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THE ERMAKOV EQUATION: A COMMENTARY

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We present a short history of the Ermakov Equation with an emphasis on its discovery by the West and the subsequent boost to research into invariants for nonlinear systems although recognizing some of the significant developments in the East. We present the modern context of the Ermakov Equation in the algebraic and singularity theory of ordinary differential equations and applications to more divers fields. The reader is referred to the previous article (Appl. Anal. Discrete Math., **2** (2008), 123–145) for an English translation of Ermakov's original paper.

1. THE LEWIS INVARIANT AND PINNEY'S SOLUTION

In 1950 the late Edmund Pinney [**2**] presented in a very succinct paper [**85**] the solution of the equation

$$\ddot{x} + \omega^2(t)\, x = \frac{1}{x^3} \tag{1}$$

in which the overdot represents differentiation with respect to the independent variable, t, which in many applications is the time (See also [**7**]). The solution which he gave is

$$x(t) = \left(Au^2 + 2Buv + Cv^2\right)^{1/2}, \tag{2}$$

where $u(t)$ and $v(t)$ are any two linearly independent solutions of the equation

$$\ddot{x} + \omega^2(t)\, x = 0 \tag{3}$$

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and the constants, A, B and C, are related according to $B^2 - AC = 1/W^2$ with W being the constant Wronskian of the two linearly independent solutions.

In 1966 the late RALPH LEWIS, while he was on sabbatical at the University of Heidelberg, commenced the calculation of an invariant for the Hamiltonian corresponding to (3), *videlicet*

$$H = \frac{1}{2}\left(p^2 + \omega^2(t)q^2\right), \quad p = \dot{q}, \tag{4}$$

using KRUSKAL's asymptotic method [**54**]. An adiabatic invariant for (4) had been proposed by LORENTZ at the Solvay Congress of 1911 [**98**], but this was not satisfactory for the application of interest to LEWIS which was the motion of a charged particle in an electromagnetic field expected to be rapidly varying in the context of plasma confinement. KRUSKAL's method involves an asymptotic expansion in terms of a parameter, ε, and for the first term in the expansion LEWIS obtained

$$I_0 = \frac{1}{2}\left((\rho p - \dot{\rho} q)^2 + \left(\frac{q}{\rho}\right)^2\right), \tag{5}$$

where $\rho(t)$ is a solution of (1) and so is given by (2). To the surprise of LEWIS the second and third terms in the asymptotic expansion were zero. He then essayed[1] a direct calculation of the first derivative of I_0 and found it to be zero! His results are found in [**65**–**67**].

Although this result could scarcely be considered to be of use in classical problems, the reduction of the solution of the corresponding time-dependent SCHRÖDINGER equation [**68**] to the solution of (3) could be regarded as a distinct advantage since any numerical computation was then deferred until almost the last line. A clear example of this is found in the calculation of BERRY's Phase for the time-dependent linear oscillator [**55**]. The utility of the result was enhanced when it was found that (1) occurred as an auxiliary equation in the calculation of invariants for nonquadratic Hamiltonian systems [**16**, **17**, **37**, **38**, **26**, **69**–**74**].

2. ERMAKOV AND HIS INVARIANT

In ERMAKOV's paper the roles of (1), (3) and (5) are interchanged by comparison with the work of LEWIS[2]. ERMAKOV introduces (1) as an equation auxiliary to (3), multiplies by an integrating factor and obtains the invariant after integration with respect to time. The integrating factor is $\rho\dot{q} - \dot{\rho}q$, in which we have replaced the momentum by $\dot{q}$, and bears a remarkable similarity to angular momentum in two dimensions if one regards ρ and q as Cartesian coordinates. This similarity was pursued by ELIEZER and GREY [**23**], but the interpretation did not endure

[1] The personal details of this story were related to one of us (PGLL) by RALPH many years ago.

