THE ESSENTIAL SKELETON OF A DEGENERATION OF ALGEBRAIC VARIETIES

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Abstract. In this paper, we explore the connections between the Minimal Model Program and the theory of Berkovich spaces. Let k be a field of characteristic zero and let X be a smooth and proper $k((t))$-variety with semi-ample canonical divisor. We prove that the essential skeleton of X coincides with the skeleton of any minimal *dlt*-model and that it is a strong deformation retract of the Berkovich analytification of X. As an application, we show that the essential skeleton of a Calabi-Yau variety over $k((t))$ is a pseudo-manifold.

Contents

1. Introduction

Let k be a field of characteristic zero and set $R = k[\![t]\!]$ and $K = k((t))$. We fix a t-adic absolute value on K by setting $|t|_K = 1/e$. Let X be a geometrically connected, smooth and proper K-variety. Then one can associate to X a K-analytic space X^{an} in the sense of [Be90]. Each point of this space can be interpreted as a real valuation on the residue field of a point of X, extending the t-adic valuation on K. Thus X^{an} is naturally related to the birational geometry of R-models of X.

An *snc*-model of X is a regular flat separated R-scheme of finite type $\mathscr{X}$, endowed with an isomorphism of K-schemes $\mathscr{X}_K \to X$, such that the special fiber $\mathscr{X}_k$ is a (not necessarily reduced) divisor with strict normal crossings. Each *snc*-model $\mathscr{X}$ of X gives rise to a so-called skeleton $\mathrm{Sk}(\mathscr{X})$, a finite simplicial space embedded in the K-analytic space X^{an}, canonically homeomorphic to the dual intersection complex $\mathcal{D}(\mathscr{X}_k)$ of $\mathscr{X}_k$ [MN13, §3]. If $\mathscr{X}$ is proper over R, then $\mathrm{Sk}(\mathscr{X})$ is a strong deformation retract of X^{an} (see Theorem 3.1.3 and (3.1.4)).

Results of this type are fundamental tools in the study of the homotopy type of K-analytic spaces, for instance in Berkovich's proof of local contractibility of smooth K-analytic spaces [Be99]. On the other hand, the fact that the space X^{an} does not depend on any choice of model implies that the homotopy type of $\mathrm{Sk}(\mathscr{X})$ does not depend on the choice of $\mathscr{X}$; see [Th07] for a similar result in the context of embedded resolutions of pairs of varieties over a perfect field.

If X is a curve of genus ≥ 1, then it is well-known that X has a minimal *snc*-model, which gives rise to a *canonical* skeleton in X^{an}. However, in higher dimensions, no such distinguished *snc*-model exists, and one can wonder if it is still possible to construct a canonical skeleton inside the space X^{an}. In this paper, we study two such constructions. Although they look quite different at first sight, we prove that they indeed yield the same result.

The first one is the so-called *essential skeleton* from [MN13, 4.6.2], a generalization of a construction of Kontsevich and Soibelman in [KS06] motivated by homological mirror symmetry. Its definition is quite natural: for every non-zero regular pluricanonical form ω on X and every proper *snc*-model $\mathscr{X}$ of X, the form ω singles out certain faces of the skeleton $\mathrm{Sk}(\mathscr{X})$ corresponding to intersections of irreducible components where ω has minimal weight in a suitable sense; see [MN13, 4.5.5] for a precise statement. Taking the union of such faces as ω varies, we obtain a simplicial subspace $\mathrm{Sk}(X)$ of $\mathrm{Sk}(\mathscr{X})$ that can be characterized intrinsically on X and thus no longer depends on any choice of an *snc*-model. This space $\mathrm{Sk}(X)$ was called the essential skeleton of X in [MN13, 4.6.2]. If X has trivial canonical sheaf, then $\mathrm{Sk}(X)$ coincides with the Kontsevich-Soibelman skeleton from [KS06] associated to any volume form ω on X.

A second construction appears in the context of the Minimal Model Program, specifically in the paper [dFKX12]. If we enlarge our class of models from *snc*-models to so-called *dlt*-models (2.2.1), then the relative minimal models over $\mathrm{Spec}(R)$ exist in any dimension, provided that the canonical divisor K_X of the generic fiber is semi-ample (see Theorem 2.2.6 – for technical reasons, we are obliged to assume that X is defined over an algebraic k-curve and to work with models over the base curve, because the results from MMP that we use have only been proven for k-schemes of finite type). Such a minimal *dlt*-model is not unique, but any two of them are crepant birational, which implies that their skeleta are the same (Corollary 3.2.7). Moreover, we prove that this canonical skeleton is still a strong deformation retract of X^{an} (Corollary 3.2.9).

Our main result, Theorem 3.3.4, states that these two constructions are equivalent: if K_X is semi-ample, then the essential skeleton $\mathrm{Sk}(X)$ coincides with the skeleton of any minimal *dlt*-model.

