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THE EQUIVARIANT DOLD THEOREM MOD k AND THE ADAMS CONJECTURE

BY

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Abstract

In this paper, we state and prove a G-equivariant version of the Dold Theorem mod k for finite groups G. We then use this theorem to prove an equivariant version of the Adams Conjecture for G cyclic, using the Becker-Gottlieb approach. The case for general G and finite structure groups is also obtained by the methods of Quillen.

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0. Basic Notions and Statement of Results

G will denote a finite ambient group, and if V is a real orthogonal finite dimensional representation of G, we shall denote the one-point compactification of V by S^V. For brevity, we shall use the term "G-module" to refer to such a representation V.

If V and W are G-modules, we say that V and W are stably G-homotopy equivalent if we can find a G-module X such that $S^{V\oplus X}$ and $S^{W\oplus X}$ are G-homotopy equivalent. (That is, there is a G-equivariant based map in each direction, with each composite homotopic to the identity through basepoint preserving equivariant maps).

In a more general setting, there is an equivariant J-homomorphism

$$J_G : K_G(X) \to \mathrm{Sph}_G(X)$$

for any finite G-CW complex X. (See, for example, [13].) Here, $K_G(-)$ denotes equivariant real orthogonal K theory and $\mathrm{Sph}_G(-)$ is the group of stable G-equivalence classes of spherical G-fibrations over X. (This group is described more fully in [16].)

The purpose of this paper is to detect at least part of the kernel of J_G for cyclic G.

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Let k be a positive integer prime to the order of G and let s be the order of k mod $|G|$. (That is, s is the smallest positive integer such that $k^s \equiv 1$ mod $|G|$). Let ψ^k denote the k^{th} equivariant Adams operation [13], and let $[\mu] \in K_G(X)$. A G-space X will be said to be G-connected if the fixed-point set X^H of X is connected for each $H \subset G$.

Our formulation of the equivariant Adams Conjecture then reads as follows:

THEOREM 0.1 (Equivariant Adams Conjecture). *Let k be a nonnegative integer, and let G be cyclic of order prime to k. Assume X is a finite G-connected G-CW complex, and let $[\mu] \in K_G(X)$.*

Then there is an integer $n \geqslant 0$ such that

$$J_G(sk^n(\psi^k - 1)[\mu]) = 0.$$

The most important technical tool used in realizing this result is an equivariant version of the Dold Theorem mod k, which we shall state below.

We shall see in §5.3 that Theorem 0.1 remains true when G is arbitrary of order prime to k and $[\mu]$ is represented by a G-vector bundle with finite structure group. By using equivariant Brauer Lift, Hauschild and May have proved the result for general finite groups of order prime to k by reducing to the case of finite structure groups.

By using different methods, McClure has proven a somewhat sharper result. All methods depend strongly on our version of the Dold Theorem mod k.

Let U denote the G-module $\mathbf{R}G^\infty$, where $\mathbf{R}G$ is the regular representation of G, and consider the group $\omega_G^0 = \lim[S^V, S^V]_G$, where V runs through all finite dimensional submodules of U, and where $[\ \ ]_G$ denotes equivariant homotopy classes. Composition of maps turns ω_G^0 into a ring, and it is well known that this ring is isomorphic with $A(G)$, and that we may define an explicit isomorphism $A(G) \to \omega_G^0$ by sending a G-set s to the Pontryagin-Thom map associated with a G-embedding of s in U. Thus we may interpret the stable degree of a G-map as an element of $A(G)$.

Let η and ξ be spherical G-fibrations over X and assume that η and ξ have corresponding fibers stably equivalent with respect to the actions by the isotropy subgroups of their projections. (Observe that, if π is any G-map, then if $x \in \text{Im}(\pi)$, G_x acts naturally on $\pi^{-1}(x)$).

Now let f: $\eta \to \xi$ be a fibrewise G-map. If $x \in X$, then the degree of the restriction of f to $\eta^{-1}(x)$ may be determined as an element of $A(G_x)$, up to multiplication by a unit in $A(G_x)$. We shall say that f has degree $M \in A(G)$ if $f|\eta^{-1}(x)$ has degree M, regarded as an element of $A(G_x)$ via the natural forgetful homomorphism, for each $x \in X$.

Our version of the equivariant Dold Theorem mod k takes the following form.

THEOREM 0.2 (Equivariant Dold Theorem mod k). *Let η and ξ be G-spherical fibrations over a finite G-CW complex X, and assume that corresponding fibers are stably equivalent with respect to the action of the appropriate (isotropy) subgroups. Let f: $\eta \to \xi$ be a fibrewise G-map of degree M in $A(G)$, and assume that M divides a power of k in $A(G)$. Then there exists an integer $n \geq 0$ and a stable fibrewise G-equivalence g: $k^n\eta \to k^n\xi$.*

Remark 0.3. (i) It would seem elegant to allow k to be an arbitrary non-zero divisor in $A(G)$, and the statement of the theorem does make sense, at least when k comes from an actual G-set (rather than a virtual one). The integer n would have to be allowed to be an element of $A(G)$ as well, and the second author has such a result for $k = 1 + G$.

(ii) The requirement that corresponding fibers "look the same" is the source of the integer s in Theorem 0.1, as its proof will show.

