The Ergodic Theory of Subadditive Stochastic Processes

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[Received October 1967. Revised February 1968]

SUMMARY

An ergodic theory is developed for the subadditive processes introduced by Hammersley and Welsh (1965) in their study of percolation theory. This is a complete generalization of the classical law of large numbers for stationary sequences.

1. SUBADDITIVE PROCESSES

IN an important paper Hammersley and Welsh (1965) introduced the concept of a subadditive stochastic process, and they have shown how such processes arise naturally in various contexts, but particularly in the study of random flows in lattices. They have shown that one may expect these processes to exhibit a certain ergodic behaviour, and have taken the first steps towards the construction of an ergodic theory like the classical one for averages of stationary sequences.

If T is any subset of the real line, a *subadditive process* x on T is a collection of (real) random variables $x_{st}(s, t \in T, s < t)$ with the property that

$$x_{st} \leqslant x_{su} + x_{ut} \tag{1}$$

for all s, t, u in T with $s < u < t$. In this paper T will be taken as the set of non-negative integers, although interesting problems arise when T is the interval $(0, \infty)$. It will be convenient to adopt the convention that $x_{tt} = 0$ for all t in T.

If (1) is replaced by

$$x_{st} \geqslant x_{su} + x_{ut}, \tag{2}$$

then x is said to be *superadditive*. Since x is superadditive if and only if $-x$ is subadditive, any theorem about subadditive processes translates at once into a corresponding result about superadditive processes.

A process which is both subadditive and superadditive satisfies

$$x_{st} = x_{su} + x_{ut} \tag{3}$$

and is described as *additive*. If, for an additive process x, we write

$$y_t = x_{t-1,t},$$

repeated application of (3) gives

$$x_{st} = y_{s+1} + y_{s+2} + \ldots + y_t, \tag{4}$$

so that additive processes are just partial sums of sequences of random variables.

18

Now the strong law of large numbers for stationary sequences (for which see, for example, Doob, 1953, Section X.2) states that, if the sequence $\{y_t\}$ is stationary and has finite expectation, then the finite limit

$$\lim_{t\to\infty} t^{-1}(y_1+y_2+\ldots+y_t) = \lim_{t\to\infty} x_{0t}/t \tag{5}$$

exists with probability one. Thus, under a stationarity assumption, the additive process x has the ergodic property that x_{0t}/t converges with probability one as $t\to\infty$. The problem is to extend this result to subadditive (and so also to superadditive) processes.

For an additive process the condition that $\{y_t\}$ be stationary is equivalent to the condition that the finite-dimensional distributions of x be invariant under the shift $x_{st}\to x_{s+1,t+1}$. This makes it natural to define stationarity for a subadditive process as follows:

Definition. A subadditive (or superadditive) process x is said to be *stationary* if its finite-dimensional distributions are the same as those of the shifted process x' defined by

$$x'_{st} = x_{s+1,t+1}. \tag{6}$$

It is to be noted that this definition is stronger than that of Hammersley and Welsh, who require only that the one-point distributions of x be the same as those of x'. For additive processes, this condition is insufficient to imply the stationarity of y_t, and their ergodic results are therefore new even in the additive case. It is possible to construct examples of processes stationary in their sense but not in ours, but these are highly artificial. All the examples quoted by Hammersley and Welsh are stationary in the stronger sense, which will be used without further comment.

Let x be a stationary subadditive process, and suppose that each random variable x_{st} has finite expectation. Because of stationarity, this can depend only on $(t-s)$;

$$E(x_{st}) = g_{t-s}. \tag{7}$$

From (1)

$$g_{\alpha+\beta}\leqslant g_\alpha+g_\beta \quad (\alpha,\beta\geqslant 1), \tag{8}$$

and a standard result in the theory of subadditive functions (Hammersley and Welsh, 1965, p. 68) shows that

$$g_t/t\to\gamma \tag{9}$$

as $t\to\infty$, where

$$\gamma = \inf_{t\geqslant 1} g_t/t \tag{10}$$

satisfies $-\infty\leqslant\gamma<\infty$. When γ is finite, it is called the *time constant* of the process x. Thus x possesses a time constant if and only if each x_{st} has a finite expectation g_{t-s} and g_t/t is bounded below.