We present two applications of this equivalence. First, as an immediate corollary of the above results, we obtain that the essential skeleton $\mathrm{Sk}(X)$ is a strong deformation retract of X_K^{an} when K_X is semi-ample (see Corollary 3.3.6). Second, in Section 4, we study the topological properties of the essential skeleton of a Calabi-Yau variety X over K. Using [KK10, Ko11], we show that $\mathrm{Sk}(X)$ is a pseudo-manifold with boundary, and even a closed pseudo-manifold when $\mathrm{Sk}(X)$ has maximal dimension and k is algebraically closed. Moreover, using logarithmic geometry, we show that $\mathrm{Sk}(X)$ only depends on the reduction modulo t^2 of any proper *snc*-model of X, which allows us to remove the technical assumption that X is defined over a curve (Theorem 4.1.4).

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Terminology and conventions. We follow [Ko13] for the definitions of various notions of a singular pair from the Minimal Model Program, including *klt, dlt* and *log canonical pairs*. In particular, we refer to [Ko13, 4.15] for the definition of *log canonical centers*. The non-archimedean analytic spaces that appear in this paper are K-analytic spaces in the sense of [Be90]. We refer to [Te13] for a gentle introduction. We will also make use of some basic logarithmic geometry; all log structures in this paper are defined with respect to the Zariski topology, and they are fine and saturated (fs). The standard introduction to logarithmic geometry is [Ka89].

2. Minimal *dlt*-models

2.1. **Models and log pullbacks.**

(2.1.1) Let k be a field of characteristic zero. We set $R = k[\![t]\!]$ and $K = k(\!(t)\!)$, and we fix a t-adic absolute value $|\cdot|_K$ on K by setting $|t|_K = 1/e$. For every K-scheme of finite type Y, we denote by Y^{an} the associated K-analytic space. For every separated R-scheme of finite type $\mathscr{Y}$ we set $\mathscr{Y}_k = \mathscr{Y} \times_R k$ and $\mathscr{Y}_K = \mathscr{Y} \times_R K$. Moreover, we will denote by $\widehat{\mathscr{Y}}$ the t-adic completion of $\mathscr{Y}$, by $\widehat{\mathscr{Y}_\eta}$ the generic fiber of $\widehat{\mathscr{Y}}$ in the category of K-analytic spaces and by

$$\mathrm{red}_{\mathscr{Y}} : \widehat{\mathscr{Y}_\eta} \to \mathscr{Y}_k$$

the canonical reduction map. The generic fiber $\widehat{\mathscr{Y}_\eta}$ is an analytic domain in $\mathscr{Y}_K^{\mathrm{an}}$, and it is equal to $\mathscr{Y}_K^{\mathrm{an}}$ if and only if $\mathscr{Y}$ is proper over R.

(2.1.2) Let $\mathscr{C}$ be a connected smooth algebraic curve over k. Let s be a k-rational point on $\mathscr{C}$ and set $C = \mathscr{C} \setminus \{s\}$. We fix a uniformizer t in $\mathcal{O}_{\mathscr{C},s}$. This choice determines an isomorphism of k-algebras $R \to \widehat{\mathcal{O}}_{\mathscr{C},s}$ and thus a morphism of k-schemes $\operatorname{Spec} R \to \mathscr{C}$.

(2.1.3) Let X be a smooth and proper scheme over C with geometrically connected fibers. A model of X over $\mathscr{C}$ is a flat separated $\mathscr{C}$-scheme of finite type $\mathscr{X}$ endowed with an isomorphism of C-schemes $\mathscr{X} \times_{\mathscr{C}} C \to X$. Note that we do not require $\mathscr{X}$ to be proper over $\mathscr{C}$. Morphisms of models are defined in the usual way. We denote by $\mathscr{X}_s$ the fiber of $\mathscr{X}$ over s, by $\mathscr{X}_R$ the base change of $\mathscr{X}$ to $\operatorname{Spec} R$ and by X_K the base change of X to $\operatorname{Spec} K$. We denote by K_X a relative canonical divisor for X over C, and for every normal model $\mathscr{X}$ of X, we denote by $K_{\mathscr{X}}$ a relative canonical divisor for $\mathscr{X}$ over $\mathscr{C}$.