1. Proof of Dold mod k

The proof of Theorem 0.2 will be set out in several steps. Observe first that it suffices to assume that M is an actual power of k in $A(G)$. Indeed, if $k^i = AM$ for some $A \in A(G)$, we may replace f by its fibrewise suspension with a G-map of degree A. This replaces f by a fibrewise G-map of degree k^i, and we are in the special case.

By the results in [10], we can find a spherical fibration $\xi^\perp$ such that $\xi \wedge_{\mathscr{F}} \xi^\perp$ is stably G-fiber-homotopy trivial; $\xi \wedge_{\mathscr{F}} \xi^\perp \cong X \times S^V$ for some G-module V. (Here, $\wedge_{\mathscr{F}}$ denotes fibrewise smash product).

Consider the fibration $\eta - \xi = \eta \wedge_{\mathscr{F}} \xi^\perp$. This fibration is a V-dimensional spherical fibration in the sense of [17]. This means that, if $x \in X$ and if F denotes the fiber over x, then F is G_x-homotopy equivalent to S^V. The notion of a G-oriented V-fibration is also defined in [17], and may be described as follows.

Let p: $E \to X$ be V-dimensional. Assume that we are given an open cover of X by invariant sets of the form $G \times {}_H U$, where $U \subset X$ is open and H-invariant, together with local fibrewise G-homotopy trivializations of the form

$$\begin{array}{ccc} (G \times {}_H U) \times S^V & \xrightarrow{\Phi} & E \\ \downarrow & & \downarrow p \\ G \times {}_H U & \xrightarrow{\phi} & X, \end{array}$$

where G acts diagonally on $(G \times {}_H U) \times S^V$. Let

$$x \in \phi(G \times {}_H U) \cap \phi'(G \times {}_{H'} U')$$

for two such trivializations ϕ and ϕ', and write $x = \phi[g, u] = \phi'[g', u']$. Consider the composite

$$T_{\phi\phi'}|_x\colon S^V \xrightarrow{\iota_x} (G \times {}_H U) \times S^V \xrightarrow{\Phi} E \xrightarrow{\Phi'^{-1}} (G \times {}_{H'} U') \times S^V \xrightarrow{\pi} S^V$$

where $\iota_x(s) = ([g, u], s)$ and $\pi([\bar{g}, \bar{u}], s) = s$. That $T_{\phi\phi'}|_x$ is G_x-equivariant is straightforward. We then say that ϕ and ϕ' are compatible if $T_{\phi\phi'}|_x$ is stably G_x-homotopic to 1 (or, equivalently, has degree 1 in $A(G_x)$) for

$$x \in \phi(G \times_H U) \cap \phi'(G \times_{H'} U').$$

A G-orientation is then a cover of X by compatible local trivializations, and the maps $\bar{\phi} \circ \iota_x$ are then the orientations for the fibers of p.

Returning to the fibrations η and ξ, we shall say that η and ξ are compatible if $\eta - \xi$ is G-orientible.

LEMMA 1.1. *Let $M \in A(G)$ be a non-zero divisor, let η and ξ have corresponding fibers equivalent* (*as above*), *and let f: $\eta \to \xi$ have degree M. Then*

(i) *η and ξ are compatible, and*

(ii) *$f \wedge 1$: $\eta \wedge_{\mathcal{F}} \xi^{-1} \to X \times S^V$ has equivariant degree exactly M on each fiber, with respect to some orientation* (*meaning a collection of homotopy trivializations as above*).

Proof. $X \times S^V$ has a standard orientation. For each $x \in X$, choose a local G-fiber-homotopy trivialization

$$\bar{\phi}: (G \times_H U) \times S^V \to E = \eta \wedge_{\mathcal{F}} \xi^{-1}$$

over a neighbourhood $G \times_H U$ of x, with $G_x = H$, $x \in U$ and U H-contractible to x. Then the composite

$$T_x: S^V \xrightarrow{\iota_x} (G \times_H U) \times S^V \xrightarrow{\bar{\phi}} E \xrightarrow{f \wedge 1} X \times S^V \xrightarrow{\pi} S^V$$

has degree $uM \in A(G_x)$ for some unit $u \in A(G_x)$.

If $y \in U$, then T_y: $S^V \to S^V$ is G_y-homotopic with T_x, so that the degree of T_y is also uM, regarded as an element of $A(G_y)$ via the forgetful map. Further, T_{gx} has degree $u^g M$, where $u^g \in A(gG_xg^{-1})$ corresponds to u under the natural isomorphism $A(G_x) \simeq A(gG_xg^{-1})$, and similarly for T_{gy} if $y \in U$.

Now assume, by fibrewise suspending, that u^{-1} may be represented by an H-equivariant homotopy equivalence $S^V \to S^V$. If we replace $\bar{\phi}$ by $\bar{\phi} \circ \bar{u}^{-1}$, where

$$\bar{u}^{-1}: (G \times_H U) \times S^V \circlearrowleft$$

is the map $([g, x], s) \mapsto ([g, x], gu^{-1}g^{-1}s)$, then this replaces T_x by a G_x-map of equivariant degree M, and similarly for T_y, T_{gx} and T_{gy} above. Thus such a system of local trivializations gives the degree of $f \wedge 1$ as exactly M.

