

# The Erdős–Ko–Rado Theorem for Integer Sequences

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## Abstract

For positive integers  $n, q, t$  we determine the maximum number of integer sequences  $(a_1, \dots, a_n)$  which satisfy  $1 \leq a_i \leq q$  for  $1 \leq i \leq n$ , and any two sequences agree in at least  $t$  positions. The result gives an affirmative answer to a conjecture of Frankl and Füredi.

## 1 Introduction

Let  $n, q, t$  be positive integers with  $q \geq 2$ ,  $n \geq t$ , and let  $[q] := \{1, 2, \dots, q\}$ . Then  $\mathcal{H} \subset [q]^n$  is a set of integer sequences  $(a_1, \dots, a_n)$ ,  $1 \leq a_i \leq q$ . We say that  $\mathcal{H}$  is  $t$ -intersecting if any two sequences intersects in at least  $t$  positions, more precisely,  $|\{i : a_i = a'_i\}| \geq t$  holds for all  $(a_1, \dots, a_n), (a'_1, \dots, a'_n) \in \mathcal{H}$ . In this paper, we determine the exact value of the following function.

$$f(n, q, t) := \max\{|\mathcal{H}| : \mathcal{H} \subset [q]^n, \mathcal{H} \text{ is } t\text{-intersecting}\}.$$

A family  $\mathcal{A} \subset 2^{[n]}$  is called  $t$ -intersecting if  $|A \cap A'| \geq t$  holds for all  $A, A' \in \mathcal{A}$ . Define a weighted size of  $\mathcal{A}$  by  $w(\mathcal{A}) := \sum_{A \in \mathcal{A}} (q-1)^{n-|A|}$ . Using a shifting technique, it is not difficult to check the following:

**Lemma 1 (Proposition 2 in [5])**  $f(n, q, t) = \max_{\mathcal{A}} w(\mathcal{A})$ , where  $\mathcal{A} \subset 2^{[n]}$  runs over all  $t$ -intersecting families.

If  $q = 2$  then  $w(\mathcal{A}) = |\mathcal{A}|$ . Thus,  $f(n, 2, t)$  is simply the maximal size of  $t$ -intersecting family  $\mathcal{A} \subset 2^{[n]}$ , which is given by the Katona Theorem. This case was solved by Kleitman [7].

Let us define a  $t$ -intersecting family  $\mathcal{A}_r \subset 2^{[n]}$  by

$$\mathcal{A}_r := \{A \subset [n] : |A \cap [t+2r]| \geq t+r\}.$$

In [5], Frankl and Füredi conjectured  $f(n, q, t) = \max_{r \geq 0} w(\mathcal{A}_r)$ . If  $q \geq t + 1$  then the conjecture claims  $f(n, q, t) = q^{n-t}$ . They showed that this is true if  $t \geq 15$ .

Now we introduce the full Erdős–Ko–Rado theorem, which was conjectured by Frankl in [4], and proved by Ahlswede and Khachatrian in [1]. Set

$$\text{AK}(n, k, t, r) := |\{B \in \binom{[n]}{k} : |B \cap [t + 2r]| \geq t + r\}|.$$

**Theorem 1 ([1])** *Let  $1 \leq t \leq k \leq n$  and  $\mathcal{B} \subset \binom{[n]}{k}$  be  $t$ -intersecting. If*

$$(k - t + 1)\left(2 + \frac{t - 1}{r + 1}\right) \leq n \leq (k - t + 1)\left(2 + \frac{t - 1}{r}\right)$$

for some  $r \in \mathbf{N}$ , then  $|\mathcal{B}| \leq \text{AK}(n, k, t, r)$ .

Using the above result, we prove the following in section 2.

