod_{j\neq i} \chi_j$ *and* $a_i^{(j)} = \prod_{i\neq k\neq j} \chi_k\, b_{ij}$ *for* $i \neq j$.

Proof. By Lemma 3.1 we get a well-defined piecewise linear function f by setting $f^{(i)} = 0$ and $f^{(j)}$ as given above for $j \neq i$. This function is reduced and assumes a positive value on v_i. Hence, it is equal to a_i. □

Lemma 3.5. *We have*

$$b_{ij} = \frac{a_i - a_j}{\prod_{i\neq k\neq j} \chi_k} \tag{3.4}$$

for $i \neq j$.

Proof. We have $a_i^{(j)} = -a_j^{(i)} = \prod_{i\neq k\neq j} \chi_k b_{ij}$ by Lemma 3.4, and $a_i^{(i)} = -a_j^{(j)} = 0$, hence, $(a_i - a_j)^{(i)} = (a_i - a_j)^{(j)}$. By Lemma 3.3, each piecewise linear function is the sum of a linear function and a multiple of a Courant function. For a Courant function a_k, the restrictions to any two maximal cones are distinct linear functions. Hence $a_i - a_j$ is in fact linear and divisible as claimed. □

We now consider higher-degree analogues of the Courant functions a_i.

Lemma 3.6. *For a subset $I \subset \{0, \ldots, n\}$ of size $m > 0$, the function $\prod_{i\in I} a_i$ is divisible by $\prod_{i\in I} \chi_i^{m-1}$.*

Proof. For a given $k \neq i$, all $a_i^{(j)}$, $i \neq j \neq k$, are divisible by χ_k. □

Hence, for $I \subset \{0, \ldots, n\}$ with $|I| = m > 0$ we may define the piecewise polynomial function

$$a_I = \frac{\prod_{i\in I} a_i}{\prod_{i\in I} \chi_i^{m-1}} \tag{3.5}$$

of polynomial degree m (and cohomological degree $2m$).

Theorem 3.7. *The functions a_I, $I \neq \emptyset$, and b_{ij}, $i \neq j$, generate $H_T^*(\mathbb{P}(\chi))$ as a ring. The only relations are* (3.4), (3.5) *and* $a_0 \cdots a_n = 0$.

Proof. Since there are no more relations between the a_I and the b_{ij} in $H_T(X_\Sigma;\mathbb{Q})$, the same is true in $H_T(X_\Sigma;\mathbb{Z})$, which injects into $H_T(X_\Sigma;\mathbb{Q})$ because it is free over $\mathbb{Z}$. It remains to show that these elements are indeed ring generators.

By Proposition 3.2, the b_{ij} generate the linear functions, which are the image of $H^2(BT)$ in $H^2_T(\mathbb{P}(\chi))$. Hence, by Lemma 2.1, it suffices to show that the ring generated by the a_I surjects onto $H^*(\mathbb{P}(\chi))$. In other words, we have to show that c_m lies in the span of $\{\iota^*(a_I) : |I| = m\}$ for each $m \geq 1$.

For $m = 1$, this is true by Lemma 3.3 because we know ι^* itself to be surjective. Moreover, Lemma 3.5 implies that all elements a_i are mapped to the same element of $H^2(\mathbb{P}(\chi))$. This must necessarily be a generator, which we can assume to be c_1 (instead of $-c_1$).

For $1 < m \leq n$, we get that

$$\iota^*(a_I) = \frac{\prod_{i\in I}\iota^*(a_i)}{\prod_{i\in I}\chi_i^{m-1}} = \frac{c_1^m}{\prod_{i\in I}\chi_i^{m-1}} = \frac{\left(\prod_{i=0}^n \chi_i^{m-1}\right) c_m}{\prod_{i\in I}\chi_i^{m-1}} = \Bigl(\prod_{i\notin I}\chi_i^{m-1}\Bigr)\, c_m. \tag{3.6}$$

Because we assume the weights to be pairwise coprime, the above multiples of c_m generate $H^{2m}(\mathbb{P}(\chi))$. □

Remark 3.8. Note that one could do better by taking only some of the a_I. For example, for $|I| = 2$, it would suffice to take a_{12}, a_{34} etc. But doing so would impose an ordering of the generators.

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