Hammersley and Welsh are able to prove, under very stringent conditions, that x_{0t}/t converges to γ as $t\to\infty$. The main conclusion of this paper is that, without any further conditions than stationarity and the existence of γ, x_{0t}/t converges, with probability one and in mean, to a random variable ξ with expectation γ. This result, when restricted to additive processes, becomes the classical Birkhoff–von Neumann ergodic theorem, of which it is a complete generalization. In most cases (see Section 4 below) it is possible to show that ξ is actually equal to γ, although the trivial case $x_{st} = (t-s)\,X$ (where X is any random variable) shows that this cannot always be true.

2. THE EASY ERGODIC THEOREM

Almost the whole of the result asserted at the end of the last section can be proved as a fairly easy consequence of the known theorems for additive processes.

Theorem 1. Let x be a stationary subadditive process with time constant γ, and let

$$\xi = \limsup_{t\to\infty} x_{0t}/t; \tag{11}$$

then ξ is almost certainly finite, and

$$E(\xi) = \gamma. \tag{12}$$

As $t\to\infty$,

$$E\left|\frac{x_{0t}}{t} - \xi\right| \to 0, \tag{13}$$

and therefore x_{0t}/t converges to ξ in probability.

Proof. For any fixed n, write $N(t)$ for the integral part of t/n, and $\nu(t) = t - N(t)\,n$. Repeated application of (1) gives

$$\begin{aligned} x_{0t} &\leqslant \sum_{r=1}^{N(t)} x_{(r-1)n,rn} + x_{N(t)n,t} \\ &\leqslant \sum_{r=1}^{N(t)} x_{(r-1)n,rn} + w_{N(t)}, \end{aligned} \tag{14}$$

where

$$w_N = \sum_{u=0}^{n-1} |x_{Nn,Nn+u}|.$$

Now the sequence $\{x_{(r-1)n,rn};\ r = 1,2,\ldots\}$ is stationary with finite expectation g_n, so that the strong law of large numbers shows that the limit

$$z_n = \lim_{N\to\infty} N^{-1} \sum_{r=1}^{N} x_{(r-1)n,rn}$$

exists with probability one, and that $E(z_n) = g_n$.
Moreover, for each $\epsilon > 0$,

$$\sum_{N=1}^{\infty} P(w_N/N \geqslant \epsilon) = \sum_{N=1}^{\infty} P(w_0 \geqslant N\epsilon) \leqslant \epsilon^{-1} E(w_0) < \infty,$$

and the Borel–Cantelli lemma thus shows that, with probability one,

$$\lim_{N\to\infty} w_N/N = 0.$$

The inequality (14) therefore gives

$$\begin{aligned} \xi &\leqslant \limsup_{t\to\infty} x_{0t}/N(t)\,n \\ &\leqslant \limsup_{N\to\infty} (Nn)^{-1} \left\{ \sum_{r=1}^{N} x_{(r-1)n,rn} + w_N \right\} \\ &= z_n/n. \end{aligned}$$

In particular, $\xi < \infty$ and $E(\xi) \leqslant g_n/n$.

Since this holds for all n, we have

$$E(\xi) \leqslant \gamma. \tag{15}$$

If we write

$$a_{st} = \sum_{r=s+1}^{t} x_{r-1,r},$$

then a is an additive process and $x_{st} \leqslant a_{st}$. Hence the process b defined by

$$b_{st} = a_{st} - x_{st}$$

is a non-negative superadditive process. If

$$B_n = \inf_{t \geqslant n} b_{0t}/t,$$

then B_n is an increasing sequence, converging as $n \to \infty$ to

$$\begin{aligned} \liminf_{t\to\infty} b_{0t}/t &= \liminf_{t\to\infty} t^{-1}(a_{0t} - x_{0t}) \\ &= z_1 - \xi. \end{aligned}$$

By monotone convergence,

$$\lim_{n\to\infty} E(B_n) = E(\lim_{n\to\infty} B_n) = E(z_1 - \xi) = g_1 - E(\xi).$$

But also

$$\lim_{n\to\infty} E(B_n) \leqslant \lim_{n\to\infty} E(b_{0n}/n) = \lim_{n\to\infty} (ng_1 - g_n)/n = g_1 - \gamma.$$

Hence

$$g_1 - E(\xi) \leqslant g_1 - \gamma,$$

which with (15) implies that $E(\xi) = \gamma$, proving (12).