(2.1.4) For every $\mathscr{C}$-model $\mathscr{X}$ of X, we denote by $\mathscr{X}^{\mathrm{snc}}$ the subset of $\mathscr{X}$ consisting of the points where $\mathscr{X}$ is regular and $\mathscr{X}_s$ is a divisor with strict normal crossings (some authors use the terminology "simple normal crossings" instead). Thus $\mathscr{X}^{\mathrm{snc}}$ is the union of X with the set of points x of $\mathscr{X}_s$ such that $\mathcal{O}_{\mathscr{X},x}$ is regular and there exist a unit u and a regular system of local parameters $(z_1,\ldots,z_n)$ in $\mathcal{O}_{\mathscr{X},x}$ and non-negative integers $N_1,\ldots,N_n$ such that

$$t = u\prod_{i=1}^{n}(z_i)^{N_i}.$$

The subset $\mathscr{X}^{\mathrm{snc}}$ is an open subscheme of $\mathscr{X}$ and it is again a $\mathscr{C}$-model of X. Moreover, if $\mathscr{X}$ is normal, then $\mathscr{X}_s^{\mathrm{snc}}$ is dense in $\mathscr{X}_s$. We say that $\mathscr{X}$ is an *snc*-model of X if $\mathscr{X} = \mathscr{X}^{\mathrm{snc}}$, that is, if $\mathscr{X}$ is regular and $\mathscr{X}_s$ is a divisor with strict normal crossings. If $\mathscr{X}$ is a model of X over $\mathscr{C}$, then a log resolution of $(\mathscr{X},\mathscr{X}_s)$ is a proper morphism of $\mathscr{C}$-models $h:\mathscr{Y}\to\mathscr{X}$ such that $\mathscr{Y}$ is an *snc*-model of X.

(2.1.5) Let $h:\mathscr{Y}\to\mathscr{X}$ be a proper morphism of normal $\mathscr{C}$-models of X. Assume that $K_{\mathscr{X}}+(\mathscr{X}_s)_{\mathrm{red}}$ is $\mathbb{Q}$-Cartier. Then the log pullback of $(\mathscr{X}_s)_{\mathrm{red}}$ to $\mathscr{Y}$ is the unique $\mathbb{Q}$-Weil divisor Δ on $\mathscr{Y}$ such that $K_{\mathscr{Y}}+\Delta$ is $\mathbb{Q}$-linearly equivalent to

$$f^*(K_{\mathscr{X}}+(\mathscr{X}_s)_{\mathrm{red}})$$

and $f_*\Delta = (\mathscr{X}_s)_{\mathrm{red}}$.

(2.1.6) We will use the following notations from [MN13]. If $\mathscr{X}$ is a normal model of X over $\mathscr{C}$, x is a point of $\widehat{\mathscr{X}_\eta}$ and D is a divisor on $\mathscr{X}$ that is supported on $\mathscr{X}_s$ and Cartier at $\mathrm{red}_{\mathscr{X}}(x)$, then we set

$$v_x(D) = -\ln|f(x)|$$

where f is any element of the local ring of $\mathscr{X}$ at $\mathrm{red}_{\mathscr{X}}(x)$ such that $D=\mathrm{div}(f)$ locally at $\mathrm{red}_{\mathscr{X}}(x)$. It is clear that $v_x(D)$ is linear in D. If $\mathscr{X}$ is regular and ω is a non-zero rational section of $\omega^{\otimes m}_{\mathscr{X}_R/R}$, for some $m>0$ (for instance, an m-pluricanonical form on X_K) then we denote by $\mathrm{div}_{\mathscr{X}}(\omega)$ the corresponding divisor on $\mathscr{X}_R$.

2.2. *dlt*-models.

(2.2.1) A *dlt*-model of X is a normal proper $\mathscr{C}$-model $\mathscr{X}$ of X such that $(\mathscr{X},(\mathscr{X}_s)_{\mathrm{red}})$ is a *dlt*-pair. This means that $(\mathscr{X},(\mathscr{X}_s)_{\mathrm{red}})$ is log canonical and that each log canonical center of $(\mathscr{X},(\mathscr{X}_s)_{\mathrm{red}})$ has non-empty intersection with $\mathscr{X}^{\mathrm{snc}}$. In particular, every proper *snc*-model of X is a *dlt*-model. An equivalent formulation of the definition is the following: $K_{\mathscr{X}}+(\mathscr{X}_s)_{\mathrm{red}}$ is $\mathbb{Q}$-Cartier, and for every log resolution $h:\mathscr{Y}\to\mathscr{X}$ of $(\mathscr{X},\mathscr{X}_s)$ and every irreducible component E of $\mathscr{Y}_s$, the multiplicity of E in the log pullback Δ of $(\mathscr{X}_s)_{\mathrm{red}}$ to $\mathscr{Y}$ is at most 1. Moreover, if it is equal to 1, then $h(E)$ must have non-empty intersection with $\mathscr{X}^{\mathrm{snc}}$. In practice, we will apply the *dlt* property *via* Lemma 3.2.3 below.

(2.2.2) We say that a *dlt*-model $\mathscr{X}$ of X is a *good minimal model* if $\mathscr{X}$ is $\mathbb{Q}$-factorial and $K_{\mathscr{X}}+(\mathscr{X}_s)_{\mathrm{red}}$ is semi-ample over $\mathscr{C}$.

