

THE EQUIVARIANT SPAN OF THE UNIT SPHERES IN REPRESENTATION SPACES

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1. Introduction

Let G be a finite group and M be a smooth G -manifold. We define $\text{Span}_G(M)$ to be the largest integer k such that M has k linearly independent smooth G -vector fields. Let V be an orthogonal G -representation space and let $S(V)$ denote the unit sphere in V . In the case where G acts freely on $S(V)$, $\text{Span}_G(S(V))$ ($=\text{Span}(S(V)/G$) has been studied by Becker [6], Iwata [13], Sjerve [23] and Yoshida [29]. In this paper, we consider $\text{Span}_G(S(V))$ when G does not act freely on $S(V)$. Our main results are Theorems 1.1 and 1.2, which are generalizations of Theorems 2.1 and 2.2 in [6] respectively. Our method is due to Becker [6].

Let H be a subgroup of G , then we write $H < G$.

Theorem 1.1. *Let G be a finite group and let V, W be unitary G -representation spaces. Suppose that*

- (i) $\dim_{\mathbb{C}} V^H = \dim_{\mathbb{C}} W^H$ for all $H < G$,
- (ii) For each $H < G$, $\dim_{\mathbb{R}} V^H \geq 2k$ if $V^H \neq \{0\}$.

Then $\text{Span}_G(S(V)) \geq k-1$ if and only if $\text{Span}_G(S(W)) \geq k-1$.

Let ξ and η be orthogonal G -vector bundles over a compact G -space. Denote by $S(\xi)$ (resp. $S(\eta)$) the unit sphere bundle of ξ (resp. η). Then $S(\xi)$ and $S(\eta)$ are said to be G -fiber homotopy equivalent if there are fiber-preserving G -maps:

$$f: S(\xi) \rightarrow S(\eta), \quad f': S(\eta) \rightarrow S(\xi)$$

such that $f \circ f'$ and $f' \circ f$ are fiber-preserving G -homotopic to the identity ([6], [19]).

Let $\mathbf{R}P^{k-1}$ denote the $(k-1)$ -dimensional real projective space with trivial G -action and let η_k denote the non-trivial line bundle over $\mathbf{R}P^{k-1}$ with trivial G -action.

Theorem 1.2. *Let G be a finite group and let V be an orthogonal G -representation space. Then we have the following:*

(i) Suppose that $\text{Span}_G(S(V)) \geq k-1$. Then there are an integer t and a G -fiber homotopy equivalence

$$f: S((\eta_k \otimes \underline{V}) \oplus \underline{\mathbf{R}}^t) \rightarrow S(\underline{V} \oplus \underline{\mathbf{R}}^t).$$

Moreover we suppose that $\dim_{\mathbf{R}} V^H \geq k+1$ if $V^H \neq \{0\}$ for each $H < G$. Then there is a G -fiber homotopy equivalence

$$f: S(\eta_k \otimes \underline{V}) \rightarrow S(\underline{V}).$$

(ii) Suppose that $\dim_{\mathbf{R}} V^H \geq 2k$ if $V^H \neq \{0\}$ for each $H < G$ and there is a G -fiber homotopy equivalence

$$f: S(\eta_k \otimes \underline{V}) \rightarrow S(\underline{V}).$$

Then $\text{Span}_G(S(V)) \geq k-1$.

Here \underline{V} denotes the trivial G -vector bundle $\mathbf{R}P^{k-1} \times V \rightarrow \mathbf{R}P^{k-1}$.

Throughout this paper G will be a finite group.

The paper is organized as follows:

In § 2, we discuss some preliminary results. In § 3, we consider equivariant duality, reducibility and coreducibility. In § 4, we consider stunted projective spaces with linear G -actions. In §§ 5 and 6, we state an equivariant version of the theorem of James. In § 7, we prove Theorem 1.1. In § 8, we prove Theorem 1.2. In § 9, we give an example.

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2. Preliminary results

First we shall fix some notations. Let X and Y be G -spaces. Let A be a G -subspace of X and let $\alpha: A \rightarrow Y$ be a G -map. Denote by $F((X, A), Y; \alpha)$ the space of all maps $f: X \rightarrow Y$ such that $f|A = \alpha$ in the compact open topology. $F((X, A), Y; \alpha)$ is a G -space with the following G -action: if $f: X \rightarrow Y$ and $g \in G$, we put

$$(g \cdot f)(x) = g(f(g^{-1}x)).$$

For $H < G$, X^H denotes the H -fixed point set in X . The set $F((X, A), Y; \alpha)^G$ is just the set of G -maps $f: X \rightarrow Y$ such that $f|A = \alpha$. Denote by $[(X, A), Y; \alpha]^G$ the set of G -homotopy classes rel A of G -maps $f: X \rightarrow Y$ such that $f|A = \alpha$. If $A = \phi$, we write $F(X, Y)$ (resp. $[X, Y]^G$) instead of $F((X, A), Y; \alpha)$ (resp. $[(X, A), Y; \alpha]^G$), for simplicity. If X, Y are G -spaces with base points, then we denote the set of G -homotopy classes relative to the base points of pointed G -maps from X to Y by $[X, Y]^G$. The base points are G -fixed points as usual. For $H < G$, (H) denotes the conjugacy class of H in G . Denote by G_x the isotropy group at $x \in X$ and we put

$$\text{Iso}(X) = \{(G_x) \mid x \in X\} .$$

For a space Z , we define $\text{conn}(Z)$ to be the largest integer n such that Z is n -connected. In particular, when Z is not path-connected (resp. $Z = \emptyset$), we put $\text{conn}(Z) = -1$ (resp. $\text{conn}(Z) = \infty$).

The following two lemmas are easily seen by the definition of G -complexes (see Bredon [8] and Waner [26]).

Lemma 2.1. *Let $f: X \rightarrow Y$ be a G -map of G -spaces such that $f^H = f \mid X^H: X^H \rightarrow Y^H$ is an n_H -equivalence for each $H < G$. Let (K, L) be a pair of G -complexes and $\alpha: L \rightarrow X$ be a G -map. Then*

$$f_*: [(K, L), X; \alpha]^G \rightarrow [(K, L), Y; f \circ \alpha]^G$$

is surjective if $\dim(K^H - L) \leq n_H$ and bijective if $\dim(K^H - L) \leq n_H - 1$ for each $(H) \in \text{Iso}(K - L)$.

Lemma 2.2. *Let (K, L) be a pair of G -complexes and X be a G -space. Let $\alpha: L \rightarrow X$ be a G -map. Then the G -fixed point morphism*

$$\phi_G: [(K, L), X; \alpha]^G \rightarrow [(K^G, L^G), X^G; \alpha^G]$$

is surjective if $\dim(K^H - L \cup K^G) \leq \text{conn}(X^H) + 1$ and bijective if $\dim(K^H - L \cup K^G) \leq \text{conn}(X^H)$ for each $(H) \in \text{Iso}(K - L \cup K^G)$.

DEFINITION 2.3. Let X be a G -space. Then X is said to be G -path-connected if and only if $\text{conn}(X^H) \geq 0$ for all $H < G$.

Let X and Y be G -spaces. We recall that the join $X * Y$ is the space obtained from the union of X, Y and $X \times Y \times [0, 1]$ by identifying

$$(x, y, 0) = x, \quad (x, y, 1) = y \quad \text{for } x \in X, y \in Y .$$

We generally omit to write in the identification map, so that the image of (x, y, t) in $X * Y$ is denoted by the same expression. A canonical G -action on $X * Y$ is given by $g \cdot (x, y, t) = (gx, gy, t)$. Let V be an orthogonal G -representation space. We see that

$$(X * Y)^H = X^H * Y^H$$

and

$$\text{conn}((X * S(V))^H) = \text{conn}(X^H) + \dim_{\mathbb{R}} V^H$$

for $H < G$. Let $i_{S(V)}: S(V) \rightarrow X * S(V)$ be an inclusion map defined by $i_{S(V)}(v) = (-, v, 1)$. We have the following theorem (cf. [18; Theorem 3.6], [20]):

Theorem 2.4. *Let K be a G -complex and X be a G -space. Let V be*

an orthogonal G -representation space. Assume that $\text{conn}(X^H) \geq 0$ for each $(H) \in \text{Iso}(K)$. Then the suspension map

$$\tau_*^V: [K, X]^G \rightarrow [(K * S(V), S(V)), X * S(V); i_{S(V)}]^G$$

is surjective if $\dim K^H \leq n_H$ and bijective if $\dim K^H \leq n_H - 1$ for each $(H) \in \text{Iso}(K)$, where

$$n_H = \min_{L < H} \begin{cases} 2 \text{conn}(X^H) + 1 & \text{if } H=L \text{ and } V^H \neq \{0\}, \\ \text{conn}(X^L) & \text{if } V^H \neq V^L, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Let $D(V)$ denote the unit disk in V . We define a G -map

$$\lambda: X \rightarrow F((D(V), S(V)), X * S(V); i_{S(V)})$$

by $\lambda(x)(tv) = (x, v, t)$ for $x \in X, v \in S(V), t \in [0, 1]$. Consider the following commutative diagram:

$$\begin{array}{ccc} & \tau_*^V & [(K * S(V), S(V)), X * S(V); i_{S(V)}]^G \\ [K, X]^G & \nearrow & \downarrow \varphi \\ & \lambda_* & [K, F((D(V), S(V)), X * S(V)); i_{S(V)}]^G, \end{array}$$

where φ is the exponential correspondence given by

$$\varphi(f)(k)(tv) = f(k, v, t) \quad \text{for } k \in K, v \in S(V), t \in [0, 1].$$

As is easily seen, φ is bijective. Using Lemma 2.2, we see that

$$\lambda^H: X^H \rightarrow F((D(V), S(V)), X * S(V); i_{S(V)})^H$$

is an n_H -equivalence for each $(H) \in \text{Iso}(K)$ by the same argument as in the proof of Theorem 3.6 in [18]. We are now in a position to apply Lemma 2.1. q.e.d.