It now follows that

$$E|t^{-1} b_{0t} - B_t| = E(t^{-1} b_{0t} - B_t) = (g_1 - t^{-1} g_t) - E(B_t) \to (g_1 - \gamma) - (g_1 - \gamma) = 0,$$

so that $t^{-1} b_{0t} - B_t \to 0$ in mean as $t \to \infty$. Since B_t is monotone and converges to $(z_1 - \xi)$, it converges in mean, and so

$$t^{-1} b_{0t} \to z_1 - \xi$$

in mean. But, by the ergodic theorem,

$$t^{-1} a_{0t} \to z_1$$

in mean, and so

$$t^{-1} x_{0t} = t^{-1}(a_{0t} - b_{0t}) \to \xi$$

in mean, and so also in probability. The proof is therefore complete.

All that is missing in Theorem 1 to make it a full generalization of the known result for additive processes is the fact that

$$\xi = \liminf_{t\to\infty} x_{0t}/t$$

with probability one. This is a much deeper result and will be accomplished with difficulty in Sections 5 and 6.

3. THE MAXIMAL ERGODIC THEOREM

The usual approach to the classical ergodic theorem proceeds by way of a "maximal ergodic theorem". Such a result exists for subadditive processes and is presented here for its own interest, although it can also be used to give an alternative, and more direct, proof of Theorem 1.

Theorem 2. Let x be a stationary subadditive process with time constant γ, and suppose that the event

$$A = \{x_{0t} \geqslant 0 \text{ for some } t \geqslant 1\}$$

has positive probability. Then

$$E(x_{01} \mid A) \geqslant 0. \tag{16}$$

Proof. If

$$m_t = \max_{1 \leqslant s \leqslant t} x_{0s},$$

then

$$\begin{aligned} m_t \leqslant m_{t+1} &= \max_{0 \leqslant s \leqslant t} x_{0,s+1} \\ &\leqslant \max_{0 \leqslant s \leqslant t} (x_{01} + x_{1,s+1}) \\ &= x_{01} + \max_{0 \leqslant s \leqslant t} x'_{0s} \\ &= x_{01} + \max(0, m'_t), \end{aligned}$$

where the primes refer to corresponding quantities for the shifted process x' defined by (6). Hence, if A_t is the event $\{m_t \geqslant 0\}$,

$$\begin{aligned} \int_{A_t} m_t \, dP &\leqslant \int_{A_t} x_{01} \, dP + \int_{A_t} \max(0, m'_t) \, dP \\ &\leqslant \int_{A_t} x_{01} \, dP + \int_{\Omega} \max(0, m'_t) \, dP \\ &= \int_{A_t} x_{01} \, dP + \int_{\Omega} \max(0, m_t) \, dP \\ &= \int_{A_t} x_{01} \, dP + \int_{A_t} m_t \, dP. \end{aligned}$$

Hence

$$\int_{A_t} x_{01} \, dP \geqslant 0,$$

and letting $t \to \infty$,

$$\int_{A} x_{01} \, dP \geqslant 0,$$

which proves (16).

The reader will note that this proof follows exactly the beautifully simple argument given by Garsia (1965) for the usual maximal ergodic theorem. Unfortunately it fails for superadditive processes, so that the usual procedure for deducing the pointwise ergodic theorem is not available.

4. THE RANDOM VARIABLE ξ

In the classical theorem for additive processes, the limit ξ is expressed as a conditional expectation relative to the σ-field of invariant events (Doob, 1953, Section X.2). In the subadditive case a similar result holds; ξ is a "conditional time constant".

Theorem 3. Let x be a stationary subadditive process with time constant γ, and let $\mathscr{I}$ be the σ-field of events defined in terms of x and invariant under the shift $x \to x'$. Then the limit ξ of Theorem 1 can be written in the form

$$\xi = \lim_{t\to\infty} t^{-1} E(x_{0t} | \mathscr{I}). \tag{17}$$

In particular, if $\mathscr{I}$ is trivial (contains only events of probability 0 and 1), then

$$\xi = \gamma. \tag{18}$$

Proof. Using primes as before to denote quantities defined with respect to the shifted process x', we have

$$\xi' = \limsup_{t\to\infty} x_{1t}/t \geqslant \limsup_{t\to\infty} t^{-1}(x_{0t} - x_{01}) = \xi.$$

But

$$E(\xi') = \gamma = E(\xi),$$

and therefore

$$P(\xi' = \xi) = 1.$$

Thus ξ is an invariant random variable, and so measurable with respect to $\mathscr{I}$.