Now consider the change-of-coordinate fiber equivalences $T_{\phi\phi'}|_x$ above. The degree d_x of $T_{\phi\phi'}|_x$ then satisfies $d_xM = M$, so that, since M is not a zero-divisor, $d_x = 1 \in A(G_x)$, it being easy to verify that non-zero divisors in $A(G)$ remain non-zero divisors in $A(H)$ for any $H \subset G$. This completes the proof.

We now have:

LEMMA 1.2. *It suffices to prove Theorem* 0.1 *for the special case* $\xi = X \times S^V$ *for some G-module V,* η *a V-dimensional oriented spherical fibration, and f of degree exactly M, with respect to a given orientation of* η*, where M is a power of k.*

LEMMA 1.3. *Let X be a disjoint union of G-spaces of the form G/H for various* $H \subset G$*, and let* $h: S^V \to S^V$ *be a G-map of equivariant degree* k^i*. Then under the hypothesis of the theorem, in the special case of Lemma* 1.1, *there exists a fibrewise G-equivalence g such that the following diagram is fibrewise stably homotopy commutative (equivariantly). Further, we may arrange that g be orientation preserving.*

$$\begin{array}{ccc} \eta & \xrightarrow{f} & X \times S^V \\ & {}_g\searrow \quad \nearrow^{1\times h} & \\ & X \times S^V & \end{array}$$

Proof. Clearly we may assume $X = G/H$. Let ϕ: $G/H \times S^V \to \eta$ be a G-homotopy trivialization which defines the orientation of η. Let $\bar{h}$: $S^V \to S^V$ denote the restriction to the fiber over the identity coset of

$$G/H \times S^V \xrightarrow{\phi} \eta \xrightarrow{f} G/H \times S^V.$$

Now $\bar{h}$ is H-equivariant and induces a G-map

$$i(\bar{h}): G/H \times S^V \to G/H \times S^V$$

by $i(\bar{h})([g], s) = ([g], g\bar{h}(g^{-1}s))$. One checks that $f\phi = i(\bar{h})$.

Since $\bar{h}$ has degree k^i, it is stably H-homotopic to h. Therefore $i(\bar{h})$ is G-homotopic to $i(h)$, and the latter is $1 \times h$, since h is G-equivariant. Thus $f \simeq i(h) \simeq 1 \times h$, where "$\simeq$" denotes G-homotopy (which in this case is automatically fiber preserving G-homotopy).

The lemma now follows by taking g to be a fiberwise G-homotopy inverse to ϕ.

Before continuing with the proof of the theorem, we pause to consider some equivariant homotopy theory.

Let $n > 0$, and let

$$f: S^n \to \operatorname{colim}(\Omega^V S^V)^H$$

be a map such that $f(*)$: $S^W \to S^W$ has equivariant degree $L \in A(H)$, for large W. We then say that f has equivariant degree L.

Denote the set of homotopy classes of such maps by $F_H(n, L)$, and note that loop addition gives a map $F_H(n, L) \times F_H(n, L') \to F_H(n, L + L')$.

Given a class $[f] \in F_H(n, L)$, we may form the m-fold Whitney sum $[f]^m \in F(n, L^m)$ by taking the class of the map f^m: $S^n \to \operatorname{colim}(\Omega^{Wm} S^{Wm})^H$ obtained by pointwise m-fold smash product.

We may also define an action on $[f]$ by $A(G)$ given by pointwise composition with the given element $M \in A(G)$, regarded as a stable G-map $S^W \to S^W$ for large W. We thus obtain an element $M[f] \in F_H(n, ML)$.

These two operations are related as follows:

LEMMA 1.4. *Let* $[f] \in F_H(n, L)$. *Then* $[f]^m = mL^{m-1}[f] + [c]$, *where* c *is a constant map in* $F_H(n, L^m(1 - m))$, *and where addition is defined pointwise.*

Proof. Write $f = g + h$ where g is a constant map in $F_H(n, L)$ and where $h = f - g$ (defined pointwise) is in $F_H(n, 0)$. Then

$$f^m = (g + h)^m = \sum_{0 \leqslant i \leqslant m} \binom{i}{m} h^i g^{m-i},$$

where the products $h^i g^{m-i}$ are maps $S^n \to \operatorname{colim}(\Omega^{V^m} S^{V^m})^H$ given by taking h pointwise on the first i summands and g on the remainder. (Note that the order in which the summands are taken is irrelevant up to homotopy). Since h has degree 0, h^i is null-homotopic if $i \geqslant 2$, whence so is $h^i g^{m-i}$. Thus

$$\begin{aligned} f^m &= g^m + mg^{m-1}h = g^m + mL^{m-1}h = g^m + mL^{m-1}(f - g) \\ &= g^m - mL^{m-1}g + mL^{m-1}f = c + mL^{m-1}f, \end{aligned}$$

as required, all equalities being taken up to homotopy.

Remark 1.5. For purposes of obstruction theory, we ignore constant terms, and write $[f]^m = mL^{m-1}[f]$.

We are now ready to prove Theorem 0.2.

Proof of Theorem 0.2. By the above lemmas, we may assume we have a fibrewise G-map $f\colon \eta \to X \times S^V$ of degree $k^t \in A(G)$, with respect to given orientations.