[2] PINNEY's contribution does not play a role here as he was not concerned with the invariant.

into higher dimensions [**36**] even though sense could be made of an Ermakov invariant in higher dimensions [**29**]. The work of Ermakov received no notice in the West until the late seventies of the last century when James Reid initiated a surge of activity through his doctoral thesis[3]. There was a flood of papers on generalisations in one direction or another of the basic result of Ermakov [**4–6**, **8**, **9**, **12**, **27**, **49**, **56**, **62**, **80**, **81**, **84**, **86–93**, **96**] and the subject of Ermakov invariants attracted the attention of many of those who worked in the area of invariants for time-dependent systems. In a more general direction Berkovich applied his method of factorisation [**10**, **11**] to problems based upon Ermakov systems. Further applications are found in Cosmology [**39–41**, **75**, **94**, **95**, **97**, **99**, **100**], partial differential equations of Mathematical Physics such as the Korteweg – de Vries and Camassa – Holm equations [**19**, **20**, **22**, **42–46**], Elasticity [**77–79**, **94**, **99**], Quantum Mechanics [**48**] and nonlinear systems in general [**15**, **18**, **25**, **47**, **50**, **51**, **63**, **64**].

Mention of the works of Berkovich on factorisation reminds one that the Ermakov equation can be found embedded in a number of more general equations for which methods to obtain solutions in closed form has been devised. The close association of Berkovich with the forbear of this journal, Publications of the Faculty of Electrical Engineering of the University of Belgrade, it's not surprising when one sees papers devoted to the resolution of nonlinear ordinary differential equations. In particular there are the papers of Kečkić [**101–103**] and Mitrinović and Kečkić [**106**] in the equations of which one can find that of Ermakov, albeit at some times after deep enquiry. More recent work by Kocić [**105**] has continued this tradition[4].

3. ALGEBRAIC PROPERTIES

The Ermakov equation, (1), is distinguished by possessing the algebra of $sl(2,R)$, isomorphic to the noncompact algebra $so(2,1)$ which geometrically represents rotations on the surface of an hyperboloid of one sheet. The connection with angular momentum in two dimensions is now obvious and the failure of the extension to three dimensions quite evident. In the context of scalar differential equations the former algebraic description is to be preferred since there really is no suggestion of rotation. The importance of the algebra, $sl(2,R)$, in the algebraic

[3] Reid was a student of mature years and studied under the supervision of John Ray at Clemson University in South Carolina. According to the story which PGLL received in 1990 James Reid found the paper in the Library of Congress. It is possible that Reid was motivated in his search by a knowledge of some of the works of Berkovich such as [**7**].

[4] The devotion of certain writers to the elucidation of even more methods and stratagems is possibly something of a mystery to outsiders. In his tribute to Professor Mitrinović on the occasion of his 70th birthday his former student Kečkić writes 'More then once Professor Mitrinović said to the present author that he made a mistake in spending so much time on such an ungratifying discipline as the integration of differential equations is known to be, and that he would have done better if he had devoted all his time to some other field (*eg* inequalities). The author of this paper wishes to his express a profound disagreement with that opinion of Professor Mitrinović.' [**104**]. Given HIS contributions to the discipline one wonders if Professor Mitrinović was in the habit of having a quiet jest.

theory of differential equations cannot be overstated. All linear and linearisable systems of ordinary differential equations of maximal symmetry possess this algebra. Not surprisingly related linear evolution equations, such as the heat equation and the SCHRÖDINGER equation, also possess this algebra. A number of papers have been devoted to these algebraic properties [**14**, **28**, **31**–**34**, **57**, **60**, **61**, **83**].

Although a scalar second-order ordinary differential equation requires just three LIE symmetries [**1**, **2**, **58**, **59**] to specify it completely, the complete symmetry group[5] of (1) is not $sl(2,R)$. NUCCI *et al* [**83**] showed that two of the three symmetries were quite nonlocal and that their derivation was nontrivial.