**Theorem 2** *Let  $q \geq 3$  and set  $r := \lfloor \frac{t-1}{q-2} \rfloor$ . Then  $f(n, q, t) = w(\mathcal{A}_r)$  for  $n \geq t + 2r$ .*

Note that

$$\begin{aligned} w(\mathcal{A}_r) &= \sum_{j=0}^{n-t-2r} \sum_{i=t+r}^{t+2r} \binom{t+2r}{i} \binom{n-t-2r}{j} (q-1)^{n-i-j} \\ &= \sum_{j=0}^{n-t-2r} \binom{n-t-2r}{j} (q-1)^{n-t-2r-j} \sum_{i=t+r}^{t+2r} \binom{t+2r}{i} (q-1)^{t+2r-i} \\ &= q^{n-t-2r} \sum_{i=0}^r \binom{t+2r}{i} (q-1)^i. \end{aligned} \tag{1}$$

In section 3, we prove the case  $q \geq t + 1$  (and  $t \geq 1$ ) directly.

Independently, Ahlswede and Khachatrian [2] obtained Theorem 2 as a diametric theorem in Hamming spaces. They used a different method. See [6] or [2] for the history of the problem.

## 2 Proof of the theorem

Throughout this section, we fix  $q$  and  $t$  and set

$$r := \lfloor \frac{t-1}{q-2} \rfloor = \frac{t-1}{q-2} - \delta.$$

Let us recall the following easy probabilistic result.

**Lemma 2 (Proposition 3 in [5])** *For every  $\epsilon > 0$  the number of sequences  $(a_1, \dots, a_n) \in [q]^n$  which contain more than  $(1 + \epsilon)(n/q)$  1's or less than  $(1 - \epsilon)(n/q)$  1's is less than  $\epsilon q^n$  for  $n > n_0(\epsilon)$ .*

Choose any sufficiently small positive  $\epsilon$ , i.e.,  $0 < \epsilon < \epsilon_0(q, t)$ , and set an open interval  $I := ((1 - \epsilon)(n/q), (1 + \epsilon)(n/q))$ . In view of Lemma 1,  $f(n, q, t)q^{-n} = w(\mathcal{A})q^{-n}$  for some  $t$ -intersecting family  $\mathcal{A}$ . Moreover Lemma 2 gives that

$$f(n, q, t)q^{-n} < w(\mathcal{B})q^{-n} + \epsilon$$

where  $\mathcal{B} := \{B \in \mathcal{A} : |B| \in I\}$ . Set  $\mathcal{B}(k) := \{B \in \mathcal{B} : |B| = k\}$ .

**Case I**  $0 < \delta < 1$ .

Note that  $\delta$  depends only on  $t$  and  $q$ .

**Lemma 3** For  $k \in I$  and sufficiently large  $n$ ,

$$(k - t + 1)\left(2 + \frac{t - 1}{r + 1}\right) \leq n \leq (k - t + 1)\left(2 + \frac{t - 1}{r}\right) \quad (2)$$

**Proof** (2) is equivalent to

$$(2 + (t - 1)/r)^{-1}n + t - 1 \leq k \leq (2 + (t - 1)/(r + 1))^{-1}n + t - 1 \quad (3)$$

Let us show the right half. Since  $k < (1 + \epsilon)(n/q)$ , it is sufficient to show

$$(1 + \epsilon)(n/q) \leq (2 + (t - 1)/(r + 1))^{-1}n + t - 1$$

or

$$(1 + \epsilon)(2 + (t - 1)/(r + 1)) < q.$$

This follows from  $q = 2 + (t - 1)/(r + \delta) > 2 + (t - 1)/(r + 1)$  and  $\epsilon < \epsilon_0(q, t)$ . One can prove the left half of (3) similarly.  $\blacksquare$

Thus, by the Ahlswede–Khachatryan theorem we have  $|\mathcal{B}(k)| \leq \text{AK}(n, k, t, r)$ . Therefore,

