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The equivariant K*-group of the Hirzebruch-Mayer SO(n)-manifold

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0. Introduction

We have calculated the equivariant K^*-group of the Hirzebruch-Mayer $O(n)$-manifold $W^{2n-1}(d)$ as follows.

Theorem. [5, § 8] *For $n\geqq 2$, the orbit space is a 2-disk the orbit type of whose interior is $(O(n-2))$ and the boundary $(O(n-1))$, and*

$$K^0_{O(n)}(W^{2n-1}(d))\cong R(O(n-1))$$

and $$K^1_{O(n)}(W^{2n-1}(d))\cong \text{Ker } \rho'_{n-1}.$$

Here $\rho'_{n-1}: R(O(n-1))\to R(O(n-2))$ *is the canonical surjection.*

In the case above the equivariant K^*-group is independent of d. Moreover, we have proved that the equivariant K^*-group of the regular $O(n)$-manifold X depends only on the orbit type decomposition of the orbit space, if $\dim X/O(n)\leqq 2$.

On the other hand, if we restrict the $O(n)$-action on the subgroup $SO(n)$, we shall get

Theorem. *For $n\geqq 3$, the orbit space is homeomorphic to a 2-disk the orbit type of whose interior is $(SO(n-2))$ and the boundary $(SO(n-1))$, and the equivariant K^*-group is calculated*

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as follows.

(a) $n=2k+1$

(ai) d: *even*

$$K^0_{SO(n)}(W^{2n-1}(d))\cong R(SO(n-1)),$$

and $$K^1_{SO(n)}(W^{2n-1}(d))\cong \mathrm{Ker}\ \rho_{n-1}$$

(aii) d: *odd*

$$K^0_{SO(n)}(W^{2n-1}(d))\cong \boldsymbol{Z}[\lambda^1\cdots,\lambda^{k-1},\lambda^k]$$
$$(\cong R(SO(n)))\subset R(SO(n-1)),$$

and $$K^1_{SO(n)}(W^{2n-1}(d))\cong \mathrm{Ker}\ \rho_{n-1}/(\lambda^k_+-\lambda^k_-).$$

Here ρ_{n-1}: $R(SO(n-1))\cong \boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1},\lambda^k_+,\lambda^k_-]/\sim\to$
$$\boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1}]\cong R(SO(n-2))$$
is the canonical surjection which carries $\lambda^k_\pm$ *to zero.*

(b) $n=2k+2$

$$0\to \mathrm{Coker}\,\rho_{n-1}\to K^0_{SO(n)}(W^{2n-1}(d))\to R(SO(n-1))\to 0$$

is exact, and

$$K^1_{SO(n)}(W^{2n-1}(d))=0.$$

Here ρ_{n-1}: $R(SO(n-1))\cong \boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1},\lambda^k]\to$
$$\boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1},\lambda^k_+,\lambda^k_-]/\sim\cong R(SO(n-2))$$
is the canonical injection which carries λ^k *to* $\lambda^k_+ +\lambda^k_-$.

In the case (a) the equivariant K^*-group does depend on the parity of d. As for the equivariant K^0-group in the case (aii), H. Matsunaga [3] has determined it recently. But since our method is simpler and more systematic, it seems worth to present the complete proof of Theorem using an appropriate $SO(n)$-CW decomposition of $W^{2n-1}(d)$.

1. G-CW structure of $W^{2n-1}(\mathrm{d})$ $(\mathrm{G}=\mathrm{O}(\mathrm{n}),\ \mathrm{SO}(\mathrm{n}))$

For $d\geqq 0$ and $n\geqq 1$, $W^{2n-1}(d)$ denote the real analytic set in $\boldsymbol{C}^{n+1}$, defined by $z_0^d+z_1^2+\cdots+z_n^2=0$ and $z_0\bar{z}_0+z_1\bar{z}_1+\cdots+z_n\bar{z}_n=2$. The action of $O(n)$ (or $SO(n)$) is defined by $Az=(z_0, A(z_1,\cdots,z_n))$, for $A\in O(n)$ (or $SO(n)$ resp.) and $z=(z_0,z_1,\cdots,z_n)\in W^{2n-1}(d)$.

Define the (closed) cells in $W^{2n-1}(d)$, by

$$\sigma_0^0 = \{(0, i, 1, 0, \cdots, 0)\}$$
$$\sigma_1^0 = \{(1, i, 0, 0, \cdots, 0)\}$$
$$\sigma_0^1 = \{(r, ip, q, 0, \cdots, 0); \sqrt{2}p = \sqrt{2-r^2+r^d},$$
$$\sqrt{2}q = \sqrt{2-r^2-r^d},\ 0 \leqq r \leqq 1\}$$
$$\sigma_1^1 = \{(e^{i\theta}, ie^{i\psi}, 0, 0, \cdots, 0);\ 2\psi = d\theta,\ 0 \leqq \theta \leqq 2\pi\}$$
$$\sigma^2 = \{(re^{i\theta}, ipe^{i\psi}, qe^{i\psi}, 0, \cdots, 0;\ 0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq 2\pi\}$$

Then, every point of the interior of σ_0^0, σ_0^1 and σ^2 has the same isotropy subgroup $O(n-2)$ (or $SO(n-2)$ resp.) and every point of σ_1^0 and σ_1^1 has the same isotropy subgroup $O(n-1)$ (or $SO(n-1)$ resp.).