The solution, (2), of (1) presented by PINNEY is equally interpreted as a first integral of the third-order equation

$$\dddot{x} + 4\omega^2\dot{x} + 4\omega\dot{\omega}x = 0 \tag{6}$$

which is the normal form of a third-order equation of maximal symmetry provided that one replaces the 1 in (1) by an arbitrary constant, say K, thereby removing the constraint on the constants in the solution. The solution of (6) is

$$x(t) = Au^2 + 2Buv + Cv^2, \tag{7}$$

where $u(t)$ and $v(t)$ are any two linearly independent solutions of (3) and A, B and C are now the three arbitrary constants of integration. Some years ago CONTE [**21**] observed that the study of any one of (3), the KUMMER-SCHWARZ equation,

$$2\dot{x}\dddot{x} - 3\ddot{x}^2 + 4\dot{x}^4 - 4\omega^2\dot{x}^2 = 0, \tag{8}$$

and the RICCATI equation

$$\dot{x} + x^2 + \omega^2 = 0 \tag{9}$$

implied the study of the other two. The KUMMER-SCHWARZ equation possesses ten contact symmetries and so is linearisable to (6) by means of a contact consolation. Subsequently GOVINDER *et al* [**35**] demonstrated that all linear and linearisable third-order ordinary differential equations could be transformed to (6) by means of a nonlocal transformation which certainly widens the class contemplated by CONTE. We observe that under the nonlocal transformation, $y^{-2}(t) = \dot{x}$, the KUMMER-SCHWARZ equation, (8), becomes the ERMAKOV equation, (1), and so the latter is also included in the class of equivalence.

Within the past three years there has been the development [**3**] of the use of recursion operators for ordinary differential equations in a manner similar to that of their well-known use for evolution partial differential equations. ERMAKOV's

[5] A concept introduced by KRAUSE [**52**, **53**] in the context of the KEPLER problem to describe the group of the minimum number of symmetries required to specify an equation completely. It should be noted that the concept was presaged in the context of partial differential equations by BLUMAN and COLE some years earlier [**13**].

equation, (1), provides a seed equation for such a sequence. Some initial results can be found in EULER *et al* [**24**] and a more complete treatment in MAHARAJ *et al* [**76**]. All elements of the sequence possess the algebra $sl(2, R)$. They also have rich properties in terms of singularity analysis in that the resonances appear in patterns[6].

At the conclusion of §2 we noted a number of studies devoted to the resolution of nonlinear ordinary differential equations for which the ERMAKOV equation is a particular instance. As far as we know, the algebraic properties of these more general equations have not been studied. Certainly in some papers [**101**–**103**, **106**, **105**] the transformations for resolution and so there is no loss of algebraic properties of the LIE point symmetries. The real question is whether the particular isolation from the general to the ERMAKOV equation has meant an increase in point symmetries. Even in this small area it would seem that the scope for future investigation exists. In this respect the West could well look for inspiration in the generalisations of methods of resolution mentioned above and to find the sources in journals such as the *Publications* and its ilk which belong to an old and well-established European tradition.

4. CONCLUDING COMMENTS

In the above we have briefly summarised some of the areas in which the ERMAKOV equation has found application in recent decades. It is quite likely that most of those applications would have been made independently of a knowledge of the work of ERMAKOV since the papers of PINNEY and LEWIS had introduced the equation into the recent literature. However, one wonders if the outburst of activity in the development of generalisations of the ERMAKOV invariant would have occurred without the direct impetus given by a knowledge of ERMAKOV's work. It is only fitting that ERMAKOV be given credit for his pioneering initiative.

Future developments involving the ERMAKOV equation and invariant can only be a matter for conjecture. One hopes that the translation[7] into English given in this issue makes the work of ERMAKOV more accessible.

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[6] In the form of (1) the ERMAKOV equation and the higher members of the differential sequence possess the so-called Weak Painlevé Property. The transformation $z = x^{-2}$ converts the equations to forms which have the standard Painlevé Property.

[7] The translator, ALEX HARIN, was a doctoral student at the University of Natal under the supervision of PGLL in the early nineties. Before coming to Natal ALEX completed his masters degree at the University of Moscow. The story of his translation from Moscow to Durban at a period when the relationships between the respective governments were somewhat less than cordial is just one more chapter in the history of ERMAKOV.

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