$$\begin{aligned} f(n, q, t)q^{-n} &< q^{-n} \sum_{k \in I} w(\mathcal{B}(k)) + \epsilon \\ &\leq q^{-n} \sum_{k \in I} \text{AK}(n, k, t, r)(q - 1)^{n-k} + \epsilon \\ &= q^{-n} \sum_{k \in I} \sum_{j=t+r}^{t+2r} \binom{t+2r}{j} \binom{n-t-2r}{k-j} (q - 1)^{n-k} + \epsilon \\ &< q^{-n} \sum_{j=t+r}^{t+2r} \binom{t+2r}{j} \sum_{k=j}^{n-t-2r+j} \binom{n-t-2r}{k-j} (q - 1)^{n-k} + \epsilon \\ &= q^{-n} \sum_{j=t+r}^{t+2r} \binom{t+2r}{j} \sum_{i=0}^{n-t-2r} \binom{n-t-2r}{i} (q - 1)^{(n-t-2r)-i} (q - 1)^{t+2r-j} + \epsilon \\ &= q^{-n} \sum_{j=t+r}^{t+2r} \binom{t+2r}{j} q^{n-t-2r} (q - 1)^{t+2r-j} + \epsilon \\ &= q^{-t-2r} \sum_{i=0}^r \binom{t+2r}{i} (q - 1)^i + \epsilon. \end{aligned}$$

Hence we have

$$g(q, t) := \lim_{n \rightarrow \infty} f(n, q, t)q^{-n} \leq q^{-t-2r} \sum_{i=0}^r \binom{t+2r}{i} (q - 1)^i. \quad (4)$$

On the other hand, (1) implies

$$g(q, t) \geq q^{-t-2r} \sum_{i=0}^r \binom{t+2r}{i} (q - 1)^i. \quad (5)$$

By (4) and (5), we finally have

$$g(q, t) = q^{-t-2r} \sum_{i=0}^r \binom{t+2r}{i} (q-1)^i.$$

Now suppose that for some  $t$ -intersecting family  $\mathcal{A} \subset 2^{[n]}$  we have  $w(\mathcal{A}) \geq q^n g(q, t) + 1$ . Since  $f(n+1, q, t) \geq qf(n, q, t)$  we have

$$f(n', q, t) \geq q^{n'-n} f(n, q, t) \geq q^{n'-n} w(\mathcal{A}) \geq q^{n'} (g(q, t) + q^{-n}),$$

which implies  $\lim_{n' \rightarrow \infty} f(n', q, t) q^{-n'} \geq g(q, t) + q^{-n} > g(q, t)$ , a contradiction. Thus we must have  $w(\mathcal{A}) \leq q^n g(q, t)$ , and actually  $w(\mathcal{A}_r) = q^n g(q, t)$ . (We need  $n \geq t+2r$  here.) This completes the proof of Case I.

**Case II**  $\delta = 0$ .

In this case, we have  $q = 2 + \frac{t-1}{r}$ .

**Lemma 4** For  $k \in I$  and sufficiently large  $n$ ,

$$(k-t+1)\left(2 + \frac{t-1}{r+1}\right) \leq n \leq (k-t+1)\left(2 + \frac{t-1}{r-1}\right).$$

In fact, one can prove

$$\left(2 + \frac{t-1}{r-1}\right)^{-1} n + t - 1 \leq (1-\epsilon) \frac{n}{q} < \frac{n}{q} + t - 1 < (1+\epsilon) \frac{n}{q} \leq \left(2 + \frac{t-1}{r+1}\right)^{-1} n + t - 1.$$

The proof is similar to the proof of Lemma 3 and we omit it. By this lemma, we have

$$|\mathcal{B}(k)| \leq \max\{\text{AK}(n, k, t, r), \text{AK}(n, k, t, r-1)\}.$$

If  $n = q(k-t+1)$  then  $\text{AK}(n, k, t, r) = \text{AK}(n, k, t, r-1)$ . Since

$$\begin{aligned} \text{AK}(n, k, t, r) &= \sum_{j=0}^r \binom{t+2r}{t+r+j} \binom{n-t-2r}{k-t-r-j} \\ &= \binom{n-t-2r}{k-t-r} \sum_{j=0}^r \binom{t+2r}{t+r+j} \prod_{i=1}^j \frac{k-t-r-i+1}{n-k-r+i}, \end{aligned}$$

we have

$$1 = \frac{\text{AK}(n, k, t, r-1)}{\text{AK}(n, k, t, r)} = \frac{(n-t-2r+2)(n-t-2r+1)}{(k-t-r+1)(n-k-r+1)} \frac{\sum_{j=0}^{r-1} \binom{t+2r-2}{t+r+j-1} \prod_{i=1}^j \frac{k-t-r-i+2}{n-k-r+i+1}}{\sum_{j=0}^r \binom{t+2r}{t+r+j} \prod_{i=1}^j \frac{k-t-r-i+1}{n-k-r+i}}.$$