In order to induct over skeleta X^n, assume that there exist

(i) $r > 0$ and a fibrewise degree 1 G-equivalence $g\colon k^r \eta \to X^{n-1} \times S^W$ in some fibrewise suspension, defined over X^{n-1}, and

(ii) a (stable) G-map $h\colon S^W \to S^W$ such that the following diagram is stably G-fibrewise homotopy commutative over X^{n-1}:

$$\begin{array}{ccc} k^r \eta & \xrightarrow{k^r f} & X^{n-1} \times S^W \\ & {}_g\searrow \quad \nearrow_{1 \times h} & \\ & X^{n-1} \times S^W & \end{array}$$

(It follows that the degree of h must be a power, k^q, of k).

The start ($n = 1$) of the induction is Lemma 1.3 and we may take $r = 0$. Thus consider the obstruction θ to extending g over a G-cell of the form $G/H \times D^n$. This obstruction is a class $[\theta] \in F_H(n-1, 1)$, and the diagram implies that $[h\theta] = 0$, that is, $k^q[\theta] = 0$.

Taking the k^q-fold Whitney sum of the whole diagram, we replace $[\theta]$ by $[\theta]^{k^q} = k^q[\theta] = 0$, by Lemma 1.3. Thus the obstruction vanishes.

Now consider the obstruction to extending the homotopy over the above G-cell. This obstruction is a class $[\psi] \in F_H(n, k^p)$, where k^p is the present degree on fibers $(= (k^q)^{k^q})$. We may alter $[\psi]$ by using any element $[\alpha]$ in $F_H(n, 1)$ to replace g by the composite

$$\eta \overset{g}{\underset{\sim}{\rightarrow}} X^n \times S^V \overset{\hat{\alpha}}{\rightarrow} X^n \times S^V,$$

where we are assuming that $X^n = X^{n-1} \cup (G/H \times D^n)$, and where $\hat{\alpha}$ is given as follows: Regard α as an H-map $(D^n, S^n) \rightarrow (\Omega^V S^V)^H$ with $\alpha(x) = 1$ for all $x \in S^n$. Then let $\hat{\alpha}$ be given by the identity over X^{n-1} and by the map

$$((gH, d), s) \mapsto ((gH, d), g\alpha(d)(g^{-1}s))$$

over $G/H \times D^n$, for $d \in D^n$ and $s \in S^V$. This operation alters $[\psi]$ by $[h\alpha] = k^p[\alpha]$.

Now take the k^p-fold Whitney sum of the whole diagram. This replaces $[\psi]$ by

$$[\psi]^{k^p} = k^p(k^p)^{k^p-1}[\psi] = (k^p)^{k^p}[\psi],$$

which may now be altered by $(k^p)^{k^p}[\alpha]$ for some $\alpha \in F_H(n, 1)$. We choose $[\alpha] = -[\psi]_1$, where $[\psi]_1$ is obtained from $[\psi]$ by translation to $F_H(n, 1)$. This replaces $(k^p)^{k^p}[\psi]$ by

$$(k^p)^{k^p}[\psi] - (k^p)^{k^p}[\psi]_1 = (k^p)^{k^p}([\psi] - [\psi]) + \text{constant}.$$

Thus the obstruction vanishes and we are done.

To end this section, we consider divisors of k^n in $A(G)$.

Let ϕ denote the set of conjugacy classes of subgroups of G, and consider the ring homomorphism

$$\theta: A(G) \rightarrow \times_{H \in \phi} Z$$

obtained by sending a G-set s to that tuple whose H^{th} coordinate is $|s^H|$. That θ is, in fact a monomorphism is shown in [7]. The following is more or less well known.

LEMMA 1.6. (i) *Coker θ is finite.*

(ii) *If $y \in \times_{H\in\phi} Z$, then there is a monic polynomial p over $A(G)$ with $p(y) = 0$.*

(iii) *If $a \in A(G)$ and there exists $b \in \times_{H\in\phi} Z$ with $ab = k^n$ for some n, then there exists $c \in A(G)$ with $ac = k^m$ for some m. (Here, k is any positive integer.)*

Proof. Part (i) is well known and may easily be checked using the finiteness of G. Part (ii) now follows immediately from (i). For (iii), write

$$b^r = x_0 + x_1 b + \cdots + x_{r-1} b^{r-1}$$

with each $x_1 \in A(G)$. Then $b^r a^{r-1} \in A(G)$, so that if $c \in A(G)$ is given by $c = b^r a^{r-1}$, then $ac = b^r a^r = k^{rn}$.

COROLLARY 1.7. *If $K \in A(G)$ with each coordinate of $\theta(K)$ a power of k, then K divides some power of k in $A(G)$.*

2. Torsion in $J_G(*)$

The strong hypothesis of Theorem 0.2, namely that corresponding fibers of the two fibrations be stably equivariantly equivalent, may be shown to hold in many situations. This amounts to saying that the differences between corresponding fibers, measured in $J_{G_x}(*)$ for appropriate x, are zero. This will not be true in general for our application to the Adams Conjecture, but it will suffice to show that these differences have the correct torsion in $J_{G_x}(*)$. The question of torsion in $J_G(*)$ for certain groups G has been studied by tomDieck in [7]. This depends heavily on the particular representations involved. In our application however, we shall be given an equivariant map between corresponding fibers. This will allow us to detect torsion geometrically for general G.