Let Φ_t be a version of the conditional probability $E(x_{0t} | \mathscr{I})$. By stationarity,

$$E(x_{st} | \mathscr{I}) = \Phi_{t-s},$$

and (1) shows that

$$\Phi_{\alpha+\beta} \leqslant \Phi_\alpha + \Phi_\beta$$

for all $\alpha, \beta \geqslant 1$ with probability one. Hence

$$\phi = \lim_{t\to\infty} \Phi_t/t$$

exists almost surely. We have to show that $\xi = \phi$ with probability one.

Let $I \in \mathscr{I}$ have positive probability. Conditional upon I, x is subadditive and stationary. For any t,

$$\Phi_{2t} \leqslant \Phi_t + \Phi_t,$$

so that Φ_t/t decreases as $t \to \infty$ through the powers of 2. Hence, with $t = 2^k$,

$$\begin{aligned} E(\phi | I) &= \lim_{k\to\infty} E(t^{-1}\Phi_t | I) \\ &= \lim_{k\to\infty} E(t^{-1} x_{0t} | I) \\ &= E(\xi | I), \end{aligned}$$

since $x_{0t}/t \to \xi$ in mean conditional on I by Theorem 1. Thus

$$\int_I \phi \, dP = \int_I \xi dP$$

for all $I \in \mathscr{I}$. Since both ϕ and ξ are $\mathscr{I}$-measurable, we have $\phi = \xi$ with probability one, and (17) is proved.

In particular, if $\mathscr{I}$ is trivial, (17) becomes

$$\xi = \lim_{t\to\infty} t^{-1} E(x_{0t}) = \gamma,$$

and the proof is complete.

Since ξ is invariant it follows at once that, for every fixed s,

$$x_{st}/t \to \xi$$

in mean as $t \to \infty$.

The theorem is of course mainly useful when one can show directly that $\mathscr{I}$ is trivial. This is so, for instance, in the problems considered by Hammersley and Welsh, in which the process is defined in terms of independent random variables to which the zero–one law can be applied. More precisely, their processes are defined by equations of the form

$$x_{st} = F_{t-s}(\ldots, u_{s-1}, u_s, u_{s+1}, \ldots), \tag{19}$$

where the functions F are fixed, and the u_s are independent collections of random variables with the same distributions. The σ-field $\mathscr{I}$ is contained in the σ-field $\mathscr{I}'$ of events defined in terms of the u_s and invariant under the shift

$$u_s \to u_{s+1}.$$

Since (Doob, 1953, p. 460) $\mathscr{I}'$ is known to be trivial, $\mathscr{I}$ is trivial, and therefore $\xi = \gamma$.

5. DECOMPOSITION OF A SUBADDITIVE PROCESS

Definition. A stationary subadditive process x is said to be *purely subadditive* if it is non-negative and has time constant $\gamma = 0$.

Theorem 4. A stationary subadditive process x with time constant γ admits a decomposition of the form

$$x_{st} = y_{st} + z_{st}, \tag{20}$$

where y is stationary and additive, with

$$E(y_{01}) = \gamma \tag{21}$$

and z is stationary and purely subadditive.

The proof is difficult and seems to require more powerful tools than do the other results of this paper; it is deferred to the next section. The theorem is important because it reduces the theory of subadditive processes to that of purely subadditive processes, which is rather simpler. For instance, Theorem 1 shows that the non-negative random variable

$$\zeta = \limsup_{t\to\infty} z_{0t}/t$$

has

$$E(\zeta) = 0,$$

so that $\zeta = 0$ with probability one. Hence, with probability one,

$$\lim_{t\to\infty} z_{0t}/t = 0.$$

We already know from the classical theory that

$$\lim_{t\to\infty} y_{0t}/t$$

exists, so that (20) establishes the existence, with probability one, of the limit

$$\lim_{t\to\infty} x_{0t}/t,$$

which must of course be the random variable ξ. We can therefore fill the gap in Theorem 1.

Theorem 5. If x is a stationary subadditive process with time constant γ, then the limit

$$\xi = \lim_{t\to\infty} x_{0t}/t$$

exists with probability one, and satisfies the conditions of Theorems 1 and 3.

This result establishes several of the conjectures in Hammersley and Welsh (1965). In particular, it shows that these authors were correct in surmising that convergence in probability of x_{0t}/t implies convergence with probability one. It will be noted that the treatment given here removes the necessity of the technique of "smothering with blankets" which Hammersley and Welsh wield with such ingenuity.