Proposition 1 *Let $K = \{g\sigma;\ g \in O(n),\ \sigma = \sigma_0^0, \sigma_1^0, \sigma_0^1, \sigma_1^1, \sigma^2\}$. Then, for $n \geqq 2$, $(W^{2n-1}(d), K)$ is an $O(n)$-*CW *complex.*

Proof: Since $G = O(n)$ is a compact Lie group and $X = W^{2n-1}(d)$ satisfies the axiom of the lst countability, we need only to check the conditions (a), (b), (d) and (e) in the definition of G-cell complex and that the induced cell decomposition of the orbit space is a CW decomposition (See [4] especially Theorem (1·10)).

By the definition of the cells we know that each cell has its characteristic map, its isotropy subgroup and its conjugate cells, that is, the conditions (b), (e) and (d) are satisfied. Define a map $p: W^{2n-1}(d) \to \boldsymbol{C}$ by $p(z_0, z_1, \cdots, z_n) = z_0$. Then, for $n \geqq 2$, it is easy to show that p induces an homeomorphism:

$$W^{2n-1}(d)/O(n) \xrightarrow{\cong} D^2 = \{z \in \boldsymbol{C};\ |z| \leqq 1\} \subset \boldsymbol{C}.$$

(See Hirzebruch-Mayer [2].)

Now remark that the collection of cells, $p(\sigma_0^0)$, $p(\sigma_1^0)$, $p(\sigma_0^1)$, $p(\sigma_1^1)$ and $p(\sigma^2)$, forms a CW decomposition of D^2. This fact shows that the induced cell structure on the orbit space is a CW decomposition and K satisfies the condition (a). q. e. d.

Proposition 1′ *Let $K' = \{g\sigma;\ g \in SO(n),\ \sigma = \sigma_0^0, \sigma_1^0, \sigma_0^1, \sigma_1^1, \sigma^2\}$. Then, for $n \geqq 3$, $(W^{2n-1}(d), K')$ is a $SO(n)$-*CW *complex.*

Proof: We only need to show that $p: W^{2n-1}(d)\to \boldsymbol{C}$ induces also the homeomorphism: $W^{2n-1}(d)/SO(n)\to D^2$.

Since $n\geqq 3$, there exist an element $A\in O(n)$ (for example $A(z_0, z_1, \cdots, z_{n-1}, z_n)=(z_0, z_1, \cdots, z_{n-1}, -z_n)$) such that $A\notin SO(n)$ and $A\sigma=\sigma$ for $\sigma=\sigma^i_j (i=0, 1, 2, j=0, 1)$.

Therefore, $W^{2n-1}(d)/O(n)=W^{2n-1}(d)/SO(n)$, because A and $SO(n)$ generate the whole group $O(n)$ and the collection of $\sigma/O(n)$ ($\sigma=\sigma^i_j, i=0, 1, 2, j=0, 1$) covers $W^{2n-1}(d)/O(n)$. q. e. d.

The boundary operations are as follows. Let $J\in SO(n)$ be the transformation $J(z_1, \cdots, z_n)=(e^{id\pi}z_1, e^{id\pi}z_2, z_3, \cdots, z_n)$.

Then,

$$\partial\sigma^1_0=(\sigma^0_1)\cup(-\sigma^0_0)$$

$$\partial\sigma^1_1=(J\sigma^0_1)\cup(-\sigma^0_1)$$

$$\partial\sigma^2=(\sigma^1_0)\cup(\sigma^1_1)\cup(-J\sigma^1_0)\cup(\bigcup_{\psi}e^{i\psi}\sigma^0_0)$$

2. Computation of $H^*_{SO(n)}(W^{2n-1}(d); K_{SO(n)})$ and proof of Theorem.

From the result of § 1, we get the cochain groups and coboundary homomorphisms as in the following diagram.

$$\begin{array}{ccccc}
C^0_G= & R(SO(n-1))(\sigma^0_1) & \oplus & R(SO(n-2))(\sigma^0_0) & \\
 & \downarrow (id)'-(id) & \searrow^{\rho} & \downarrow id & \\
C^1_G= & R(SO(n-1)(\sigma^1_1) & \oplus & R(SO(n-2)(\sigma^1_0) & \\
 & \downarrow \rho & \swarrow & (id)-(id)' & \\
C^2_G= & R(SO(n-2)(\sigma^2) & & &
\end{array}$$

where $C^*_G=C^*_{SO(n)}(W^{2n-1}(d); K_{SO(n)})$, $\rho=\rho_{n-1}: R(SO(n-1))\to R(SO(n-2))$ is the restriction map and $(id)'=J^*$.