Let $J_G(X)$ denote, as usual, the image of J_G: $K_G(X) \to \mathrm{Sph}_G(X)$, so that $J_G(*) = RO(G)/\sim$, the ring of virtual representations modulo stable G-homotopy equivalence.

PROPOSITION 2.1. *Let $V - W$ represent an element $\gamma \in J_G(*)$, and assume that we are given a G-map $f: S^V \to S^W$ such that*

$$\deg f^H = \begin{cases} \pm 1 & \text{if } V^H \neq V, \\ \pm k, \text{ prime to } |G|, & \text{if } V^H = V. \end{cases}$$

(Thus we are assuming $\dim V^H = \dim W^H$ *for each $H < G$.) Let s denote the order of $|k| \bmod |G|$. Then $s\gamma = 0$ in $J_G(*)$.*

Proof. Let K be a maximal subgroup of G such that $V^K = V$. Then K is normal in G, and K contains every subgroup H of G which fixes V. Regard V as a G/K-module, and choose a new G/K-module R such that R contains an orbit of type G/K. Then consider the map $sf \oplus 1$: $S^{sV \oplus R} \to S^{sW \oplus R}$, which we assume trivial outside a little neighbourhood of the origin. We may choose free G/K orbits in $sV \oplus R$, away from any proper fixed-point set and the little neighbourhood of the origin just mentioned, and use these to alter the degree of $sf \oplus 1$ by arbitrary multiples of $|G|$ by using the Pontryagin-Thom construction, the point being that we are using free G/K orbits and that any G/K-equivariant map we construct is automatically

G-equivariant. By the definition of s, we may alter this degree to ± 1. This does not affect the degree on any fixed-point subset, which is also ± 1 by the choice of K. Thus the resulting map is a G-equivalence, as required.

COROLLARY 2.2. *Suppose that we are given a fibrewise G-map $f: \eta \to \xi$ of based spherical G-fibrations over a G-space X such that the restriction f_x of f to the fiber above $x \in X$ satisfies*

$$\deg(f_x)^H = \begin{cases} \pm 1 & \text{if } V^H \neq V, \\ \pm k, \text{ prime to } |G|, & \text{if } V^H = V, \end{cases}$$

for $V = \eta^{-1}(x)$ and subgroups H of G_x. Then $sf: s\eta \to s\xi$ has corresponding fibers equivariantly homotopy equivalent.

Remark 2.3. When G is cyclic and $|G|$ is prime, then the above s is the smallest possible choice.

3. One and Two Dimensional Bundles

Here we prove Theorem 0.1 in the case of one and two dimensional G-vector bundles over a finite G-connected G-CW complex. This is done by generalizing the arguments of Adams in [1, §4].

In order to proceed, we consider the principle structure of G-vector bundles. Let A be any topological group. Recall [8] that a (G, A) bundle is an A-bundle on which G acts through A-bundle maps. (In the case of G-vector bundles, A usually denotes an orthogonal or unitary group.)

If $\mu: E \to X$ is a (G, A) bundle, we may define an associated principle G-bundle $P(\mu): P(E) \to X$ by taking $P(E)$ to be the space of A-bundle inclusions $F \to E$ of the fiber, and allowing G to act on $P(E)$ via its action on E. Note that the natural A-action on $P(E)$ is compatible with that of G.

In this way, we obtain a natural decomposition

$$E = P(E) \times {}_A F$$

of any (G, A)-bundle E. In particular, when $F = \mathbf{R}^n$ and $A = 0(n)$, we obtain $E = P(E) \times {}_{0(n)}\mathbf{R}^n$. Observe that, in the terminology of [8], $P(E)$ is a principle $(G, 0(n))$ bundle.

If $\psi: 0(n) \to 0(n')$ is a homomorphism, and if

$$\mu: P(E) \times {}_{0(n)}\mathbf{R}^n \to P(E)/0(n)$$

is a $(G, 0(n))$ bundle, then one has an induced bundle

$$\psi(\mu): P(E) \times {}_{0(n)}\mathbf{R}^{n'} \to P(E)/0(n),$$

where we let $0(n)$ act on $\mathbf{R}^{n'}$ via ψ. If ψ is virtual, one obtains a virtual bundle $\psi(\mu)$ in the evident manner. In particular, if ψ^k is the virtual representation for the k^{th} Adams operation, then $\psi^k(\mu)$ coincides with the k^{th} Adams operation on μ.

Now let μ be a $(G, 0(1))$ vector bundle (of dimension one). If k is even, then $|G|$ must be odd, whence μ must have trivial isotropy action on fibers. It follows that μ is classified by a map

$$X = P(E)/0(1) \to \mathbf{R}P^n,$$

where $n = \dim X$ and G acts trivially on $\mathbf{R}P^n$. In this case, Theorem 0.1 follows formally from the nonequivariant case, and we may take $s = 1$. If k is odd, then $\psi^k(\mu) = \mu$, since then ψ^k and 1 coincide as representations.