The above ratio tends to

$$\frac{q^2}{(q-1)} \frac{\sum_{j=0}^{r-1} \binom{t+2r-2}{t+r+j-1} (q-1)^{-j}}{\sum_{j=0}^r \binom{t+2r}{t+r+j} (q-1)^{-j}} = \frac{q^2}{(q-1)} \frac{\sum_{i=1}^r \binom{t+2r-2}{i-1} (q-1)^i}{\sum_{i=0}^r \binom{t+2r}{i} (q-1)^i}$$

as  $n \rightarrow \infty$  for fixed  $q, t$  and  $n = q(k - t + 1)$ . This proves

$$q^2 \sum_{i=1}^r \binom{t+2r-2}{i-1} (q-1)^i = (q-1) \sum_{i=0}^r \binom{t+2r}{i} (q-1)^i \quad (6)$$

Now choose  $k \in I$ . (Here we do not assume  $n = q(k - t + 1)$ .) Then,

$$\begin{aligned} \frac{\text{AK}(n, k, t, r-1)}{\text{AK}(n, k, t, r)} &= \frac{(n-t-2r+2)(n-t-2r+1) \sum_{j=0}^{r-1} \binom{t+2r-2}{t+r+j-1} \prod_{i=1}^j \frac{k-t-r-i+2}{n-k-r+i+1}}{(k-t-r+1)(n-k-r+1) \sum_{j=0}^r \binom{t+2r}{t+r+j} \prod_{i=1}^j \frac{k-t-r-i+1}{n-k-r+i}} \\ &< \frac{n^2}{(1-\epsilon)(n/q)(1-(1+\epsilon)/q)n} \frac{\sum_{j=0}^{r-1} \binom{t+2r-2}{t+r+j-1} \prod_{i=1}^j \frac{(1+\epsilon)(n/q)}{(1-(1+\epsilon)/q)n}}{\sum_{j=0}^r \binom{t+2r}{t+r+j} \prod_{i=1}^j \frac{(1-\epsilon)(n/q)}{(1-(1-\epsilon)/q)n}} \\ &= \frac{q^2}{(1-\epsilon)(q-1-\epsilon)} \frac{\sum_{i=1}^r \binom{t+2r-2}{i-1} \left(\frac{q-1-\epsilon}{1+\epsilon}\right)^i}{\sum_{i=0}^r \binom{t+2r}{i} \left(\frac{q-1+\epsilon}{1-\epsilon}\right)^i}. \end{aligned}$$

By (6), the above ratio tends to 1 as  $\epsilon \rightarrow 0$ . Thus for any  $\epsilon' > 0$  we can conclude that

$$\text{AK}(n, k, t, r-1) < (1+\epsilon') \text{AK}(n, k, t, r)$$

if we choose  $\epsilon$  sufficiently small and  $n$  sufficiently large, and  $k \in I$ . Finally we have

$$\begin{aligned} f(n, q, t) q^{-n} &< q^{-n} \sum_{k \in I} \max\{\text{AK}(n, k, t, r), \text{AK}(n, k, t, r-1)\} (q-1)^{n-k} + \epsilon \\ &< (1+\epsilon') q^{-n} \sum_{k \in I} \text{AK}(n, k, t, r) (q-1)^{n-k} + \epsilon \\ &< (1+\epsilon') q^{-t-2r} \sum_{i=0}^r \binom{t+2r}{i} (q-1)^i + \epsilon. \end{aligned}$$

Using the same argument in Case I, we have

$$g(q, t) := \lim_{n \rightarrow \infty} f(n, q, t) q^{-n} = q^{-t-2r} \sum_{i=0}^r \binom{t+2r}{i} (q-1)^i,$$

and  $f(n, q, t) = q^n g(q, t)$ , which completes the proof of the theorem.  $\blacksquare$

### 3 Another approach

In this section we give a direct proof for the case  $q \geq t+1$  using tools developed in [1].