3. Equivariant duality, reducibility and coreducibility

In this section, we recall the definitions of equivariant duality, reducibility and coreducibility (see [18] and [26]) and consider an equivariant version of Atiyah's duality theorem. Let X and Y be pointed G -spaces. The reduced join $X \wedge Y$ has a natural G -action induced from the diagonal action on $X \times Y$. For an orthogonal G -representation space V, Σ^V denotes the one-point compactification of V and $\Sigma^V X = \Sigma^V \wedge X$ is called Σ^V -suspension of X . We remark that Σ^V is a pointed finite G -complex ([12]).

DEFINITION 3.1. Let X and X^* be G -path-connected pointed finite G -

complexes. Let U be an orthogonal G -representation space. Then a pointed G -map

$$\mu: \Sigma^U \rightarrow X \wedge X^*$$

is said to be a (U) -duality G -map if $\mu^H: \Sigma^{U^H} \rightarrow X^H \wedge X^{*H}$ is a duality map in the usual sense ([6], [24]) for each $H < G$.

DEFINITION 3.2. Let X be a G -path-connected pointed finite G -complex and V be an orthogonal G -representation space.

(i) A pointed G -map $f: \Sigma^V \rightarrow X$ is said to be a (V) -reduction G -map if $f^H: \Sigma^{V^H} \rightarrow X^H$ is a reduction map in the usual sense ([3]) for each $H < G$, and then X is called G - (V) -reducible.

(ii) A pointed G -map $f: X \rightarrow \Sigma^V$ is said to be a (V) -coreduction G -map if $f^H: X^H \rightarrow \Sigma^{V^H}$ is a coreduction map in the usual sense ([3]) for each $H < G$, and then X is called G - (V) -coreducible.

Let M be a path-connected closed smooth manifold with trivial G -action. Let ξ be a smooth G -vector bundle over M . The fibers ξ_x for $x \in M$ are orthogonal G -representation spaces. Since M is path-connected, ξ_x does not depend on the choice of $x \in M$. So we put $V = \xi_x$. Assume that $V^G \neq \{0\}$. Then $T(\xi)$ is a G -path-connected pointed finite G -complex ([12]), where $T(\xi)$ denotes the Thom space of ξ .

Proposition 3.3. *If $T(\xi)$ is G - V -coreducible, then there is a G -fiber homotopy equivalence $f: S(\xi \oplus \underline{\mathbf{R}}^1) \rightarrow S(\underline{V} \oplus \underline{\mathbf{R}}^1)$. Conversely, if there is a G -fiber homotopy equivalence $f: S(\xi) \rightarrow S(\underline{V})$, then $T(\xi)$ is G - V -coreducible.*

Using Equivariant Dold Theorem (Kawakubo [19; Theorem 2.1]) and Equivariant J.H.C. Whitehead Theorem (Bredon [8; Chap. II Corollary (5.5)]), the proof is almost parallel to that of Proposition 2.8 in [3]. So we omit it.

Let ω , ξ_1 and ξ_2 be smooth G -vector bundles over M . We put $V = \omega_x$, $W_1 = (\xi_1)_x$ and $W_2 = (\xi_2)_x$ for $x \in M$. Assume that $V^G \neq \{0\}$, $W_1^G \neq \{0\}$ and $W_2^G \neq \{0\}$. Then $T(\omega)$, $T(\xi_1)$ and $T(\xi_2)$ are G -path-connected pointed finite G -complexes.

Lemma 3.4. *If there are a reduction G -map $\alpha: \Sigma^V \rightarrow T(\omega)$ and a coreduction G -map $\beta: T(\xi_1 \oplus \xi_2) \rightarrow \Sigma^{W_1 \oplus W_2}$, then there is a duality G -map*

$$\mu: \Sigma^{W_1 \oplus W_2 \oplus V} \rightarrow T(\xi_1) \wedge T(\xi_2 \oplus \omega).$$

Using Equivariant J.H.C. Whitehead Theorem ([8]), the proof is quite similar to that of (13.2) in [6]. So we omit it.

4. Linear actions on stunted projective spaces

Let V be an orthogonal G -representation space and $\varepsilon_{\mathbf{R}}$ be the non-trivial orthogonal 1-dimensional \mathbf{Z}_2 -representation space. Then $\varepsilon_{\mathbf{R}} \otimes V$ is an orthogonal $(\mathbf{Z}_2 \times G)$ -representation space.

- DEFINITION 4.1. (i) $\mathbf{R}P(V) = S(\varepsilon_{\mathbf{R}} \otimes V) / (\mathbf{Z}_2 \times \{e\})$,
 (ii) For $m \geq k$, $P_k(V \oplus \mathbf{R}^m) = \mathbf{R}P(V \oplus \mathbf{R}^m) / \mathbf{R}P(V \oplus \mathbf{R}^{m-k})$.

Then $P_k(V \oplus \mathbf{R}^m)$ is a pointed finite G -complex ([12]). We see that, if $m > k$, then for $H < G$

$$P_k(V \oplus \mathbf{R}^m)^H = P_k(V^H \oplus \mathbf{R}^m),$$

$$\dim P_k(V \oplus \mathbf{R}^m)^H = \dim_{\mathbf{R}} V^H + m - 1$$

and

$$\text{conn}(P_k(V \oplus \mathbf{R}^m)^H) = \dim_{\mathbf{R}} V^H + m - k - 1.$$

In particular, if $m > k$, then $P_k(V \oplus \mathbf{R}^m)$ is G -path-connected. Atiyah [3] identifies the Thom space of a multiple of η_k as a stunted projective space. As G -spaces this identification takes the form

$$T(\eta_k \otimes (\underline{V} \oplus \underline{\mathbf{R}}^{m-k})) = P_k(V \oplus \mathbf{R}^m).$$

Let $a_k(\mathbf{R})$ ($k > 0$) be the integer defined by [4; § 5]. We recall that the group $\tilde{J}(\mathbf{R}P^{k-1})$ is cyclic of order $a_k(\mathbf{R})$ ([1], [2]). We remark that $a_k(\mathbf{R}) \geq k$ for $k > 0$.

Lemma 4.2. *Let m, n and k be integers such that $m \equiv 0 \pmod{a_k(\mathbf{R})}$, $n \equiv k \pmod{a_k(\mathbf{R})}$ and $n > m \geq 2k \geq 4$. Let U be an arbitrary orthogonal G -representation space. Then we have the following:*

- (i) *If $\Sigma^U P_k(V \oplus \mathbf{R}^m)$ is G - $U \oplus V \oplus \mathbf{R}^{m-1}$ -reducible, then there is a duality G -map*

$$\mu_1: \Sigma^{\mathbf{R}^{m-1}U \oplus V \oplus \mathbf{R}^{n-k}} \rightarrow P_k(\mathbf{R}^m) \wedge \Sigma^U P_k(V \oplus \mathbf{R}^n),$$

- (ii) *If $\Sigma^U P_k(V \oplus \mathbf{R}^n)$ is G - $U \oplus V \oplus \mathbf{R}^{n-k}$ -coreducible, then there is a duality G -map*

$$\mu_2: \Sigma^{U \oplus V \oplus \mathbf{R}^{m-1} \oplus \mathbf{R}^{n-k}} \rightarrow \Sigma^U P_k(V \oplus \mathbf{R}^m) \wedge P_k(\mathbf{R}^n).$$

Proof. We remark that

$$T(U \oplus (\eta_k \otimes (\underline{V} \oplus \underline{\mathbf{R}}^{m-k}))) = \Sigma^U P_k(V \oplus \mathbf{R}^m),$$

$$T(U \oplus (\eta_k \otimes (\underline{V} \oplus \underline{\mathbf{R}}^{n-k}))) = \Sigma^U P_k(V \oplus \mathbf{R}^n).$$

First we show (i). By assumption, there is a reduction G -map

$$\alpha: \Sigma^{U \oplus V \oplus \mathbf{R}^{m-1}} \rightarrow T(U \oplus (\eta_k \otimes (\underline{V} \oplus \underline{\mathbf{R}}^{m-k}))).$$

Set

$$\begin{aligned} \omega &= \underline{U} \oplus (\eta_k \otimes (\underline{V} \oplus \underline{\mathbf{R}}^{m-k})), \quad \xi_1 = \eta_k \otimes \underline{\mathbf{R}}^{m-k}, \\ \xi_2 &= \eta_k \otimes \underline{\mathbf{R}}^{n-m}. \end{aligned}$$

Since $\xi_1 \oplus \xi_2$ is trivial, there is a coreduction (G -)map

$$\beta: T(\xi_1 \oplus \xi_2) \rightarrow \Sigma^{\mathbf{R}^{n-k}}.$$

Applying Lemma 3.4 to $\alpha, \beta, \omega, \xi_1$ and ξ_2 , we have a duality G -map μ_1 .