(2.2.3) For every *dlt*-model $\mathscr{X}$ of X, we can define the dual complex $\mathcal{D}((\mathscr{X}_s)_{\mathrm{red}})$ for the *dlt*-pair $(\mathscr{X}, (\mathscr{X}_s)_{\mathrm{red}})$ by gluing cells corresponding to irreducible components of intersections of irreducible components of $\mathscr{X}_s$, as in Definition 8 in [dFKX12]. When k is not algebraically closed, we note that we only glue cells corresponding to irreducible components (instead of geometrically irreducible components). In other words, $\mathcal{D}((\mathscr{X}_s)_{\mathrm{red}})$ is the quotient of the $\mathrm{Gal}(\bar{k}/k)$-equivariant dual complex constructed in [dFKX12, §31].

(2.2.4) For every *dlt*-model $\mathscr{X}$ of X, the log canonical centers of $(\mathscr{X}, (\mathscr{X}_s)_{\mathrm{red}})$ are the irreducible components of intersections of irreducible components of $(\mathscr{X}_s)_{\mathrm{red}}$, by [Ko13, 4.16]. These are also precisely the closures in $\mathscr{X}_s$ of the connected components of intersections of irreducible components of $\mathscr{X}_s^{\mathrm{snc}}$ (since these connected components are the log canonical centers of $(\mathscr{X}^{\mathrm{snc}}, (\mathscr{X}_s^{\mathrm{snc}})_{\mathrm{red}})$). Thus, the dual intersection complex $\mathcal{D}((\mathscr{X}_s)_{\mathrm{red}})$ is the same as the dual intersection complex of the strict normal crossings divisor $\mathscr{X}_s^{\mathrm{snc}}$, and the cells of this complex correspond bijectively to the log canonical centers of $(\mathscr{X}, (\mathscr{X}_s)_{\mathrm{red}})$. See Section 2 in [dFKX12] for more background.

(2.2.5) Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be two *dlt*-models of X over $\mathscr{C}$. We say that $\mathscr{X}_1$ and $\mathscr{X}_2$ are crepant birational if there exist a normal proper $\mathscr{C}$-model $\mathscr{Y}$ of X and morphisms of $\mathscr{C}$-models $f_i : \mathscr{Y} \to \mathscr{X}_i$ for $i = 1, 2$ such that the log pullbacks of $(\mathscr{X}_{1,s})_{\mathrm{red}}$ and $(\mathscr{X}_{2,s})_{\mathrm{red}}$ coincide (see [Ko13, 2.23]). Note that we can always assume that $\mathscr{Y}$ is an *snc*-model, by taking a log resolution of $(\mathscr{Y}, \mathscr{Y}_s)$. The following theorem collects two fundamental results from the Minimal Model Program.

Theorem 2.2.6.

(1) *The C-scheme X has a good minimal dlt-model if and only if K_X is semi-ample over C.*
(2) *Any two good minimal dlt-models of X are crepant birational.*

Proof. (1) The condition that K_X is semi-ample over C is obviously necessary, since for every *dlt*-model $\mathscr{X}$ of X, the divisor K_X is $\mathbb{Q}$-linearly equivalent to the restriction of $K_{\mathscr{X}} + (\mathscr{X}_s)_{\mathrm{red}}$ to X. Conversely, assume that K_X is semi-ample over C, and let $\mathscr{Y}$ be a proper *snc*-model of X. Then applying [HX13, 2.12] to the *dlt*-pair $(\mathscr{Y}, (\mathscr{Y}_s)_{\mathrm{red}})$, we see that X has a good minimal *dlt*-model. Condition (1) of [HX13, 2.12] follows from our assumption, and condition (2) follows from the following observation. Let m be a positive integer such that $m(K_{\mathscr{Y}} + (\mathscr{Y}_s)_{\mathrm{red}})$ is Cartier. Over a sufficiently small open neighbourhood of s in $\mathscr{C}$, we have an isomorphism of $\mathcal{O}_{\mathscr{C}}$-algebras

$$R(\mathscr{Y}/\mathscr{C}, m(K_{\mathscr{Y}} + (\mathscr{Y}_s)_{\mathrm{red}})) \cong R(\mathscr{Y}/\mathscr{C}, m(K_{\mathscr{Y}} + (\mathscr{Y}_s)_{\mathrm{red}}) - \mathscr{Y}_s),$$

where $R(\mathscr{Y}/\mathscr{C}, L) := \bigoplus_{j\geq 0} \pi_*(\mathcal{O}_{\mathscr{Y}}(jL))$ with $\pi : \mathscr{Y} \to \mathscr{C}$ the structural morphism. Thus it suffices to show that

$$\mathcal{A} = R(\mathscr{Y}/\mathscr{C}, m(K_{\mathscr{Y}} + (\mathscr{Y}_s)_{\mathrm{red}}) - \mathscr{Y}_s)$$

is a finitely generated $\mathcal{O}_{\mathscr{C}}$-algebra. If we denote by M the maximum of the multiplicities of the components in $\mathscr{Y}_s$ then we may assume that $m > M$, so that

$$(\mathscr{Y}, (\mathscr{Y}_s)_{\mathrm{red}} - \frac{1}{m}\mathscr{Y}_s)$$

is a *klt* pair. Hence, the finite generation of $\mathcal{A}$ follows from [BCHM10].