Let  $\mathcal{A} \subset 2^{[n]}$ . A family  $\mathcal{G} \subset 2^{[n]}$  is called a kernel of  $\mathcal{A}$  if  $\mathcal{A} = \bigcup_{G \in \mathcal{G}} \mathcal{U}(G)$  where  $\mathcal{U}(G) := \{F \subset [n] : G \subset F\}$ . A rank of  $\mathcal{A}$  is defined by

$$\text{rank}(\mathcal{A}) := \min\{|\bigcup_{G \in \mathcal{G}} G| : \mathcal{G} \text{ is a kernel of } \mathcal{A}\}.$$

**Theorem 3** *Let  $\mathcal{A} \subset 2^{[n]}$  be a shifted  $t$ -intersecting family with  $w(\mathcal{A}) = f(n, q, t)$ . Then  $\text{rank}(\mathcal{A}) \leq t+2r$ , where  $r := \lfloor \frac{t-1}{q-2} \rfloor$ .*

Since the proof is almost the same as the proof of Lemma 6 in [1], we omit the details.

**Proof** (Outline) Choose a shifted, inclusion minimal (i.e., antichain) kernel  $\mathcal{G} \subset 2^{[n]}$  of  $\mathcal{A}$  satisfying  $\text{rank}(\mathcal{A}) = |\bigcup_{G \in \mathcal{G}} G|$ . Assume that  $\delta > 0$  and  $M := t + 2r + \delta = \text{rank}(\mathcal{A})$ . Let  $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$ ,  $\mathcal{G}_0 := \{G \in \mathcal{G} : M \in G\}$ ,  $\mathcal{G}_1 := \mathcal{G} - \mathcal{G}_0$ , and let

$$\mathcal{G}_0 = \mathcal{R}_{t+1} \cup \cdots \cup \mathcal{R}_{M-1},$$

where  $\mathcal{R}_i := \mathcal{G}_0 \cap \binom{[M]}{i}$ . Set

$$\mathcal{R}'_i := \{E - \{M\} : E \in \mathcal{R}_i\} \subset \binom{[M-1]}{i-1}.$$

Then,  $E \in \mathcal{R}'_i$ ,  $E' \in \mathcal{R}'_j$  and  $i + j \neq M + t$  imply  $|E \cap E'| \geq t$ . Thus we may assume that  $\mathcal{R}_i \neq \emptyset$ ,  $\mathcal{R}_j \neq \emptyset$ ,  $i + j = M + t$  for some  $i, j$ .

**Case I**  $i \neq j$ .

Define

$$\begin{aligned} \mathcal{F}_1 &:= \mathcal{G}_1 \cup (\mathcal{G}_0 - (\mathcal{R}_i \cup \mathcal{R}_j)) \cup \mathcal{R}'_i, \\ \mathcal{F}_2 &:= \mathcal{G}_1 \cup (\mathcal{G}_0 - (\mathcal{R}_i \cup \mathcal{R}_j)) \cup \mathcal{R}'_j, \\ \mathcal{B}_i &:= \mathcal{U}(\mathcal{F}_i). \end{aligned}$$

Then we have

$$\begin{aligned} \mathcal{A} - \mathcal{B}_1 &= \{R \cup S : R \in \mathcal{R}_j, S \in 2^{[M+1, n]}\}, \\ \mathcal{B}_1 - \mathcal{A} &= \{R \cup S : R \in \mathcal{R}'_i, S \in 2^{[M+1, n]}\}, \end{aligned}$$

and hence

$$\begin{aligned} w(\mathcal{A} - \mathcal{B}_1) &= |\mathcal{R}_j|(q-1)^{M-j}q^{n-M}, \\ w(\mathcal{B}_1 - \mathcal{A}) &= |\mathcal{R}'_i|(q-1)^{M-i+1}q^{n-M}. \end{aligned}$$

If  $w(\mathcal{A}) \geq w(\mathcal{B}_1)$  and  $w(\mathcal{A}) \geq w(\mathcal{B}_2)$  then

$$\begin{aligned} |\mathcal{R}_j|(q-1)^{M-j} &\geq |\mathcal{R}'_i|(q-1)^{M-i+1}, \\ |\mathcal{R}'_i|(q-1)^{M-i} &\geq |\mathcal{R}_j|(q-1)^{M-j+1}. \end{aligned}$$

Thus  $1 \geq (q-1)^2$ , a contradiction.