Next we show (ii). By assumption, there is a coreduction G -map

$$\beta: T(\underline{U} \oplus (\eta_k \otimes (\underline{V} \oplus \underline{\mathbf{R}}^{n-k}))) \rightarrow \Sigma^{U \oplus V \oplus \mathbf{R}^{n-k}}.$$

Since $m \equiv 0 \pmod{a_k(\mathbf{R})}$ and $m \geq 2k$, there is a reduction (G -)map

$$\alpha: \Sigma^{\mathbf{R}^{m-1}} \rightarrow T(\eta_k \otimes \underline{\mathbf{R}}^{m-k}).$$

Set

$$\begin{aligned} \omega &= \eta_k \otimes \underline{\mathbf{R}}^{m-k}, \quad \xi_1 = \underline{U} \oplus (\eta_k \otimes (\underline{V} \oplus \underline{\mathbf{R}}^{m-k})), \\ \xi_2 &= \eta_k \otimes \underline{\mathbf{R}}^{n-m}. \end{aligned}$$

Applying Lemma 3.4, we have a duality G -map μ_2 . q.e.d.

Lemma 4.3. *Let m and k be integers such that $m > k > 0$. Let V be an orthogonal G -representation space. Assume that $P_k(V \oplus \mathbf{R}^m)$ is either G - $V \oplus \mathbf{R}^{m-1}$ -reducible or G - $V \oplus \mathbf{R}^{m-k}$ -coreducible. Then we have*

$$\dim_{\mathbf{R}} V^K - \dim_{\mathbf{R}} V^H \geq k$$

if $V^K \neq V^H$ for $K < H < G$.

Proof. Let $K < H < G$ such that $V^K \neq V^H$. First we assume that $P_k(V \oplus \mathbf{R}^m)$ is G - $V \oplus \mathbf{R}^{m-1}$ -reducible. Then, by definition, $P_k(V^H \oplus \mathbf{R}^m)$ and $P_k(V^K \oplus \mathbf{R}^m)$ are reducible. It follows from Atiyah [3; Theorem 6.2] that $\dim_{\mathbf{R}} V^H + m \equiv 0 \pmod{a_k(\mathbf{R})}$ and $\dim_{\mathbf{R}} V^K + m \equiv 0 \pmod{a_k(\mathbf{R})}$. Thus we see that $\dim_{\mathbf{R}} V^K - \dim_{\mathbf{R}} V^H \equiv 0 \pmod{a_k(\mathbf{R})}$. Now we have

$$\dim_{\mathbf{R}} V^K - \dim_{\mathbf{R}} V^H \geq a_k(\mathbf{R}) \geq k.$$

Next we assume that $P_k(V \oplus \mathbf{R}^m)$ is G - $V \oplus \mathbf{R}^{m-k}$ -coreducible. Then $P_k(V^H \oplus \mathbf{R}^m)$ and $P_k(V^K \oplus \mathbf{R}^m)$ are coreducible. By Atiyah [3; Proposition 2.8], we have

$$\begin{aligned} J(\eta_k \otimes (\underline{V}^H \oplus \underline{\mathbf{R}}^{m-k}) - (\underline{V}^H \oplus \underline{\mathbf{R}}^{m-k})) &= 0 \quad \text{in } \check{J}(\mathbf{R}P^{k-1}), \\ J(\eta_k \otimes (\underline{V}^K \oplus \underline{\mathbf{R}}^{m-k}) - (\underline{V}^K \oplus \underline{\mathbf{R}}^{m-k})) &= 0 \quad \text{in } \check{J}(\mathbf{R}P^{k-1}). \end{aligned}$$

Thus we obtain that $\dim_{\mathbf{R}} V^K - \dim_{\mathbf{R}} V^H \equiv 0 \pmod{a_k(\mathbf{R})}$. Now we see that

$$\dim_{\mathbf{R}} V^K - \dim_{\mathbf{R}} V^H \geq a_k(\mathbf{R}) \geq k. \quad \text{q.e.d.}$$

Proposition 4.4. *Let m, n and k be integers such that $m \equiv 0 \pmod{a_k(\mathbf{R})}$, $n \equiv k \pmod{a_k(\mathbf{R})}$ and $n > m \geq 2k \geq 4$. Let V be an orthogonal G -representation space. Then the following two conditions are equivalent:*

- (i) $P_k(V \oplus \mathbf{R}^m)$ is G - $V \oplus \mathbf{R}^{m-1}$ -reducible,
- (ii) $P_k(V \oplus \mathbf{R}^n)$ is G - $V \oplus \mathbf{R}^{n-k}$ -coreducible.

Proof. First we show that (i) implies (ii). By Lemma 4.2, there is a duality G -map

$$\mu_1: \Sigma^{\mathbf{R}^{m-1} \oplus V \oplus \mathbf{R}^{n-k}} \rightarrow P_k(\mathbf{R}^m) \wedge P_k(V \oplus \mathbf{R}^n).$$

We put $U = V \oplus \mathbf{R}^l$. For $s > 0$, we define a homomorphism

$$\begin{aligned} \bar{\Gamma}_s(\mu_1): [\Sigma^{sU} P_k(V \oplus \mathbf{R}^n), \Sigma^{sU} \Sigma^{V \oplus \mathbf{R}^{n-k}}]_0^G \\ \rightarrow [\Sigma^{sU} \Sigma^{\mathbf{R}^{m-1} \oplus V \oplus \mathbf{R}^{n-k}}, \Sigma^{sU} P_k(\mathbf{R}^m) \wedge \Sigma^{V \oplus \mathbf{R}^{n-k}}]_0^G \end{aligned}$$

by the following: if $f: \Sigma^{sU} P_k(V \oplus \mathbf{R}^n) \rightarrow \Sigma^{sU} \Sigma^{V \oplus \mathbf{R}^{n-k}}$ is a pointed G -map, then $\bar{\Gamma}_s(\mu_1)([f])$ is represented by the composition

$$\begin{aligned} \Sigma^{sU} \Sigma^{\mathbf{R}^{m-1} \oplus V \oplus \mathbf{R}^{n-k}} \xrightarrow{1 \wedge \mu_1} \Sigma^{sU} P_k(\mathbf{R}^m) \wedge P_k(V \oplus \mathbf{R}^n) \xrightarrow{T_1} \\ P_k(\mathbf{R}^m) \wedge \Sigma^{sU} P_k(V \oplus \mathbf{R}^n) \xrightarrow{1 \wedge f} P_k(\mathbf{R}^m) \wedge \Sigma^{sU} \Sigma^{V \oplus \mathbf{R}^{n-k}} \xrightarrow{T_2} \Sigma^{sU} P_k(\mathbf{R}^m) \wedge \Sigma^{V \oplus \mathbf{R}^{n-k}}, \end{aligned}$$

where T_1 and T_2 are the switching maps. Then we have the following:

Assertion 4.4.1. *If $s > \dim_{\mathbf{R}} V + m + n + 1$, then $\bar{\Gamma}_s(\mu_1)$ is an isomorphism.*

The proof is quite similar to that of Assertion 4.1.1 in [18]. So we omit it.

On the other hand, the standard identification

$$\nu_1: \Sigma^{\mathbf{R}^{m-1} \oplus V \oplus \mathbf{R}^{n-k}} \rightarrow \Sigma^{\mathbf{R}^{m-1}} \wedge \Sigma^{V \oplus \mathbf{R}^{n-k}}$$

is a duality G -map. We define a homomorphism

$$\Gamma_s(\nu_1): [\Sigma^{sU} \Sigma^{\mathbf{R}^{m-1}}, \Sigma^{sU} P_k(\mathbf{R}^m)]_0^G \rightarrow [\Sigma^{sU} \Sigma^{\mathbf{R}^{m-1} \oplus V \oplus \mathbf{R}^{n-k}}, \Sigma^{sU} P_k(\mathbf{R}^m) \wedge \Sigma^{V \oplus \mathbf{R}^{n-k}}]_0^G$$

by the following: if $f: \Sigma^{sU} \Sigma^{\mathbf{R}^{m-1}} \rightarrow \Sigma^{sU} P_k(\mathbf{R}^m)$ is a pointed G -map, then $\Gamma_s(\nu_1)([f]) = [f']$, where f' is the composition

$$\Sigma^{sU} \Sigma^{\mathbf{R}^{m-1} \oplus V \oplus \mathbf{R}^{n-k}} \xrightarrow{1 \wedge \nu_1} \Sigma^{sU} \Sigma^{\mathbf{R}^{m-1}} \wedge \Sigma^{V \oplus \mathbf{R}^{n-k}} \xrightarrow{f \wedge 1} \Sigma^{sU} P_k(\mathbf{R}^m) \wedge \Sigma^{V \oplus \mathbf{R}^{n-k}}.$$

For $s > \dim_{\mathbf{R}} V + m + n + 1$, we put

$$\begin{aligned} D_s(\nu_1, \mu_1) = \bar{\Gamma}_s(\mu_1)^{-1} \circ \Gamma_s(\nu_1): [\Sigma^{sU} \Sigma^{\mathbf{R}^{m-1}}, \Sigma^{sU} P_k(\mathbf{R}^m)]_0^G \\ \rightarrow [\Sigma^{sU} P_k(V \oplus \mathbf{R}^n), \Sigma^{sU} \Sigma^{V \oplus \mathbf{R}^{n-k}}]_0^G. \end{aligned}$$