It should not be supposed that the decomposition (20) is uniquely determined by x. For example, let $v_1, v_2, \ldots$ be independent random variables with the standard normal distribution, and write

$$w_{st} = v_{s+1} + v_{s+2} + \ldots + v_t.$$

Then w is additive, with time constant 0. If

$$x_{st} = \max(w_{st}, 0),$$

it is easily seen that x is subadditive, with

$$g_t = E(x_{0t}) = (t/2\pi)^{\frac{1}{2}},$$

so that $\gamma = 0$. Two different decompositions satisfying the conditions of Theorem 4 are those with

$$y_{st} = 0, \quad z_{st} = x_{st}$$

and

$$y_{st} = w_{st}, \quad z_{st} = \max(-w_{st}, 0).$$

6. Proof of Theorem 4

Consider the set Ω of all functions $\omega = \omega(s,t)$ defined for $s, t \in T$, $s \leqslant t$ and such that

$$\omega(t,t) = 0, \quad \omega(s,t) \leqslant \omega(s,u) + \omega(u,t) \tag{22}$$

for all $s \leqslant u \leqslant t$. Then the given process is, in effect, a random element of Ω. More precisely, if x_{st} is the co-ordinate function defined on Ω by

$$x_{st}(\omega) = \omega(s,t),$$

and if $\mathscr{F}$ is the smallest σ-field with respect to which all the functions x_{st} are measurable, then the given process defines a probability measure P on $(\Omega, \mathscr{F})$ such that the random variables x_{st} on $(\Omega, \mathscr{F}, P)$ have the finite-dimensional distributions of the process.

For any ω in Ω, we may define a new element $\theta\omega$ by

$$(\theta\omega)(s,t) = \omega(s+1, t+1);$$

θ is then a function of Ω into itself. Since

$$x_{st}(\theta\omega) = x_{s+1,t+1}(\omega),$$

the stationarity of x is equivalent to the statement that θ preserves the measure P, namely that

$$P(\theta^{-1} A) = P(A) \quad (A \in \mathscr{F}). \tag{23}$$

We shall write $L = L_1(\Omega, \mathscr{F}, P)$ for the Banach space of all real integrable functions on Ω, with the norm

$$\|f\| = \int_\Omega |f| \, dP = E|f|.$$

For any f in L, a new element Tf of L is defined by

$$(Tf)(\omega) = f(\theta\omega);$$

T is then a bounded linear operator (in fact an isometry) of L into itself.

Lemma. There exists f in L such that, for all n,

$$f + Tf + \ldots + T^{n-1} f \leqslant x_{0n}, \tag{24}$$

and such that

$$E(f) = \gamma.$$

Notice that γ is the largest possible value of $E(f)$ for f satisfying (24), since applying E to (24) gives

$$nE(f) \leqslant g_n,$$

and letting $n \to \infty$,

$$E(f) \leqslant \gamma.$$

Assuming for the moment that the lemma has been proved, set

$$y_{st} = T^s f + T^{s+1} f + \ldots + T^{t-1} f.$$

Then y is stationary and additive, and applying T^s to (24) shows that $y_{st} \leqslant x_{st}$, so that $z_{st} = x_{st} - y_{st}$ is stationary, non-negative and subadditive. Moreover,

$$E(y_{01}) = E(f) = \gamma,$$

and the time constant of z is

$$\lim_{t\to\infty} t^{-1} E(x_{0t} - y_{0t}) = \lim_{t\to\infty} t^{-1}(g_t - \gamma t) = 0.$$

Hence Theorem 4 follows from the lemma.

Consider the element f_m of L defined by

$$f_m = m^{-1} \sum_{r=1}^{m} (x_{0r} - x_{1r}).$$

For any n,

$$f_m+Tf_m+\ldots+T^{n-1}f_m = m^{-1}\sum_{k=0}^{n-1}\sum_{r=1}^{m}(x_{k,k+r}-x_{k+1,k+r})$$

$$= m^{-1}\sum_{t=1}^{m+n-1}\sum_{s=a}^{b-1}(x_{st}-x_{s+1,t})$$

$$= m^{-1}\sum_{t=1}^{m+n-1}(x_{at}-x_{bt}),$$

where $a = \max(t-m,0)$, $b = \min(t,n)$. Hence we have

$$f_m+Tf_m+\ldots+T^{n-1}f_m \leqslant m^{-1}\sum_{t=1}^{m+n-1}x_{ab}. \tag{25}$$

Notice that, as $m\to\infty$ for fixed n, the right-hand side of (25) converges to x_{0n}. Moreover,

$$E(f_m) = m^{-1}\sum_{r=1}^{m}(g_r-g_{r-1}) = g_m/m, \tag{26}$$

so that $E(f_m)\to\gamma$ as $m\to\infty$. Accordingly, if the sequence $\{f_m\}$ has, in a suitable topology, a limit point f in L, this will satisfy the conditions of the lemma. The remainder of the proof consists of the (unfortunately rather involved) compactness arguments needed to establish the existence of this limit point.