**Case II**  $i = j = \frac{M+t}{2} = t + r + \frac{\delta}{2}$ .

In this case  $\delta$  is even and  $\delta \geq 2$ . Using the same argument in Case I, we may assume that  $\mathcal{R}_\alpha = \emptyset$  for all  $\alpha \neq i$ , and  $\mathcal{G} = \mathcal{R}_i \cup \mathcal{G}_1$ . The average degree  $\bar{d}$  of  $\mathcal{R}'_i \subset \binom{[M-1]}{i-1}$  is given by  $\bar{d} = (i-1)|\mathcal{R}'_i|/(M-1)$ . Therefore we can find  $\ell \in [M-1]$  such that  $\deg_{\mathcal{R}'_i}(\ell) \leq \bar{d}$ . Define a  $t$ -intersecting family  $\mathcal{T}$  as follows:

$$\mathcal{T} := \{E \in \mathcal{R}'_i : \ell \notin E\} \subset \binom{[M-1] - \{\ell\}}{i-1}.$$

Then  $|\mathcal{T}| \geq |\mathcal{R}'_i| - \bar{d} = \frac{M-i}{M-1}|\mathcal{R}_i|$ . Let  $\mathcal{A} = \mathcal{D}_1 \cup \mathcal{D}_2$  where  $\mathcal{D}_1 := \mathcal{U}(\mathcal{G}_1)$ ,  $\mathcal{D}_2 := \mathcal{U}(\mathcal{R}_i) - \mathcal{D}_1$ , and let  $\mathcal{U}(\mathcal{T} \cup \mathcal{G}_1) = \mathcal{D}_1 \cup \mathcal{D}_3$  where  $\mathcal{D}_3 := \mathcal{U}(\mathcal{T}) - \mathcal{D}_1$ . Then we have

$$\begin{aligned} w(\mathcal{D}_2) &= |\mathcal{R}_i|(q-1)^{M-i}q^{n-M}, \\ w(\mathcal{D}_3) &= |\mathcal{T}|(q-1)^{M-i}q^{n-M+1} \geq \frac{M-i}{M-1}|\mathcal{R}_i|(q-1)^{M-i}q^{n-M+1}. \end{aligned}$$

If  $w(\mathcal{D}_2) \geq w(\mathcal{D}_3)$  then  $1 \geq \frac{M-i}{M-1} \cdot q$ . Since  $M = t + 2r + \delta$  and  $i = t + r + \frac{\delta}{2}$ , we have

$$t + 2r + \delta - 1 \geq \frac{2r + \delta}{2}q,$$

or equivalently,

$$r \leq \frac{t-1-(q/2-1)\delta}{q-2} = \frac{t-1}{q-2} - \frac{\delta}{2}.$$

Since  $\frac{\delta}{2} \geq 1$  we have  $r \leq \frac{t-1}{q-2} - 1$ , which contradicts a definition of  $r$ . ■

**Corollary 1** *If  $q \geq t + 1$  then  $f(n, q, t) = q^{n-t}$ .*

**Proof** Suppose that  $\mathcal{A} \subset 2^{[n]}$  is  $t$ -intersecting and  $w(\mathcal{A}) = f(n, q, t)$ . By Theorem 3, we may assume  $\text{rank}(\mathcal{A}) \leq t + 2r$ ,  $r := \lfloor \frac{t-1}{q-2} \rfloor$ . If  $q \geq t + 2$  then  $r = 0$ , and  $f(n, q, t) \leq w(\mathcal{A}_0) = q^{n-t}$ .

If  $q = t + 1$  then  $r = 1$  and  $f(n, q, t) \leq \max\{w(\mathcal{A}_0), w(\mathcal{A}_1)\}$ . In this case we have  $w(\mathcal{A}_0) = w(\mathcal{A}_1) = q^{n-t}$ . ■

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