Since $m \equiv 0 \pmod{a_k(\mathbf{R})}$ and $m \geq 2k$, there is a reduction (G -)map $f_1: \Sigma^{\mathbf{R}^{m-1}} \rightarrow P_k(\mathbf{R}^m)$. Let $f_2: \Sigma^{sU} P_k(V \oplus \mathbf{R}^n) \rightarrow \Sigma^{sU} \Sigma^{V \oplus \mathbf{R}^{n-k}}$ be a pointed G -map such that $D_s(v_1, \mu_1)([1 \wedge f_1]) = [f_2]$. As is easily seen, f_2 is a coreduction G -map. Here we consider the suspension map

$$\sigma_*^{sU}: [P_k(V \oplus \mathbf{R}^n), \Sigma^{V \oplus \mathbf{R}^{n-k}}]_0^G \rightarrow [\Sigma^{sU} P_k(V \oplus \mathbf{R}^n), \Sigma^{sU} \Sigma^{V \oplus \mathbf{R}^{n-k}}]_0^G.$$

Let $K < H < G$ such that $(sU)^H \neq (sU)^K$. Since $U = V \oplus \mathbf{R}^1$, we see that $V^H \neq V^K$. Applying Lemma 4.3, we have

$$\begin{cases} \dim(P_k(V \oplus \mathbf{R}^n)^H) = \dim_{\mathbf{R}} V^H + n - 1, \\ 2 \operatorname{conn}((\Sigma^{V \oplus \mathbf{R}^{n-k}})^H) + 1 = 2(\dim_{\mathbf{R}} V^H + n - k - 1) + 1 \geq \dim_{\mathbf{R}} V^H + n - 1, \\ \operatorname{conn}((\Sigma^{V \oplus \mathbf{R}^{n-k}})^K) = \dim_{\mathbf{R}} V^K + n - k - 1 \geq \dim_{\mathbf{R}} V^H + n - 1. \end{cases}$$

By the suspension theorem [18; Theorem 3.6], we see that σ_*^{sU} is surjective. Let $f_3: P_k(V \oplus \mathbf{R}^n) \rightarrow \Sigma^{V \oplus \mathbf{R}^{n-k}}$ be a pointed G -map such that $\sigma_*^{sU}([f_3]) = [f_2]$. Then it is easy to see that f_3 is also a coreduction G -map. That is, $P_k(V \oplus \mathbf{R}^n)$ is G - $V \oplus \mathbf{R}^{n-k}$ -coreducible.

Similarly, using μ_2 in Lemma 4.2, we see that (ii) implies (i). q.e.d.

5. An equivariant version of the theorem of James

First we fix some notations. Let $V_k(V)$ denote the Stiefel manifold of orthogonal k -frames in an orthogonal G -representation space V with G -action defined by

$$g \cdot (v_1, \dots, v_k) = (gv_1, \dots, gv_k).$$

Then $V_k(V)$ is a smooth G -manifold. If $\dim_{\mathbf{R}} V^H \geq k$ for some $H < G$, then we see that

$$V_k(V)^H = V_k(V^H)$$

and

$$\operatorname{conn}(V_k(V)^H) = \dim_{\mathbf{R}} V^H - k - 1.$$

Let

$$q_k: V_k(V) \rightarrow S(V)$$

send (v_1, \dots, v_k) to v_k . We remark that $q_k: V_k(V) \rightarrow S(V)$ is a smooth G -fiber bundle in the sense of Bierstone [7]. We remark the following:

Lemma 5.1. *Span $_G(S(V)) \geq k - 1$ if and only if $q_k: V_k(V) \rightarrow S(V)$ has a smooth G -cross-section.*

Let $m > k > 0$. There is a well-known mapping

$$\tau_k: P_k(V \oplus \mathbf{R}^m) \rightarrow V_k(V \oplus \mathbf{R}^m)$$

by

$$\tau_k([x]) = (e_{n+m-k+1} - 2(e_{n+m-k+1}, x)x, \dots, e_{n+m} - 2(e_{n+m}, x)x),$$

where $n = \dim_{\mathbf{R}} V$ and e_i denotes the i -th unit vector in $V \oplus \mathbf{R}^m$. We see that τ_k is a G -map and for $H < G$

$$\tau_k^H: P_k(V \oplus \mathbf{R}^m)^H \rightarrow V_k(V \oplus \mathbf{R}^m)^H$$

is a $2(\dim_{\mathbf{R}} V^H + m - k)$ -equivalence (see James [16; Lemma 8.1]). We remark that $\tau_1: P_1(V \oplus \mathbf{R}^m) \rightarrow S(V \oplus \mathbf{R}^m) (= V_1(V \oplus \mathbf{R}^m))$ is a G -homeomorphism. Let

$$p: S(V \oplus \mathbf{R}^m) \rightarrow P_k(V \oplus \mathbf{R}^m)$$

and

$$\pi': P_k(V \oplus \mathbf{R}^m) \rightarrow P_1(V \oplus \mathbf{R}^m)$$

be the natural projection and the collapsing map respectively. For $S(V \oplus \mathbf{R}^m)$, we choose a base point $x_0 \in S(\mathbf{R}^{m-k}) (\subset S(V \oplus \mathbf{R}^{m-k}) \subset S(V \oplus \mathbf{R}^m))$. There is a pointed G -map $u: P_1(V \oplus \mathbf{R}^m) \rightarrow S(V \oplus \mathbf{R}^m)$ such that u and τ_1 are G -homotopic. We put

$$\pi = u \circ \pi': P_k(V \oplus \mathbf{R}^m) \rightarrow S(V \oplus \mathbf{R}^m).$$

Then p and π are pointed G -maps.

Lemma 5.2. *Let $m > k > 0$. Let $f: S(V \oplus \mathbf{R}^m) \rightarrow P_k(V \oplus \mathbf{R}^m)$ be a pointed G -map. Then f is a reduction G -map if and only if the composition*

$$S(V \oplus \mathbf{R}^m)^H \xrightarrow{f^H} P_k(V \oplus \mathbf{R}^m)^H \xrightarrow{\pi^H} S(V \oplus \mathbf{R}^m)^H$$

is an ordinary homotopy equivalence (i.e. has degree ± 1) for each $(H) \in \text{Iso}(S(V \oplus \mathbf{R}^m))$.

The proof is easy.

A G -homeomorphism

$$h: S(V) * S(\mathbf{R}^m) \rightarrow S(V \oplus \mathbf{R}^m)$$

is given by $h(x, y, t) = (x \cdot \cos(\pi t/2), y \cdot \sin(\pi t/2))$. In [14], James defined the intrinsic map

$$\mu: V_k(V) * V_k(\mathbf{R}^m) \rightarrow V_k(V \oplus \mathbf{R}^m).$$

We see that μ is a G -map and the following diagram commutes:

$$\begin{array}{ccc} V_k(V) * V_k(\mathbf{R}^m) & \xrightarrow{\mu} & V_k(V \oplus \mathbf{R}^m) \\ \downarrow q_k * q_k & & \downarrow q_k \\ S(V) * S(\mathbf{R}^m) & \xrightarrow{h} & S(V \oplus \mathbf{R}^m). \end{array}$$

Now we prove the following theorem, which is a generalization of Proposition 11.5 in [6] (see also Theorem 8.2 in [16]):

Theorem 5.3. *Let m and k be integers such that $m \equiv 0 \pmod{a_k(\mathbf{R})}$ and $m \geq 2k \geq 4$. If $\text{Span}_G(S(V)) \geq k-1$, then $P_k(V \oplus \mathbf{R}^m)$ is G - $V \oplus \mathbf{R}^{m-1}$ -reducible.*

Proof. Since $m \equiv 0 \pmod{a_k(\mathbf{R})}$ and $m \geq 2k$, there is a reduction (G -)map $\rho: S(\mathbf{R}^m) \rightarrow P_k(\mathbf{R}^m)$. It follows from Lemma 5.1 that there is a G -cross-section of q_k

$$\Delta: S(V) \rightarrow V_k(V).$$

Then we define a G -map

$$\gamma: S(V \oplus \mathbf{R}^m) \rightarrow V_k(V \oplus \mathbf{R}^m)$$

by the composition

$$\begin{aligned} S(V \oplus \mathbf{R}^m) &\xrightarrow{h^{-1}} S(V) * S(\mathbf{R}^m) \xrightarrow{\Delta * \rho} V_k(V) * P_k(\mathbf{R}^m) \xrightarrow{1 * \tau_k} \\ &V_k(V) * V_k(\mathbf{R}^m) \xrightarrow{\mu} V_k(V \oplus \mathbf{R}^m). \end{aligned}$$

Consider a map

$$\tau_{k*}: [S(V \oplus \mathbf{R}^m), P_k(V \oplus \mathbf{R}^m)]^G \rightarrow [S(V \oplus \mathbf{R}^m), V_k(V \oplus \mathbf{R}^m)]^G.$$

Since $\tau_k^H: P_k(V \oplus \mathbf{R}^m)^H \rightarrow V_k(V \oplus \mathbf{R}^m)^H$ is a $2(\dim_{\mathbf{R}} V^H + m - k)$ -equivalence for each $H < G$, it follows from Lemma 2.1 that τ_{k*} is bijective. Moreover we see that

$$[S(V \oplus \mathbf{R}^m), P_k(V \oplus \mathbf{R}^m)]^G \cong [S(V \oplus \mathbf{R}^m), P_k(V \oplus \mathbf{R}^m)]_0^G.$$

Hence there is a pointed G -map

$$\lambda: S(V \oplus \mathbf{R}^m) \rightarrow P_k(V \oplus \mathbf{R}^m)$$

such that $\tau_{k*}([\lambda]) = [\gamma]$. As is easily seen, the composition

$$S(V \oplus \mathbf{R}^m)^H \xrightarrow{\lambda^H} P_k(V \oplus \mathbf{R}^m)^H \xrightarrow{\pi^H} S(V \oplus \mathbf{R}^m)^H$$

is an ordinary homotopy equivalence for each $H < G$. By Lemma 5.2, λ is a reduction G -map. That is, $P_k(V \oplus \mathbf{R}^m)$ is G - $V \oplus \mathbf{R}^{m-1}$ -reducible. q.e.d.