The dual L^* of L is the space $L_\infty(\Omega, \mathscr{F}, P)$ of bounded measurable functions on Ω, and acts on L by the formula

$$(\phi,f) = \int_\Omega \phi f\,dP \quad (f\in L, \phi\in L^*).$$

The dual L^{**} of L^* is the space of bounded finitely additive set functions on $(\Omega, \mathscr{F})$ vanishing on null sets, acting on L^* by

$$(\mu,\phi) = \int_\Omega \phi\,d\mu \quad (\phi\in L^*, \mu\in L^{**}).$$

The natural imbedding $\kappa: L\to L^{**}$ is defined by

$$(\kappa f)(A) = \int_A f\,dP \quad (f\in L, A\in\mathscr{F}).$$

(For these and related results see, for instance, Dunford and Schwartz, 1964, Section V.8.)

From (25) with $n = 1$, $f_m\leqslant x_{01}$, so that

$$\|f_m\|\leqslant\|x_{01}\|+\|x_{01}-f_m\| = \|x_{01}\|+E(x_{01}-f_m)$$

$$= \|x_{01}\|+g_1-g_m/m\leqslant\|x_{01}\|+g_1-\gamma = M \quad \text{(say)}.$$

Hence the elements κf_m of L^{**} lie in the set $\{\mu\in L^{**}; \|\mu\|\leqslant M\}$, which by the Bourbaki–Alaoglu theorem (Dunford and Schwartz, 1964, p. 424) is compact in the weak* topology of L^{**}. Thus the sequence (κf_m) has a weak* limit point μ in L^{**}.

The operator S defined on L** by

$$(S\mu)(A) = \mu(\theta^{-1} A) \quad (A \in \mathscr{F})$$

is weak* continuous, and

$$S\kappa = \kappa T.$$

From (25),

$$\kappa f_m + S\kappa f_m + \ldots + S^{n-1} \kappa f_m \leqslant m^{-1} \sum_{t=1}^{m+n-1} \kappa x_{ab},$$

and since the right-hand side converges to κx_{0n},

$$\mu + S\mu + \ldots + S^{n-1} \mu \leqslant \kappa x_{0n}. \tag{27}$$

From (26),

$$(\kappa f_m, 1) = (1, f_m) = E(f_m) = g_m/m \to \gamma,$$

so that

$$(\mu, 1) = \gamma,$$

that is,

$$\mu(\Omega) = \gamma. \tag{28}$$

From (27) with $n = 1$, $(\kappa x_{01} - \mu)$ is a non-negative finitely additive set function, and therefore admits, by a theorem of Yosida and Hewitt (1952, Theorem 1.23), a unique decomposition as the sum of a measure and a non-negative purely finitely additive set function. Hence

$$\mu = \lambda - \pi,$$

where λ is a signed measure and π is non-negative and purely finitely additive. From (27),

$$\lambda + S\lambda + \ldots + S^{n-1} \lambda \leqslant \kappa x_{0n} + \pi_n,$$

where

$$\pi_n = \pi + S\pi + \ldots + S^{n-1} \pi,$$

being a sum of purely finitely additive set functions, is purely finitely additive (Yosida and Hewitt, 1952, Theorem 1.17). Hence

$$\lambda + S\lambda + \ldots + S^{n-1} \lambda \leqslant \kappa x_{0n}.$$

In particular

$$\lambda(\Omega) + S\lambda(\Omega) + \ldots + S^{n-1} \lambda(\Omega) \leqslant \kappa x_{0n}(\Omega) = g_n,$$

so that

$$n\lambda(\Omega) \leqslant g_n.$$

Letting $n \to \infty$,

$$\lambda(\Omega) \leqslant \gamma = \mu(\Omega) = \lambda(\Omega) - \pi(\Omega).$$

Hence $\pi = 0$, and $\mu = \lambda$ is a signed measure.

Moreover, since $\mu \in L^{**}$, μ vanishes on null sets, and so has a density f with respect to P. Hence $\mu = \kappa f$, and (27) shows that f satisfies (24). Finally, from (28),

$$E(f) = (\kappa f)(\Omega) = \mu(\Omega) = \gamma,$$

and the proof is complete.

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