Consider now the two-dimensional case. Thus let μ be a $(G, 0(2))$ bundle of dimension 2. If k is odd, then one has the k^{th} power map $f\colon \mu \to \psi^k(\mu)$, given on fibers $\mathbf{R}^2 = \mathbf{C}$ via $z \to z^k$. Then f is G-equivariant. Let $g\colon V \to W$ denote the restriction of f to a fiber V of μ with associated isotropy subgroup H. Then, one has the following:

PROPOSITION 3.1. *For each $K \subset H$,*

(i) $$\dim V^K = \dim W^K;$$

(ii) $$\deg g^K = \begin{cases} \pm k & \text{if } V^K = V, \\ \pm 1 & \text{otherwise}. \end{cases}$$

Proof. Suppose that $V^K = V$. Then clearly $W^K = W$, and $\deg g^K = 1$. If $\dim V^K = 1$, then $\dim W^K \geqslant 1$. But it is clear that the whole of W cannot be fixed by K, since k is prime to the order of G. Thus $\dim W^K = 1$, and since k is odd, $\deg g^k = 1$. If $\dim V^K = 0$, then the result is immediate.

We may now apply Corollaries 2.2 and 1.6 to conclude that

$$sf\colon s\mu \to s\psi^k(\mu)$$

satisfies the hypothesis of Dold mod k, and hence that $J_G sk^m(1 - \psi^k)(\mu) = 0$, as required.

We now consider the case k even. Here, define a representation

$$\mu_k\colon 0(2) \to 0(2)$$

just as in [1, §4]. One then has a k^{th} power map $f\colon \mu \to \mu_k(\mu)$. The restriction of f to a fiber V with isotropy H now satisfies the conclusion of Proposition 3.1. Indeed, one may repeat the proof verbatim, excluding the case $\dim V^K = 1$, since G must have odd order. We then conclude that $J_G sk^m(1 - \mu_k)(\mu) = 0$. By Adams' calculation, $\psi^k(\mu)$ and $\mu_k(\mu)$ differ by $\lambda(\mu) - 1$, where λ is the determinant representation $0(2) \to 0(1)$. Since $|G|$ is odd, $\lambda(\mu)$ is classified by a map into $\mathbf{R}P^n$, and so, by the argument in [1, 4.1], $J_G k^e(\lambda(\mu) - 1) = 0$ for some e, and the result now follows.

Remark 3.2. The assumption that the degree of sf is constant in the sense of §0 follows from the G-connectivity of the base space, although it seems clear that the connectivity requirement is not needed for this.

4. Cyclic Group Actions and the Becker-Gottlieb Reduction

When G is a cyclic group, one may generalize the arguments in [5] to reduce Theorem 0.1 to the two-dimensional case. Let $\mu\colon E \to X$ be a $(G, 0(n))$ vector bundle of dimension n, and write μ as the natural projection

$$P(E) \times_{0(n)} \mathbf{R}^n \to P(E)/0(n).$$

Since the actions of G and $0(n)$ are compatible, we may form the G-bundle

$$\gamma\colon P(E) \to P(E)/N(T)$$

where $N(T)$ is the normalizer of the (usual) maximal torus in $0(n)$. We then have a commutative diagram

$$\begin{array}{ccc} P(E) \times_{N(T)} \mathbf{R}^n & \xrightarrow{\tilde{\lambda}} & P(E) \times_{0(n)} \mathbf{R}^n \\ \downarrow & & \downarrow \\ P(E)/N(T) & \xrightarrow{\lambda} & P(E)/0(n) = X \end{array}$$

which is clearly a diagram of induced bundles. Now assume that X is a finite G-CW complex, and let

$$t\colon X^+ \to P(E)/N(T)^+$$

denote the equivariant transfer described in [15]. (Here, t is a map $\Sigma^W X^+ \to \Sigma^W P(E)/N(T)^+$ for some G-module W.)

Proposition 4.1. *Let X be a G-connected finite G-CW complex with G cyclic. Then the composite $\lambda \circ t$ is a stable G-equivalence. (That is, $\Sigma^W \lambda \circ t$ is a G-equivalence for large enough W).*

Proof. It suffices to show that $(\lambda \circ t)^H$ is a stable equivalence for each subgroup $H \subset G$. By the theory of equivariant transfer [15], the map $(\lambda \circ t)^H$ induces multiplication by $\chi(0(n)/N(T))^H$ in generalized cohomology. Thus, if $\chi(0(n)/N(T))^H = 1$, then $(\lambda \circ t)^H$ is a stable equivalence.

By a result of Borel [7], the rational cohomology of $0(n)/T$ is concentrated in even dimensions. This, together with the fact that $\chi(0(n)/N(T)) = 1$, implies that

$$H^m(0(n)/N(T); Q) = \begin{cases} Q & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

Thus, any map $f\colon 0(n)/N(T) \to 0(n)/N(T)$ has Lefshetz number 1.

Let h generate H. Then $(0(n)/N(T))^H = (0(n)/N(T))^h$, the fixed set of the element h. By averaging the metric, we may assume that H acts through isometries. Now if $f\colon M \to M$ is an isometry, then the Lefschetz number of f is equal to the Euler characteristic of its fixed point set, $\Lambda(f) = \chi(M^f)$. We therefore have

$$\chi(0(n)/N(T)^H) = \chi(0(n)/N(T)^h) = \Lambda(h) = 1,$$

as required.

COROLLARY 4.2. *If h^* is any generalized equivariant cohomology theory, then*

$$\lambda^*: h^*(B^+) \to h^*(P(E)/N(T)^+)$$

is injective.