(2) It is already observed in Definition 15 of [dFKX12] that this follows from the proof of [KM98, 3.52]. □

3. The essential skeleton

3.1. Retraction to the skeleton of an *snc*-model.

(3.1.1) Let $\mathscr{Y}$ be a connected regular flat separated R-scheme of finite type such that the special fiber $\mathscr{Y}_k$ is a divisor with strict normal crossings. Then, as explained in [MN13, §3.1], one can associate to $\mathscr{Y}$ its skeleton $\mathrm{Sk}(\mathscr{Y})$, which is a topological subspace of the generic fiber $\widehat{\mathscr{Y}_\eta}$ of the formal t-adic completion of $\mathscr{Y}$. It is the set of points of $\widehat{\mathscr{Y}_\eta}$ that correspond to a real valuation on the function field of $\mathscr{Y}_K$ that is monomial with respect to the strict normal crossings divisor $\mathscr{Y}_k$. The skeleton $\mathrm{Sk}(\mathscr{Y})$ is canonically homeomorphic to the dual intersection complex $\mathcal{D}((\mathscr{Y}_k)_{\mathrm{red}})$ of $\mathscr{Y}_k$, and there exists a canonical continuous retraction

$$\rho_{\mathscr{Y}} : \widehat{\mathscr{Y}_\eta} \to \mathrm{Sk}(\mathscr{Y}).$$

Moreover, $\mathrm{Sk}(\mathscr{Y})$ carries a canonical piecewise $\mathbb{Z}$-affine structure [MN13, §3.2].

(3.1.2) We keep the notations from (2.1.3). For each $\mathscr{C}$-model $\mathscr{X}$ of X, we define the skeleton of $\mathscr{X}$ by

$$\mathrm{Sk}(\mathscr{X}) = \mathrm{Sk}(\mathscr{X}_R^{\mathrm{snc}}) \subset (\widehat{\mathscr{X}_R^{\mathrm{snc}}})_\eta \subset X_K^{\mathrm{an}}$$

and we write $\rho_{\mathscr{X}}$ for $\rho_{\mathscr{X}_R^{\mathrm{snc}}}$. If $\mathscr{X}$ is a proper *snc*-model of X, one has the following crucial property.

Theorem 3.1.3. *If $\mathscr{X}$ is a proper snc-model of X over $\mathscr{C}$, then there exists a continuous map*

$$H : [0,1] \times X_K^{\mathrm{an}} \to X_K^{\mathrm{an}}$$

such that $H(0,\cdot)$ is the identity, $H(t,x) = x$ for all x in $\mathrm{Sk}(\mathscr{X})$ and all t in $[0,1]$, and $H(1,\cdot) = \rho_{\mathscr{X}}$. Thus $\mathrm{Sk}(\mathscr{X})$ is a strong deformation retract of X_K^{an}.

Proof. A closely related result is proven in [Th07, 3.26]. We will explain how our statement can be deduced from that result. Following the notation in [Th07], we denote by $\mathscr{X}^{\beth}$ the k-analytic space associated to the toroidal embedding $X \hookrightarrow \mathscr{X}$, where k is endowed with the trivial absolute value. By definition, $\mathscr{X}^{\beth}$ is the generic fiber of the formal t-adic completion $\widehat{\mathscr{X}}$ of $\mathscr{X}$, viewed as a special formal k-scheme by forgetting the $k[\![t]\!]$-structure [Be96, §1].

The relation between $\mathscr{X}^{\beth}$ and X_K^{an} is explained in detail at the beginning of Section 4 in [Ni11]; let us recall the main idea. Considering the morphism of special formal k-schemes $\widehat{\mathscr{X}} \to \mathrm{Spf}\, k[\![t]\!]$ and passing to the generic fibers, we obtain a morphism of k-analytic spaces from $\mathscr{X}^{\beth}$ to the open unit disc D over k. We can identify the underlying topological space of D with $[0,1[$ by means of the homeomorphism

$$D \to [0,1[: x \mapsto |t(x)|.$$

The residue field of D at the point $1/e$ in $[0,1[$ is K with our chosen t-adic absolute value $|\cdot|_K$, and the K-analytic space X_K^{an} is canonically isomorphic to the fiber of $\mathscr{X}^{\beth}$ over $1/e$. Thus we can view X_K^{an} as the subspace of $\mathscr{X}^{\beth}$ consisting of the points x such that $|t(x)| = 1/e$.