6. A converse of Theorem 5.3

Let m and k be integers such that $m \equiv 0 \pmod{a_k(\mathbf{R})}$ and $m \geq 2k \geq 4$. Let $\kappa: S(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m)$ be a 1-section of q_k . That is, the composition

$$S(\mathbf{R}^m) \xrightarrow{\kappa} V_k(\mathbf{R}^m) \xrightarrow{q_k} S(\mathbf{R}^m)$$

has degree 1. For $n > k$, we define

$$\theta_\kappa: V_k(\mathbf{R}^n)*S(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^{n+m})$$

by the composition

$$V_k(\mathbf{R}^n)*S(\mathbf{R}^m) \xrightarrow{1*\kappa} V_k(\mathbf{R}^n)*V_k(\mathbf{R}^m) \xrightarrow{\mu} V_k(\mathbf{R}^{n+m}),$$

where μ is the intrinsic map (see § 5). By Theorem 3.1 in [15], θ_κ is a $(2n-2k+m-1)$ -equivalence. The following Theorem is a converse of Theorem 5.3.

Theorem 6.1. *Let m and k be integers such that $m \equiv 0 \pmod{a_k(\mathbf{R})}$ and $m \geq 2k \geq 4$. Let V be an orthogonal G -representation space. Assume that*

- (i) *For each $H < G$, $\dim_{\mathbf{R}} V^H \geq 2k$ if $V^H \neq \{0\}$,*
- (ii) *$P_k(V \oplus \mathbf{R}^m)$ is G - $V \oplus \mathbf{R}^{m-1}$ -reducible.*

Then $\text{Span}_G(S(V)) \geq k-1$.

Proof. First we show the following Assertion 6.1.1.

Assertion 6.1.1. *There is a G -map*

$$\gamma_0: S(V \oplus \mathbf{R}^m) \rightarrow V_k(V \oplus \mathbf{R}^m)$$

such that γ_0 satisfies the following:

$$(6.1.2) \quad \gamma_0(S(\mathbf{R}^m)) \subset V_k(\mathbf{R}^m) (\subset V_k(V \oplus \mathbf{R}^m)),$$

(6.1.3) *the composition*

$$S(\mathbf{R}^m) \xrightarrow{\gamma_0|S(\mathbf{R}^m)} V_k(\mathbf{R}^m) \xrightarrow{q_k} S(\mathbf{R}^m)$$

has degree 1,

(6.1.4) *the composition*

$$S(V \oplus \mathbf{R}^m)^H \xrightarrow{\gamma_0^H} V_k(V \oplus \mathbf{R}^m)^H \xrightarrow{q_k^H} S(V \oplus \mathbf{R}^m)^H$$

has degree 1 for each $H < G$.

Proof of Assertion 6.1.1. By assumption, we have a reduction G -map

$$\lambda': S(V \oplus \mathbf{R}^m) \rightarrow P_k(V \oplus \mathbf{R}^m).$$

Let $\pi: P_k(V \oplus \mathbf{R}^m) \rightarrow S(V \oplus \mathbf{R}^m)$ be the pointed G -map as in § 5. We put

$$\lambda = \lambda' \circ (\pi \circ \lambda') : S(V \oplus \mathbf{R}^m) \rightarrow P_k(V \oplus \mathbf{R}^m).$$

Then λ is also a reduction G -map such that

$$\text{deg}(\pi \circ \lambda)^H = 1 \quad \text{for all } H < G.$$

We put

$$\gamma_1 = \tau_k \circ \lambda : S(V \oplus \mathbf{R}^m) \rightarrow V_k(V \oplus \mathbf{R}^m).$$

We consider a G -map

$$\gamma_2 = \gamma_1|S(\mathbf{R}^m): S(\mathbf{R}^m) \rightarrow V_k(V \oplus \mathbf{R}^m).$$

First we assume that $V^G \neq \{0\}$. Since $m \equiv 0 \pmod{a_k(\mathbf{R})}$ and $m \geq 2k$, there is a (G -)cross-section of q_k

$$\Delta: S(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m) \subset V_k(V \oplus \mathbf{R}^m).$$

Since $\text{conn}(V_k(V \oplus \mathbf{R}^m)^G) \geq \dim S(\mathbf{R}^m)$, γ_2 and Δ are G -homotopic. Remark that $(S(V \oplus \mathbf{R}^m), S(\mathbf{R}^m))$ has the G -homotopy extension property. We have a G -map

$$\gamma_0: S(V \oplus \mathbf{R}^m) \rightarrow V_k(V \oplus \mathbf{R}^m)$$

such that γ_0 and γ_1 are G -homotopic and $\gamma_0|S(\mathbf{R}^m) = \Delta$. As is easily seen, γ_0 satisfies our required properties.

Next we assume that $V^G = \{0\}$. In this case, $\gamma_2 = \gamma_1^G: S(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m)$ is a 1-section of q_k . Therefore we put $\gamma_0 = \gamma_1$.

This completes the proof of Assertion 6.1.1.

We put $\gamma_3 = \gamma_0|S(\mathbf{R}^m): S(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m) (\subset V_k(V \oplus \mathbf{R}^m))$. Consider a map

$$\begin{aligned} \theta_{\gamma_3^*}: [(S(V \oplus \mathbf{R}^m), S(\mathbf{R}^m)), V_k(V) * S(\mathbf{R}^m); i_{S(\mathbf{R}^m)}]^G \\ \rightarrow [(S(V \oplus \mathbf{R}^m), S(\mathbf{R}^m)), V_k(V \oplus \mathbf{R}^m); \gamma_3]^G. \end{aligned}$$

Since γ_3 is a 1-section, $\theta_{\gamma_3^*}^H: V_k(V)^H * S(\mathbf{R}^m) \rightarrow V_k(V \oplus \mathbf{R}^m)^H$ is a $(2 \dim_{\mathbf{R}} V^H - 2k + m - 1)$ -equivalence for each $(H) \in \text{Iso}(S(V \oplus \mathbf{R}^m) - S(\mathbf{R}^m))$. Applying Lemma 2.1, $\theta_{\gamma_3^*}$ is surjective. Therefore we have a G -map

$$\gamma_4: S(V \oplus \mathbf{R}^m) \rightarrow V_k(V) * S(\mathbf{R}^m)$$

such that $\theta_{\gamma_3^*}([\gamma_4]) = [\gamma_0]$ and $\gamma_4|S(\mathbf{R}^m) = i_{S(\mathbf{R}^m)}$. As is easily seen, the composition

$$S(V)^H * S(\mathbf{R}^m) \xrightarrow{h^H} S(V \oplus \mathbf{R}^m)^H \xrightarrow{\gamma_4^H} V_k(V)^H * S(\mathbf{R}^m) \xrightarrow{q_k^H * 1} S(V)^H * S(\mathbf{R}^m)$$

has degree 1 for each $H < G$, where h is as in § 5. Consider the following suspension map

$$\tau_{*}^{\mathbf{R}^m}: [S(V), V_k(V)]^G \rightarrow [(S(V) * S(\mathbf{R}^m), S(\mathbf{R}^m)), V_k(V) * S(\mathbf{R}^m); i_{S(\mathbf{R}^m)}]^G.$$

Since $\dim S(V)^H \leq 2 \text{conn}(V_k(V^H)) + 1$ for each $(H) \in \text{Iso}(S(V))$, it follows from Theorem 2.4 that $\tau_{*}^{\mathbf{R}^m}$ is surjective. Then we have a G -map

$$\gamma_5: S(V) \rightarrow V_k(V)$$

such that $\tau_{*}^{\mathbf{R}^m}([\gamma_5]) = [\gamma_4 \circ h]$. As is easily seen, the composition

$$S(V)^H \xrightarrow{\gamma_5^H} V_k(V)^H \xrightarrow{q_k^H} S(V)^H$$

has degree 1 for each $(H) \in \text{Iso}(S(V))$. Let $K < H < G$ such that $V^K \neq V^H$. Using Lemma 4.3, we have $\dim_{\mathbf{R}} V^K - \dim_{\mathbf{R}} V^H \geq k \geq 2$. Thus it follows from Rubinsztein [22; Theorem 8.4] that $q_k \circ \gamma_5$ is G -homotopic to the identity. Since $q_k: V_k(V) \rightarrow S(V)$ is a smooth G -fiber bundle in the sense of Bierstone [7], q_k has the smooth G -homotopy lifting property. Using Wasserman [27; Corollary 1.12], we see that q_k has a smooth G -cross-section. Now, by Lemma 5.1, we have $\text{Span}_{\mathbf{C}}(S(V)) \geq k - 1$. q.e.d.