In view of the work in [11] and [15], the contravariant functors $\mathrm{Sph}_G(-)$ and $KO_G(-)$ are the zeroth terms of equivariant cohomology theories, whence λ^*: $\mathrm{Sph}_G(X) \to \mathrm{Sph}_G(Y)$ is a monomorphism, where $Y = P(E)/N(T)$. Thus, to prove that a given element of $KO_G(X)$ is in the kernel of J_G, it suffices to prove that its restriction to $KO_G(P(E)/N(T))$ is in ker J_G. Since pulling back a $(G, 0(n))$ bundle over λ reduces the structure group to $N(T)$, we may therefore restrict attention to $(G, N(T))$ bundles when proving Theorem 0.1, as in the non-equivariant case [5]. Since we are stabilizing, we may also assume that the given bundle is $2n$ dimensional.

In order to reduce from the case of $N(T) = \Sigma_n \int 0(2)$-bundles to the $0(2)$ case, one may mimic the technique in [5, §7] directly. There, the authors consider a subgroup H of $N(T)$ and a homomorphism ϕ: $H \to 0(2)$ such that the following is true. If α: $E \to X$ is the given $N(T)$-bundle and if ξ is the G-bundle $P(E) \times {}_H\mathbf{R}^2 \to P(E)/H$ with H acting on $\mathbf{R}^2$ via ϕ, then $\alpha = i(\xi)$, where i is fiberwise induction associated with the inclusion $H \to N(T)$. (That is, α has the form $P(E) \times {}_{N(T)}i(\mathbf{R}^2) \to P(E)/N(T)$.) Here, as always, the equivariance is formal in view of the compatibility of the $0(n)$- and G-actions on $P(E)$.

Thus to prove Theorem 0.1, it suffices to show that it is true for G-bundles of the form $i(\xi)$, where ξ is as above. This would be an easy consequence of Quillen's argument in [12] and the Dold mod k Theorem, provided we assume that $\psi^k i = i\psi^k$ in $KO_G(X)[k^{-1}]$. This is proved in the next section.

5. Compatibility of ψ^k and Induction

If $p: U \to B$ is a finite covering space in the equivariant sense, and if q: $E \to U$ is an orthogonal G-vector bundle, we may construct the "induced" vector bundle $p_!(q)$: $p_!(E) \to B$ by the usual (and natural) geometric construction. (See, for example, [4].) This gives a well-defined homomorphism

$$p_! : K_G(U) \to K_G(B).$$

This map may also be described in terms of the G-transfer and Thom isomorphism as follows:

If V is a Spin$(8n)$ representation of G, then, by [3], there exists a Thom isomorphism

$$T_V: \overline{K}_G(B^+) \to \overline{K}_G(B^+ \wedge S^V).$$

Here, reduced equivariant K-theory is interpreted as the based cohomology theory associated with K_G, the notation $\widetilde{K}_G$ being reserved for "reduced" equivariant K-theory (that is, virtual G-bundles of (non-equivariant) virtual dimension zero). The isomorphism T_V is obtained in [3] by multiplication with a Thom class $t_V \in \overline{K}_G(S^V)$. If $\tau(p)$: $B^+ \wedge S^V \to U^+ \wedge S^V$ is the geometrically defined G-transfer (for large enough V), then $p_!$ may be given by the composite

$$p_! : \overline{K}_G(U^+) \xrightarrow{T_V} \overline{K}_G(U^+ \wedge S^V) \to \overline{K}_G(B^+ \wedge S^V) \xrightarrow{T_{V}-1} \overline{K}_G(B^+),$$

the equivalence of the two definitions of $p_!$ being possible to see geometrically.

Now let $(k, |G|) = 1$, as usual. We first establish:

PROPOSITION 5.1. *Computing in* $\overline{K}_G(B^+)[k^{-1}]$, *we have* $\psi^k p_! = p_!\psi^k$.

Proof. Using the operations ψ^k and the isomorphisms T_V, one may obtain a cannabalistic class $\rho_V^k \in RO(G)$, given by

$$\rho_V^k = T_V^{-1}\psi^k T_V(1),$$

where $1 \in \overline{K}_G(S^0) = RO(G)$ is the multiplicative unit. From this, one obtains directly

$$\rho_V^k \psi^k(T_V^{-1}(x)) = T_V^{-1}(\psi^k(x)), \tag{1}$$

for $x \in \overline{K}_G(B^+ \wedge S^V)$. (Intuitively, ρ_V^k is the "commutator" of ψ^k and T_V.) Assume now that the classes ρ_V^k become units when we invert k in $RO(G)$. Then we have, in $RO(G)[k^{-1}]$,

$$\begin{aligned} p_!(\psi^k(x)) &= T_V^{-1}(\tau(p)^*(T_V(\psi^k(x))) \\ &= T_V^{-1}(\tau(p)^*(\rho_V^k)^{-1}\psi^k(T_V(x))) \quad \text{(by (1))}, \\ &= (\rho_V^k)^{-1}T_V^{-1}\psi^k(\tau(p)^*(T_V(x))) \\ &= (\rho_V^k)^{-1}\rho_V^k\rho^k(T_V^{-1}(\tau(p)^*(T_V(x))) \quad \text{(by (1)))}, \\ &= \psi^k(p_!(x)). \end{aligned}$$

Thus it remains to show:

LEMMA 5.2. *The classes* ρ_V^k *are units in* $RO(G)[k^{-1}]$.