In [Th07, 3.13], Thuillier constructs a retraction $p_{\mathscr{X}}$ of $\mathscr{X}^{\beth}$ onto a certain subspace $\mathcal{S}(\mathscr{X})$, the skeleton of the toroidal embedding. Moreover, in [Th07, 3.26], he shows that $p_{\mathscr{X}}$ can be extended to a strong deformation retraction H of $\mathscr{X}^{\beth}$ onto $\mathcal{S}(\mathscr{X})$. Going through the definitions, one observes that $p_{\mathscr{X}}$ and H commute with the morphism $\mathscr{X}^{\beth}\to D$ and that the restriction of

$$p_{\mathscr{X}}:\mathscr{X}^{\beth}\to\mathcal{S}(\mathscr{X})$$

over the point $1/e$ of D is precisely the retraction

$$\rho_{\mathscr{X}}:X_K^{\mathrm{an}}\to\mathrm{Sk}(\mathscr{X}).$$

Thus by restricting H over $1/e\in D$, we obtain a map that satisfies all the properties in the statement. □

(3.1.4) Theorem 3.1.3 can be extended to the case where X is defined over K instead of C and $\mathscr{X}$ is a proper *snc*-model of X over R. The general proof technique is the same as in [Th07], but one replaces the formalism of toroidal embeddings by the more flexible language of logarithmic geometry. Details will appear in [Ni13]. We will only use this generalization in the proof of Theorem 4.2.4.

3.2. The skeleton of a good minimal *dlt*-model.

(3.2.1) In the following subsections, we will make use of the *weight function*

$$\mathrm{wt}_{\omega}:X_K^{\mathrm{an}}\to\mathbb{R}\cup\{+\infty\}$$

associated to a non-zero m-pluricanonical form on X_K, for any $m>0$. Its construction and main properties are described in [MN13, 4.4.5]. For us, its most important features are the following: if $\mathscr{X}$ is an *snc*-model of X over $\mathscr{C}$ and x is a point of $\mathrm{Sk}(\mathscr{X})$, then

$$\mathrm{wt}_{\omega}(x)=v_x(\mathrm{div}_{\mathscr{X}}(\omega)+m(\mathscr{X}_s)_{\mathrm{red}})$$

(here we use the notation recalled in (2.1.6)). Moreover, for every point y of $\widehat{\mathscr{X}_{\eta}}$, we have

$$\mathrm{wt}_{\omega}(y)\geq\mathrm{wt}_{\omega}(\rho_{\mathscr{X}}(y))$$

with equality if and only if y lies on $\mathrm{Sk}(\mathscr{X})$. In [MN13, 4.4.5] there is no properness assumption on X_K; this allows us to deal with rational pluricanonical forms by removing the locus of poles from X_K.

(3.2.2) It will often be useful to interpret the weight function in terms of logarithmic differential forms. Let $\mathscr{Y}$ be a regular separated R-scheme of finite type such that $\mathscr{Y}_k$ is a divisor with strict normal crossings. We write S^+ for the log scheme associated to $R\setminus\{0\}\to R$ and $\mathscr{Y}^+$ for the log scheme obtained by endowing $\mathscr{Y}$ with the divisorial log structure associated to $\mathscr{Y}_k$. Then $\mathscr{Y}^+$ is log smooth over S^+. If we denote by $j:\mathscr{Y}_K\to\mathscr{Y}$ the natural open immersion, then a simple computation shows that the sub-$\mathcal{O}_{\mathscr{Y}}$-module $\omega_{\mathscr{Y}^+/S^+}$ of $j_*\omega_{\mathscr{Y}_K/K}$ is equal to $\omega_{\mathscr{Y}/R}((\mathscr{Y}_k)_{\mathrm{red}}-\mathscr{Y}_k)$ (it suffices to check that these line bundles coincide at the generic points of the special fiber $\mathscr{Y}_k$). Thus if ω is an m-pluricanonical form on X_K and $\mathscr{X}$ is an *snc*-model of X over $\mathscr{C}$, then

$$\mathrm{wt}_{\omega}(x)=v_x(\mathrm{div}_{\mathscr{X}^+}(\omega))+m$$

for every point x of $\mathrm{Sk}(\mathscr{X})$, where we denote by $\mathrm{div}_{\mathscr{X}^+}(\omega)$ the divisor on $\mathscr{X}_R$ associated to ω viewed as a rational section of the line bundle $\omega^{\otimes m}_{\mathscr{X}_R^+/S^+}$.