7. Proof of Theorem 1.1

Let V and W be unitary G -representation spaces such that $\dim_{\mathbf{C}} V^H = \dim_{\mathbf{C}} W^H$ for all $H < G$. By Lee-Wasserman [21; Proposition 3.17], there are direct sum decompositions

$$\begin{cases} V = V_1 \oplus V_2 \oplus \cdots \oplus V_r, \\ W = W_1 \oplus W_2 \oplus \cdots \oplus W_r \end{cases}$$

such that V_i and W_i ($1 \leq i \leq r$) are irreducible unitary G -representation spaces and V_i is conjugate to W_i by a field automorphism of \mathbf{C} for $1 \leq i \leq r$. That is, there are integers $n(i)$ ($1 \leq i \leq r$) such that $(n(i), |G|) = 1$ and $W_i = \psi^{n(i)}(V_i)$ for $1 \leq i \leq r$, where ψ^s denotes the equivariant s -th Adams operation and $|G|$ denotes the order of G . Since $\psi^{s+|G|} = \psi^s$, we may assume that $n(i)$ ($1 \leq i \leq r$) are odd integers. Let $\varepsilon_{\mathbf{C}}$ be the non-trivial unitary 1-dimensional \mathbf{Z}_2 -representation space. Then $\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} V$ and $\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} W$ are unitary $(\mathbf{Z}_2 \times G)$ -representation spaces and

$$\begin{cases} \varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} V = (\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} V_1) \oplus (\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} V_2) \oplus \cdots \oplus (\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} V_r), \\ \varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} W = (\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} W_1) \oplus (\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} W_2) \oplus \cdots \oplus (\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} W_r) \end{cases}$$

are decompositions of $\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} V$ and $\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} W$ into direct sums of irreducible unitary $(\mathbf{Z}_2 \times G)$ -representation spaces respectively. Since $n(i)$ ($1 \leq i \leq r$) are odd, there are integers $\bar{n}(i)$ ($1 \leq i \leq r$) such that $(\bar{n}(i), 2|G|) = 1$ and $n(i) \cdot \bar{n}(i) \equiv 1 \pmod{2|G|}$. Then we have

$$\begin{cases} \varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} V_i = \psi^{\bar{n}(i)}(\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} W_i) & \text{for } 1 \leq i \leq r, \\ \varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} W_i = \psi^{n(i)}(\varepsilon_{\mathbf{C}} \otimes_{\mathbf{C}} V_i) & \text{for } 1 \leq i \leq r. \end{cases}$$

The following lemma is due to Tornehave [25] (see also [11]).

Lemma 7.1. *There are $(\mathbf{Z}_2 \times G)$ -maps*

$$\begin{cases} \varphi_i: S(\varepsilon_c \otimes_c V_i) \rightarrow S(\varepsilon_c \otimes_c W_i), \\ \psi_i: S(\varepsilon_c \otimes_c W_i) \rightarrow S(\varepsilon_c \otimes_c V_i) \end{cases}$$

for $1 \leq i \leq r$ such that

$$\deg \varphi_i^K = n(i)^{d_i(K)} \quad \text{and} \quad \deg \psi_i^K = \bar{n}(i)^{d_i(K)}$$

for each $K < \mathbf{Z}_2 \times G$, where $d_i(K) = \dim_c(\varepsilon_c \otimes_c V_i)^K (= \dim_c(\varepsilon_c \otimes_c W_i)^K)$.

We put

$$(7.2) \quad \begin{cases} \varphi = \varphi_1 * \cdots * \varphi_r: S(\varepsilon_c \otimes_c V) \rightarrow S(\varepsilon_c \otimes_c W), \\ \psi = \psi_1 * \cdots * \psi_r: S(\varepsilon_c \otimes_c W) \rightarrow S(\varepsilon_c \otimes_c V). \end{cases}$$

Then, for each $K < \mathbf{Z}_2 \times G$, we have

$$\deg(\psi \circ \varphi)^K \equiv 1 \pmod{2|G|} \quad \text{and} \quad \deg(\varphi \circ \psi)^K \equiv 1 \pmod{2|G|}.$$

Let U be a unitary G -representation space and $m \geq 2$. We define a homomorphism

$$\Psi: [\Sigma^{U \oplus \mathbf{R}^{m-1}}, \Sigma^{U \oplus \mathbf{R}^{m-1}}]_0^G \rightarrow \prod_{(H) \in \text{Iso}(\Sigma^{U \oplus \mathbf{R}^{m-1}})} \mathbf{Z}$$

by the following: if $f: \Sigma^{U \oplus \mathbf{R}^{m-1}} \rightarrow \Sigma^{U \oplus \mathbf{R}^{m-1}}$ is a pointed G -map, then $\Psi([f]) = \prod_{(H) \in \text{Iso}(\Sigma^{U \oplus \mathbf{R}^{m-1}})} \deg f^H$ (for details see Rubinsztein [22]). By the same argument as in tom Dieck [10; Proposition 1.2.3], we have the following:

Lemma 7.3. *Let $x \in \prod_{(H) \in \text{Iso}(\Sigma^{U \oplus \mathbf{R}^{m-1}})} \mathbf{Z}$ be an arbitrary element. Then $|G| x \in \text{Im } \Psi$.*

Proposition 7.4. *Let $m > k \geq 2$. Let V and W be unitary G -representation spaces such that $\dim_c V^H = \dim_c W^H$ for all $H < G$. Then the following two conditions are equivalent:*

(i) *There is a reduction G -map*

$$f: \Sigma^{V \oplus \mathbf{R}^{m-1}} \rightarrow P_k(V \oplus \mathbf{R}^m),$$

(ii) *There is a reduction G -map*

$$g: \Sigma^{W \oplus \mathbf{R}^{m-1}} \rightarrow P_k(W \oplus \mathbf{R}^m).$$

Proof. It suffices to show that (i) implies (ii). Let

$$\begin{cases} \varphi: S(\varepsilon_c \otimes_c V) \rightarrow S(\varepsilon_c \otimes_c W), \\ \psi: S(W) \rightarrow S(V) \end{cases}$$

be a $(\mathbf{Z}_2 \times G)$ -map and a $G(\subset \mathbf{Z}_2 \times G)$ -map as in (7.2) respectively. We put a $(\mathbf{Z}_2 \times G)$ -map

$$\varphi_1 = \varphi^* 1_{S(\varepsilon_{\mathbf{R}} \otimes \mathbf{R}^m)}: S((\varepsilon_C \otimes V) \oplus (\varepsilon_{\mathbf{R}} \otimes \mathbf{R}^m)) \rightarrow S((\varepsilon_C \otimes W) \oplus (\varepsilon_{\mathbf{R}} \otimes \mathbf{R}^m))$$

and a pointed G -map

$$\psi_1 = \psi^* 1_{S(\mathbf{R}^m)}: S(W \oplus \mathbf{R}^m) \rightarrow S(V \oplus \mathbf{R}^m).$$

Remark that φ_1 induces a pointed G -map

$$\varphi_2: P_k(V \oplus \mathbf{R}^m) \rightarrow P_k(W \oplus \mathbf{R}^m)$$

such that the following diagram commutes:

$$\begin{array}{ccc} S((\varepsilon_C \otimes V) \oplus (\varepsilon_{\mathbf{R}} \otimes \mathbf{R}^m)) & \xrightarrow{\varphi_1} & S((\varepsilon_C \otimes W) \oplus (\varepsilon_{\mathbf{R}} \otimes \mathbf{R}^m)) \\ \downarrow p_1 & & \downarrow p_2 \\ P_k(V \oplus \mathbf{R}^m) & \xrightarrow{\varphi_2} & P_k(W \oplus \mathbf{R}^m), \end{array}$$

where p_1 and p_2 are the natural projections as in § 5. We define a pointed G -map

$$g_1: \Sigma^{W \oplus \mathbf{R}^{m-1}} \rightarrow P_k(W \oplus \mathbf{R}^m)$$

by the composition

$$\begin{array}{ccccc} \Sigma^{W \oplus \mathbf{R}^{m-1}} & \xrightarrow{d_2} & S(W \oplus \mathbf{R}^m) & \xrightarrow{\psi_1} & S(V \oplus \mathbf{R}^m) & \xrightarrow{d_1} \\ & & & & \downarrow f & \\ & & \Sigma^{V \oplus \mathbf{R}^{m-1}} & \xrightarrow{\varphi_2} & P_k(W \oplus \mathbf{R}^m), & \end{array}$$

where d_1 and d_2 are pointed G -homeomorphisms. Let $\pi_1: P_k(V \oplus \mathbf{R}^m) \rightarrow S(V \oplus \mathbf{R}^m)$ and $\pi_2: P_k(W \oplus \mathbf{R}^m) \rightarrow S(W \oplus \mathbf{R}^m)$ be the natural collapsing maps as in § 5. Let

$$g_2: \Sigma^{W \oplus \mathbf{R}^{m-1}} \rightarrow \Sigma^{W \oplus \mathbf{R}^{m-1}}$$

be a G -map defined by the composition

$$\Sigma^{W \oplus \mathbf{R}^{m-1}} \xrightarrow{g_1} P_k(W \oplus \mathbf{R}^m) \xrightarrow{\pi_2} S(W \oplus \mathbf{R}^m) \xrightarrow{d_2^{-1}} \Sigma^{W \oplus \mathbf{R}^{m-1}}.$$

Then it is easy to see that

$$\deg g_2^H \equiv \deg(d_1 \circ \pi_1 \circ f)^H \pmod{2|G|} \quad \text{for each } H < G.$$

Since f is a reduction G -map, we remark that $\deg(d_1 \circ \pi_1 \circ f)^H = \pm 1$ for each $H < G$. Let $a(H)$ be an integer such that

$$\deg g_2^H = \deg(d_1 \circ \pi_1 \circ f)^H + 2a(H)|G|$$

for each $(H) \in \text{Iso}(\Sigma^{W \oplus \mathbf{R}^{m-1}})$. By Lemma 7.3, there is a pointed G -map

$$g_3: \Sigma^{W \oplus \mathbf{R}^{m-1}} \rightarrow \Sigma^{W \oplus \mathbf{R}^{m-1}}$$

such that $\text{deg } g_3^H = a(H)|G|$ for each $(H) \in \text{Iso}(\Sigma^{W \oplus \mathbf{R}^{m-1}})$. We define a pointed G -map

$$g_4: \Sigma^{W \oplus \mathbf{R}^{m-1}} \rightarrow P_k(W \oplus \mathbf{R}^m)$$

by the composition

$$\Sigma^{W \oplus \mathbf{R}^{m-1}} \xrightarrow{g_3} \Sigma^{W \oplus \mathbf{R}^{m-1}} \xrightarrow{d_2} S(W \oplus \mathbf{R}^m) \xrightarrow{p_2} P_k(W \oplus \mathbf{R}^m).$$