Proof. The character function gives an inclusion i of $RO(G)$ in the ring $C(G, Z[\xi_n])$ of class functions $G \to Z[\xi_n]$, where $n = |G|$ and ξ_n is a primitive n^{th} root of unity. By [4], the inclusion i is a ring map and thus passes to a ring map when we localize. Thus it suffices to show that, for each $g \in G$, $i\rho_V^k(g)$ is a unit in $Z[k^{-1}][\xi_n]$.

Now assume that our representation V is given by γ: $G \to \text{Spin}(8n)$. Then we may assume that, for a given $g \in G$, $\gamma(g)$ lies in the standard maximal torus $T \subset \text{Spin}(8n)$, by conjugation.

By [2, 5.9], we have the identity

$$i\rho_V^k(z_1, \ldots, z_{4n}) = \prod_{1 \leq r \leq 4n} \frac{z_r^{k/2} - z_r^{-k/2}}{z_r^{1/2} - z_r^{-1/2}},$$

for $(z_1, \ldots, z_{4n}) \in T$.

Thus it suffices to show that each factor is invertible whenever z_r is an n^{th} root of unity, and $(k, n) = 1$.

But

$$\frac{z^{k/2} - z^{-k/2}}{z^{1/2} - z^{-1/2}} = z^{-1/2(k+1)} \cdot \frac{z^k - 1}{z - 1},$$

where the second factor is an n^{th} root of unity in $Z[k^{-1}][\xi_n]$ because of the following case by case consideration:

(i) If $z = 1$, then

$$\frac{z^k - 1}{z - 1} = 1 + z + \cdots + z^{k-1} = k;$$

(ii) If z is a primitive n^{th} root of unity, then there is a j with $kj \equiv 1 \pmod{n}$, whence

$$\left(\frac{z^k - 1}{z - 1}\right)^{-1} = \frac{z - 1}{z^k - 1} = \frac{(z^k)^j - 1}{z^k - 1} = 1 + z^k + \cdots + (z^k)^{j-1};$$

(iii) If z is neither, then z is a primitive d^{th} root of unity, where $d \mid |G|$, and the argument is the same as (ii).

This completes the proof.

Finally, we may relate $p_!$ to induction over structure groups by observing that, if $A < B < 0(n)$ are specified subgroups, and if $q\colon E \to X$ is a (G, A) vector bundle, then the induced bundle

$$i(q)\colon i(E) \to P(E)/B$$

coincides with the construction $p_!(q)$, for $p\colon P(E)/A \to P(E)/B$.

Remark 5.3. In the case of finite structural group and arbitrary finite ambient group G, we may apply the arguments of Quillen in [12] to obtain a proof of Theorem 0.1 in this case, using the Dold theorem mod k above.

REFERENCES

1. J. F. ADAMS, *On the groups J(X) I,* Topology, vol. 2 (1963), pp. 181–195.
2. ———, *On the groups J(X) II,* Topology, vol. 3 (1963), p. 159.
3. M. ATIYAH, *Bott Periodicity and index of elliptic operators,* Quart. J. Math. Ser. (2), vol. 19 (1968), pp. 113–140.

4. ———, *Characters and cohomology of finite groups,* Inst. Hautes Etudes Sci., Publ. Math. 9, vol. 26 (1961), pp. 23–64.
5. J. C. BECKER and D. H. GOTTLIEB, *The transfer map and fiber bundles,* Topology, vol. 14 (1975), pp. 1–12.
6. A. BOREL, *Sur la cohomologie des espaces fibrés principaux et des spaces homogènes de groupes de Lie,* Ann. of Math., vol. 57 (1953), p. 189.
7. T. TOMDIECK, *Transformation groups and representation theory,* Springer Lecture Notes, vol. 34, Springer-Verlag, New York, 1967.
8. T. LASHOF and M. ROTHENBERG, *G-smoothing theory,* Proc. Symp. Pure Math., vol. 32-I (1978), pp. 211–266.
9. J. P. MAY and H. HAUSCHILD, *Equivariant Brouer lift,* University of Chicago, in preparation.
10. J. P. MAY, *E_∞ ring spaces and E_∞ ring spectra,* Lecture Notes in Math., vol. 577, Springer-Verlag, New York, 1977.
11. J. P. MAY, H. HAUSCHILD and S. WANER, *Equivariant infinite loop spaces,* University of Chicago, in preparation.
12. D. QUILLEN, *The Adams conjecture,* Topology, vol. 10 (1971), pp. 67–80.
13. G. B. SEGAL, *Equivariant stable homotopy theory,* Actes Congress Math, vol. 2 (1970), pp. 59–63.
14. S. WANER, *Equivariant homotopy theory and Milnor's Theorem,* Trans. Amer. Math. Soc., vol. 258 (1980), pp. 351–368.
15. ———, *Equivariant fibrations and transfer,* Trans. Amer. Math. Soc., vol. 258 (1980), pp. 369–384.
16. ———, *Equivariant classifying spaces and fibrations,* Trans. Amer. Math. Soc., vol. 258 (1980), pp. 385–405.
17. ———, *Oriented equivariant spherical fibrations,* Trans. Amer. Math. Soc., vol. 271 (1982), pp. 313–323.

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