Lemma 3.2.3. *Let $\mathscr{X}$ be a dlt-model of X and let $h : \mathscr{Y} \to \mathscr{X}$ be a log resolution of $(\mathscr{X}, \mathscr{X}_s)$. Denote by Δ the log pullback of $(\mathscr{X}_s)_{\mathrm{red}}$ to $\mathscr{Y}$. Let x be a point of $\mathrm{Sk}(\mathscr{Y})$ such that $\mathrm{red}_{\mathscr{X}}(x)$ does not lie in $\mathscr{X}^{\mathrm{snc}}$. Then $\Delta < (\mathscr{Y}_s)_{\mathrm{red}}$ locally at $\mathrm{red}_{\mathscr{Y}}(x)$.*

Proof. By the definition of a *dlt*-model, we know that $\Delta \leq (\mathscr{Y}_s)_{\mathrm{red}}$. Thus it suffices to show that these divisors are different locally at $\mathrm{red}_{\mathscr{Y}}(x)$. Since x lies on $\mathrm{Sk}(\mathscr{Y})$, its reduction $\mathrm{red}_{\mathscr{Y}}(x)$ is a generic point of the intersection of the irreducible components of $\mathscr{Y}_s$ that contain $\mathrm{red}_{\mathscr{Y}}(x)$. Thus if we denote by $h' : \mathscr{Y}' \to \mathscr{Y}$ the blow-up of $\mathscr{Y}$ at the closure of $\mathrm{red}_{\mathscr{Y}}(x)$, then $\mathscr{Y}'$ is again an *snc*-model of X.

We denote by Δ' the log pullback of Δ to $\mathscr{Y}'$. The image of the exceptional divisor E of h' in $\mathscr{X}$ is the closure of $\mathrm{red}_{\mathscr{X}}(x) = h(\mathrm{red}_{\mathscr{Y}}(x))$ and thus disjoint from $\mathscr{X}^{\mathrm{snc}}$. By the definition of a *dlt*-model, we know that the multiplicity of E in Δ' is strictly smaller than 1. Since the log pullback of $(\mathscr{Y}_s)_{\mathrm{red}}$ to $\mathscr{Y}'$ is equal to $(\mathscr{Y}'_s)_{\mathrm{red}}$, we see that $\Delta < (\mathscr{Y}_s)_{\mathrm{red}}$ locally at $\mathrm{red}_{\mathscr{Y}}(x)$. □

Proposition 3.2.4. *Let $\mathscr{X}$ be a dlt-model of X over $\mathscr{C}$, let $\mathscr{Y}$ be a proper snc-model of X over $\mathscr{C}$ and let $h : \mathscr{Y} \to \mathscr{X}$ be a morphism of $\mathscr{C}$-models. Denote by Δ the log pullback of $(\mathscr{X}_s)_{\mathrm{red}}$ to $\mathscr{Y}$. If we set*

$$S = \{x \in \mathrm{Sk}(\mathscr{Y}) \mid v_x(\Delta) = v_x((\mathscr{Y}_s)_{\mathrm{red}})\}$$

then $\mathrm{Sk}(\mathscr{X}) = S$.

Proof. Applying [MN13, 3.1.7] to the proper morphism $h^{-1}(\mathscr{X}^{\mathrm{snc}}) \to \mathscr{X}^{\mathrm{snc}}$, we see that $\mathrm{Sk}(\mathscr{X})$ is contained in $\mathrm{Sk}(\mathscr{Y})$. Moreover, it follows from Lemma 3.2.3 that for every point x of S, the reduction $\mathrm{red}_{\mathscr{X}}(x)$ must be contained in $\mathscr{X}^{\mathrm{snc}}$. Now let x be any point in $\mathrm{Sk}(\mathscr{Y})$ such that $\mathrm{red}_{\mathscr{X}}(x)$ lies in $\mathscr{X}^{\mathrm{snc}}$. We must show that $v_x(\Delta) = v_x((\mathscr{Y}_s)_{\mathrm{red}})$ if and only if x lies in $\mathrm{Sk}(\mathscr{X})$, or, equivalently, x is equal to its projection

$$x' = \rho_{\mathscr{X}}(x)$$

to the skeleton of $\mathscr{X}$. Let ω be a local generator of $\omega_{\mathscr{X}^{\mathrm{snc}}/\mathscr{C}}$ at $\mathrm{red}_{\mathscr{X}}(x)$. It induces a rational section of the canonical bundle $\omega_{X_K/K}$ by base change. By [MN13, 4.4.5], we know that $x = x'$ if and only if

$$\mathrm{wt}_{\omega}(x) = \mathrm{wt}_{\omega}(x').$$

Since the divisor of ω is zero in a neighbourhood of $\mathrm{red}_{\mathscr{X}}(x')$, we have

$$\mathrm{wt}_{\omega}(x') = v_{x'}((\mathscr{X}_s)_{\mathrm{red}}) = v_x((\mathscr{X}_s)_{\mathrm{red}}).$$

On the other hand, computing $\mathrm{wt}_{\omega}(x)$ on the model $\mathscr{Y}$ we get

$$\mathrm{wt}_{\omega}(x) = v_x(\mathrm{div}_{\mathscr{Y}}(\omega) + (\mathscr{Y}_s)_{\mathrm{red}}) = v_x((\mathscr{X}_s)_{\mathrm{red}}) + v_x((\mathscr{Y}_s)_{\mathrm{red}} - \Delta).$$

Thus we see that $\mathrm{Sk}(\mathscr{X}) = S$. □

Corollary 3.2.5. *Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be two dlt-models of X over $\mathscr{C}$. If $\mathscr{X}_1$ and $\mathscr{X}_2$ are crepant birational, then* $\mathrm{Sk}(\mathscr{X}_1) = \mathrm{Sk}(\mathscr{X}_2)$.