Then we see that the composition

$$(\Sigma^{W \oplus \mathbf{R}^{m-1}})^H \xrightarrow{g_4^H} P_k(W \oplus \mathbf{R}^m)^H \xrightarrow{\pi_2^H} S(W \oplus \mathbf{R}^m)^H \xrightarrow{(d_2^{-1})^H} (\Sigma^{W \oplus \mathbf{R}^{m-1}})^H$$

has degree $2a(H)|G|$ for each $(H) \in \text{Iso}(\Sigma^{W \oplus \mathbf{R}^{m-1}})$. Since $m \geq 2$, pointed G -homotopy classes of pointed G -maps from $\Sigma^{W \oplus \mathbf{R}^{m-1}}$ to $P_k(W \oplus \mathbf{R}^m)$ form a group. Then we put

$$g = g_1 - g_4: \Sigma^{W \oplus \mathbf{R}^{m-1}} \rightarrow P_k(W \oplus \mathbf{R}^m).$$

It is easy to see that the composition

$$(\Sigma^{W \oplus \mathbf{R}^{m-1}})^H \xrightarrow{g^H} P_k(W \oplus \mathbf{R}^m)^H \xrightarrow{\pi_2^H} S(W \oplus \mathbf{R}^m)^H \xrightarrow{(d_2^{-1})^H} (\Sigma^{W \oplus \mathbf{R}^{m-1}})^H$$

has $\text{deg}(d_1 \circ \pi_1 \circ f)^H (= \pm 1)$ for each $(H) \in \text{Iso}(\Sigma^{W \oplus \mathbf{R}^{m-1}})$. It follows from Lemma 5.2 that g is a reduction G -map. q.e.d.

Proof of Theorem 1.1. We may assume that $k \geq 2$. Let m be an integer such that $m \equiv 0 \pmod{a_k(\mathbf{R})}$ and $m \geq 2k$. If $\text{Span}_G(S(V)) \geq k-1$, it follows from Theorem 5.3 that $P_k(V \oplus \mathbf{R}^m)$ is G - $V \oplus \mathbf{R}^{m-1}$ -reducible. According to Proposition 7.4, $P_k(W \oplus \mathbf{R}^m)$ is G - $W \oplus \mathbf{R}^{m-1}$ -reducible. By Theorem 6.1, $\text{Span}_G(S(W)) \geq k-1$.

The converse is quite similar. q.e.d.

8. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Lemma 8.1. *Let U be an orthogonal G -representation space such that $\dim_{\mathbf{R}} U^H \geq k+1$ if $U^H \neq \{0\}$ for each $H < G$. Assume that there are an integer m and a G -fiber homotopy equivalence*

$$f: S((\eta_k \otimes \underline{U}) \oplus \underline{\mathbf{R}}^m) \rightarrow S(\underline{U} \oplus \underline{\mathbf{R}}^m).$$

Then we have a G -fiber homotopy equivalence

$$\tilde{f}: S(\eta_k \otimes \underline{U}) \rightarrow S(\underline{U}).$$

Proof. First we show that the following Assertion 8.1.1.

Assertion 8.1.1. *There are an integer $n (\geq m)$ and a G -map*

$$f_1: S((\eta_k \otimes \underline{U}) \oplus \underline{\mathbf{R}}^n) \rightarrow S(U \oplus \mathbf{R}^n)$$

such that a restriction

$$f_1|S((\eta_k \otimes \underline{U}) \oplus \underline{\mathbf{R}}^n)_x: S((\eta_k \otimes \underline{U}) \oplus \underline{\mathbf{R}}^n)_x \rightarrow S(U \oplus \mathbf{R}^n)$$

for $x \in \mathbf{R}P^{k-1}$ is a G -homotopy equivalence and a restriction $f_1|S(\underline{\mathbf{R}}^n)$ is the natural projection $S(\underline{\mathbf{R}}^n) \rightarrow S(\mathbf{R}^n) \subset S(U \oplus \mathbf{R}^n)$.

Proof of Assertion 8.1.1. We put $f_2 = p_1 \circ f: S((\eta_k \otimes \underline{U}) \oplus \underline{\mathbf{R}}^m) \rightarrow S(U \oplus \mathbf{R}^m)$, where $p_1: S(\underline{U} \oplus \underline{\mathbf{R}}^m) \rightarrow S(U \oplus \mathbf{R}^m)$ is the natural projection.

Suppose first that $U^G \neq \{0\}$. By assumption, we see that $\text{conn}(S(U \oplus \mathbf{R}^m)^G) \geq \dim S(\underline{\mathbf{R}}^m)$. Then $f_2|S(\underline{\mathbf{R}}^m): S(\underline{\mathbf{R}}^m) \rightarrow S(U \oplus \mathbf{R}^m)$ and the natural projection $p_2: S(\underline{\mathbf{R}}^m) \rightarrow S(\mathbf{R}^m) \subset S(U \oplus \mathbf{R}^m)$ are G -homotopic. Since $(S((\eta_k \otimes \underline{U}) \oplus \underline{\mathbf{R}}^m), S(\underline{\mathbf{R}}^m))$ has the G -homotopy extension property, we have a G -map

$$f_1: S((\eta_k \otimes \underline{U}) \oplus \underline{\mathbf{R}}^m) \rightarrow S(U \oplus \mathbf{R}^m)$$

such that f_1 and f_2 are G -homotopic and $f_1|S(\underline{\mathbf{R}}^m) = p_2$. We put $n = m$. It is easy to see that f_1 has our required properties.

Suppose second that $U^G = \{0\}$. Remark that $f_2^G: S(\underline{\mathbf{R}}^m) \rightarrow S(\mathbf{R}^m)$ is a map such that $(f_2^G)_x: S(\mathbf{R}^m) \rightarrow S(\mathbf{R}^m)$ is a homotopy equivalence for $x \in \mathbf{R}P^{k-1}$. It is well-known that there is a map $h: S(\underline{\mathbf{R}}^{m+m'}) \rightarrow S(\mathbf{R}^{m+m'})$ such that $f_2^G \tilde{*} h: S(\underline{\mathbf{R}}^{m+m'}) \rightarrow S(\mathbf{R}^{m+m'})$ is homotopic to the natural projection $p_3: S(\underline{\mathbf{R}}^{m+m'}) \rightarrow S(\mathbf{R}^{m+m'})$, where $\tilde{*}$ denotes the fiberwise join. We put

$$f_3 = f_2 \tilde{*} h: S((\eta_k \otimes \underline{U}) \oplus \underline{\mathbf{R}}^{m+m'}) \rightarrow S(U \oplus \mathbf{R}^{m+m'}).$$

Then $f_3|S(\underline{\mathbf{R}}^{m+m'}) = f_3^G$ is (G) -homotopic to p_3 . By the same argument as in the case when $U^G \neq \{0\}$, we have a G -map

$$f_1: S((\eta_k \otimes \underline{U}) \oplus \underline{\mathbf{R}}^{m+m'}) \rightarrow S(U \oplus \mathbf{R}^{m+m'})$$

such that f_1 is G -homotopic to f_3 and $f_1|S(\underline{\mathbf{R}}^{m+m'}) = p_3$. We put $n = m + m'$. Then f_1 has our required properties.

This completes the proof of Assertion 8.1.1.

We see that f_1 induces a G -map

$$f_4: S(\eta_k \otimes \underline{U}) * S(\mathbf{R}^n) \rightarrow S(U) * S(\mathbf{R}^n)$$

such that the following diagram commutes:

$$\begin{array}{ccc}
 S(\eta_k \otimes \underline{U}) * S(\underline{\mathbf{R}}^n) & \xrightarrow{f_1} & S(U) * S(\underline{\mathbf{R}}^n), \\
 \downarrow q & & \uparrow f_4 \\
 S(\eta_k \otimes \underline{U}) * S(\underline{\mathbf{R}}^n) & &
 \end{array}$$

where q is the natural projection. Then $f_4|_{S(\underline{\mathbf{R}}^n)} = i_{S(\underline{\mathbf{R}}^n)}: S(\underline{\mathbf{R}}^n) \rightarrow S(U) * S(\underline{\mathbf{R}}^n)$. For each $(H) \in \text{Iso}(S(\eta_k \otimes \underline{U})) (= \text{Iso}(S(U)))$, we see that $\dim S(\eta_k \otimes \underline{U})^H \leq 2 \text{conn}(S(U)^H) + 1$. It follows from Theorem 2.4 that we obtain a G -map

$$f_5: S(\eta_k \otimes \underline{U}) \rightarrow S(U)$$

such that $f_5 * 1_{S(\underline{\mathbf{R}}^n)}$ is G -homotopic to f_4 . By Equivariant Dold Theorem ([19]), it is easy to see that

$$\bar{f} = p_4 \times f_5: S(\eta_k \otimes \underline{U}) \rightarrow \mathbf{R}P^{k-1} \times S(U)$$

gives a G -fiber homotopy equivalence, where $p_4: S(\eta_k \otimes \underline{U}) \rightarrow \mathbf{R}P^{k-1}$ is the natural projection. q.e.d.