We recall the ring structure of $R(SO(m))$. (See [1] for example.) Let T^k be a maximal torus of $SO(2k)$ and $SO(2k+1)$. Then, $R(T^k)\cong \boldsymbol{Z}[y_1, \cdots, y_k, y_1^{-1}, \cdots, y_k^{-1}]$, the Weyl group $W(SO(2k))$ acts as the group generated by permutations of the y_i and transformations $y_i\to y_i^{\varepsilon(i)}$, $\varepsilon(i)=\pm 1$, $\prod\varepsilon(i)=1$, and the Weyl group $W(SO(2k+1))$ acts as the group generated by permutations of the y_i and the transformations $y_i\to y_i^{\varepsilon(i)}$, $\varepsilon(i)=\pm 1$. Let $\lambda^1, \cdots, \lambda^k$ denote the elementary symmetric functions in the $2k$-variables $y_1, \cdots, y_k, y_1^{-1}$,

$\cdots,y_k^{-1}$ and $\lambda_\pm^k=\sum y_{i(1)}^{\varepsilon(1)}y_{i(2)}^{\varepsilon(2)}\cdots y_{i(k)}^{\varepsilon(k)}$ $(i(1)\leqq i(2)\leqq\cdots\leqq i(k),\ \varepsilon(1)\varepsilon(2)\cdots\varepsilon(k)=\pm 1)$. Then, $R(SO(2k+1))\cong \boldsymbol{Z}[\lambda^1,\cdots,\lambda^k]$ and $R(SO(2k))$ is the ring generated by $\lambda^1,\cdots,\lambda^{k-1},\lambda_+^k$ and λ_-^k with one relation $(\lambda_+^k+\lambda^{k-2}+\cdots)(\lambda_-^k+\lambda^{k-2}+\cdots)=(\lambda^{k-1}+\lambda^{k-3}+\cdots)^2$. (We write $R(SO(2k))\cong \boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1},\lambda_+^k,\lambda_-^k]/\sim$.)

Case (a) $n=2k+1$

$$\rho_{n-1}: R(SO(n-1))\cong \boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1},\lambda_+^k,\lambda_-^k]/\sim \to \boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1}]\cong R(SO(n-2))$$

is a natural surjection which carries $\lambda_\pm^k$ to zero.

If $d=$even, then $J=id$. Therefore, we get

$$H_G^0\cong R(SO(n-1)) \text{ and } H_G^1\cong \operatorname{Ker}\rho_{n-1}.$$

If $d=$odd, then $J^*=(id)'$ is the identity on $R(SO(n-2))$ and exchanges $\lambda_\pm^k$ for $\lambda_\mp^k$ on $R(SO(n-1))$. Therefore, we get

$$H_G^0\cong \boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1},\lambda_+^k+\lambda_-^k] \text{ and } H_G^1=\operatorname{Ker}\rho_{n-1}/(\lambda_+^k-\lambda_-^k)$$

Case(b) $n=2k+2$

$$\rho_{n-1}: R(SO(n-1))\cong \boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1},\lambda^k]\to \boldsymbol{Z}[\lambda^1,\cdots,\lambda^{k-1},\lambda_+^k,\lambda_-^k]/\sim\cong R(SO(n-2))$$

is a natural injection which carries λ^k to $\lambda_+^k+\lambda_-^k$.

If $d=$even, then $J=id$. Therefore. we get

$$H_G^0\cong R(SO(n-1)),\ H_G^1=0 \text{ and } H_G^2\cong \operatorname{Coker}\rho_{n-1}.$$

If $d=$odd, then $J^*=(id)'$ is the identity on $R(SO(n-1))$ and exchanges $\lambda_\pm^k$ for $\lambda_\mp^k$ on $R(SO(n-2))$. Since
id: $R(SO)n-2))(\sigma_0^0)\to R(SO(n-2))(\sigma_0^1)$ is onto, $(id)-(id)'$ on $R(SO(n-2))(\sigma_0^1)$ has no effect on H_G^*. Therefore, the result is the same as in the case in which $d=$even.

By the theorem (8.2) of [5] the Atiyah-Hirzebruch spectral sequence collapses in these cases. Therefore, we complete the proof of Theorem.

As for $K_{O(n)}^*(W^{2n-1}(d))$, we get also the result in the same way as in the case (ai), because $J^*=id$ independent of the parity of d.

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