Proof of Theorem 1.2. We may assume that $k \geq 2$. Let m and n be integers such that $m \equiv 0 \pmod{a_k(\mathbf{R})}$, $n \equiv k \pmod{a_k(\mathbf{R})}$ and $n > m \geq 2k$.

First we show (i). By Theorem 5.3, $P_k(V \oplus \mathbf{R}^m)$ is G - $V \oplus \mathbf{R}^{m-1}$ -reducible. Applying Proposition 4.4, $P_k(V \oplus \mathbf{R}^n)$ is G - $V \oplus \mathbf{R}^{n-k}$ -coreducible. It follows from Proposition 3.3 that we have a G -fiber homotopy equivalence

$$f_1: S((\eta_k \otimes (\underline{V} \oplus \underline{\mathbf{R}}^{n-k})) \oplus \underline{\mathbf{R}}^1) \rightarrow S(\underline{V} \oplus \underline{\mathbf{R}}^{n-k} \oplus \underline{\mathbf{R}}^1).$$

Since $n \equiv k \pmod{a_k(\mathbf{R})}$ and $n > 2k$, we have a G -fiber homotopy equivalence

$$f_2: S((\eta_k \otimes \underline{V}) \oplus \underline{\mathbf{R}}^{n-k+1}) \rightarrow S(\underline{V} \oplus \underline{\mathbf{R}}^{n-k+1}).$$

The first result follows. The second result follows from Lemma 8.1

Next we show (ii). Since $n \equiv k \pmod{a_k(\mathbf{R})}$ and $n > 2k$, we have a G -fiber homotopy equivalence

$$f_3: S(\eta_k \otimes (\underline{V} \oplus \underline{\mathbf{R}}^{n-k})) \rightarrow S(\underline{V} \oplus \underline{\mathbf{R}}^{n-k}).$$

By Proposition 3.3, $P_k(V \oplus \mathbf{R}^n)$ is G - $V \oplus \mathbf{R}^{n-k}$ -coreducible. Applying Proposition 4.4, $P_k(V \oplus \mathbf{R}^m)$ is G - $V \oplus \mathbf{R}^{m-1}$ -reducible. It follows from Theorem 6.1 that $\text{Span}_G(S(V)) \geq k - 1$. q.e.d.

9. An example

Let G be a metacyclic group

$$\{a, b \mid a^m = b^q = e, bab^{-1} = a^r\},$$

where m is a positive odd integer, q is an odd prime integer, $(r-1, m)=1$ and r is a primitive q -th root of 1 mod m . Let $\mathbf{Z}_m = \langle a \rangle < G$ and let $t^h (h \in \mathbf{Z})$ be the unitary 1-dimensional \mathbf{Z}_m -representation space with a acting on \mathbf{C}^1 as multiplication with $\exp(2\pi h\sqrt{-1}/m)$. Let T_h denote the induced representation space $\text{Ind}_{\mathbf{Z}_m}^G(t^h)$ of the \mathbf{Z}_m -representation space t^h . Then T_h is a unitary q -dimensional G -representation space (for details see [9; § 47] or [17]). We put

$$V_n = T_{h_1} \oplus T_{h_2} \oplus \cdots \oplus T_{h_n},$$

where $(h_i, m)=1$ for $1 \leq i \leq n$.

EXAMPLE 9.1. If $n \geq 9$, then $\text{Span}_G(S(V_n)) = \rho(2n, \mathbf{R}) - 1$.

Here $\rho(s, \mathbf{R})$ denotes the largest integer k such that $s \equiv 0 \pmod{a_k(\mathbf{R})}$ ([1]).

Proof of Example 9.1. Since $\dim_{\mathbf{R}} V_n = 2nq$ and q is odd, $\text{Span}(S(V_n)) = \rho(2nq, \mathbf{R}) - 1 = \rho(2n, \mathbf{R}) - 1$. Thus we have

$$(9.1.1) \quad \text{Span}_G(S(V_n)) \leq \text{Span}(S(V_n)) = \rho(2n, \mathbf{R}) - 1.$$

By Becker [6; Theorems 1.1 and 2.2], there is a \mathbf{Z}_m -fiber homotopy equivalence

$$f_1: S(\eta_{\rho(2n, \mathbf{R})} \otimes_{\mathbf{R}} \underline{nt}) \rightarrow S(\underline{nt}).$$

By the same argument as in [5; II. Proposition 2.2], we have a G -fiber homotopy equivalence

$$f_2: S(\eta_{\rho(2n, \mathbf{R})} \otimes_{\mathbf{R}} \underline{nT_1}) \rightarrow S(\underline{nT_1}).$$

Since $n \geq 9$, we see that $\dim_{\mathbf{R}} nT_1^H \geq 2\rho(2n, \mathbf{R})$ if $nT_1^H \neq \{0\}$ for each $H < G$. Applying Theorem 1.2, we have $\text{Span}_G(S(nT_1)) \geq \rho(2n, \mathbf{R}) - 1$. It is easy to see that $\dim_{\mathbf{C}} V_n^H = \dim_{\mathbf{C}} nT_1^H$ for all $H < G$. Thus it follows from Theorem 1.1 that we have

$$(9.1.2) \quad \text{Span}_G(S(V_n)) \geq \rho(2n, \mathbf{R}) - 1.$$

Combining (9.1.1) and (9.1.2), we have $\text{Span}_G(S(V_n)) = \rho(2n, \mathbf{R}) - 1$. q.e.d.

Added in proof. Professor P. May kindly informed me that Dr. U. Namboodiri has obtained similar results [30].

References

- [1] J.F. Adams: *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.
- [2] ———: *On the groups J(X)-II*, Topology **3** (1965), 137–171.
- [3] M.F. Atiyah: *Thom complexes*, Proc. London Math. Soc. **11** (1961), 291–310.

- [4] ———, R. Bott and A. Shapiro: *Clifford modules*, *Topology* **3** (1964), Suppl. 1, 3–38.
- [5] ——— and D.O. Tall: *Group representations, λ -rings and the J -homomorphism*, *Topology* **8** (1969), 253–297.
- [6] J.C. Becker: *The Span of spherical space forms*, *Amer. J. Math.* **94** (1972), 991–1026.
- [7] E. Bierstone: *The equivariant covering homotopy property for differentiable G -fiber bundles*, *J. Differential Geom.* **8** (1973), 615–622.
- [8] G.E. Bredon: *Equivariant cohomology theories*, *Lecture Notes in Math.* 34, Springer-Verlag, 1967.
- [9] C.W. Curtis and I. Reiner: *Representation theory of finite groups and associative algebras*, Interscience, 1962.
- [10] T. tom Dieck: *Transformation groups and representation theory*, *Lecture Notes in Math.* 766, Springer-Verlag, 1979.
- [11] ——— and T. Petrie: *Geometric modules over the Burnside ring*, *Invent. Math.* **47** (1978), 273–287.
- [12] S. Illman: *Smooth equivariant triangulations of G -manifolds for G a finite group*, *Math. Ann.* **233** (1978), 199–220.
- [13] K. Iwata: *Span of lens spaces*, *Proc. Amer. Math. Soc.* **26** (1970), 687–688.
- [14] I.M. James: *The intrinsic join: A study of the homotopy groups of Stiefel manifolds*, *Proc. London Math. Soc.* **8** (1958), 507–535.
- [15] ———: *Cross-sections of Stiefel manifolds*, *Proc. London Math. Soc.* **8** (1958), 536–547.
- [16] ———: *Spaces associated with Stiefel manifolds*, *Proc. London Math. Soc.* **9** (1959), 115–140.
- [17] S. Kakutani: *On the groups $J_{Z_{m,q}}(*)$* , *Osaka J. Math.* **17** (1980), 513–524.
- [18] ——— and M. Murayama: *Equivariant duality, reducibility and coreducibility*, preprint 1981.
- [19] K. Kawakubo: *Compact Lie group actions and fiber homotopy type*, *J. Math. Soc. Japan* **33** (1981), 295–321.
- [20] K. Komiya and M. Morimoto: *Equivariant desuspension of G -maps*, *Osaka J. Math.* **18** (1981), 525–532.
- [21] C.N. Lee and A.G. Wasserman: *On the groups $JO(G)$* , *Memoirs of Amer. Math. Soc.* 159 (1975).
- [22] R.L. Rubinsztein: *On the equivariant homotopy of spheres*, *Dissertationes Math.* 134 (1976).
- [23] D. Sjerve: *Vector bundles over orbit manifolds*, *Trans. Amer. Math. Soc.* **138** (1969), 97–106.
- [24] E.H. Spanier: *Function spaces and duality*, *Ann. of Math.* **70** (1959), 338–378.
- [25] J. Tornehave: *Equivariant maps of spheres with conjugate orthogonal actions*, Aarhus Universitet preprint series No. 24 (1976/77).
- [26] S. Waner: *Equivariant homotopy theory and Milnor's theorem*, *Trans. Amer. Math. Soc.* **258** (1980), 351–368.
- [27] A.G. Wasserman: *Equivariant differential topology*, *Topology* **8** (1969), 127–150.
- [28] K. Wirthmüller: *Equivariant S -duality*, *Arch. Math.* **26** (1975), 427–431.

- [29] T. Yoshida: *A remark on vector fields on lens spaces*, J. Sci. Hiroshima Univ. Ser. A-I **31** (1967), 13–15.
- [30] U. Namboodiri: *Equivariant vector fields on spheres*, Dr. Phil. Thesis, Univ. of Chicago, 1982.

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