Proof. This follows immediately from Proposition 3.2.4. □

(3.2.6) Corollary 3.2.5 implies, in particular, that the skeleta $\mathrm{Sk}(\mathscr{X}_1) = \mathrm{Sk}(\mathscr{X}_2)$ are isomorphic as topological spaces with piecewise affine structure, by [MN13, §3.2]. Since $\mathrm{Sk}(\mathscr{X}_i)$ is canonically homeomorphic to the dual complex associated to the reduced special fiber of $\mathscr{X}_i$, for $i = 1, 2$, this also follows from Proposition 11 in [dFKX12], whose proof relies on Weak Factorization. The proofs of Corollary 3.2.5 and [MN13, §3.2] do not use Weak Factorization.

Corollary 3.2.7. *If K_X is semi-ample, then the skeleton of a good minimal dlt-model of X does not depend on the choice of the good minimal dlt-model.*

Proof. This follows from Theorem 2.2.6(2) and Corollary 3.2.5. □

Theorem 3.2.8. *Assume that K_X is semi-ample over C. If $\mathscr{X}$ is a good minimal dlt-model of X and $\mathscr{Y}$ is any dlt-model of X, then $\mathrm{Sk}(\mathscr{X})$ is contained in $\mathrm{Sk}(\mathscr{Y})$. Moreover, $\mathrm{Sk}(\mathscr{X})$ can be obtained from $\mathrm{Sk}(\mathscr{Y})$ (as a topological subspace of $\mathrm{Sk}(\mathscr{Y})$ with piecewise affine structure) by a finite number of elementary collapses.*

Proof. For the definition of an elementary collapse in a simplicial topological space, we refer to Definition 18 in [dFKX12]. By Corollary 3.2.7, we can assume that the good minimal *dlt*-model $\mathscr{X}$ is the result of running MMP for $(\mathscr{Y}, (\mathscr{Y}_s)_{\mathrm{red}})$. Now the statement follows from Corollary 22 in [dFKX12]. When k is not algebraically closed, see also §31 in [dFKX12]. □

Corollary 3.2.9. *If $\mathscr{X}$ is a good minimal dlt-model of X, then $\mathrm{Sk}(\mathscr{X})$ is a strong deformation retract of X_K^{an}.*

Proof. Let $\mathscr{Y} \to \mathscr{X}$ be a log resolution of $(\mathscr{X}, \mathscr{X}_s)$. By Theorem 3.2.8, the skeleton $\mathrm{Sk}(\mathscr{X})$ is a strong deformation retract of $\mathrm{Sk}(\mathscr{Y})$. By Theorem 3.1.3, $\mathrm{Sk}(\mathscr{Y})$ is a strong deformation retract of X_K^{an}. □

3.3. **Kontsevich-Soibelman skeleta.**

(3.3.1) In [MN13, §4.5], Mustaţă and the first-named author associated to every non-zero regular pluricanonical form ω on X_K a skeleton $\mathrm{Sk}(X_K, \omega)$ in X_K^{an}, generalizing a construction of Kontsevich and Soibelman [KS06]. The skeleton $\mathrm{Sk}(X_K, \omega)$ is precisely the locus of points of X_K^{an} where the weight function wt_ω reaches its minimal value. If $\mathscr{X}$ is any *snc*-model of X over $\mathscr{C}$, then $\mathrm{Sk}(X_K, \omega)$ is a union of closed faces of $\mathrm{Sk}(\mathscr{X})$, which can be explicitly computed [MN13, 4.5.5]. Taking the union of the skeleta $\mathrm{Sk}(X_K, \omega)$ over all non-zero pluricanonical forms ω on X_K, one obtains a topological subspace $\mathrm{Sk}(X_K)$ of X_K^{an} that was called the essential skeleton of X_K in [MN13, 4.6.2]. It is an interesting birational invariant of X_K. In this subsection, we will compare the essential skeleton to the skeleton of a good minimal *dlt*-model of X.

Proposition 3.3.2. *Assume that K_X is semi-ample over C and let $\mathscr{X}$ be a dlt-model of X. For every integer $m > 0$ and every non-zero m-pluricanonical form ω on X_K, we have*

$$\mathrm{Sk}(X_K, \omega) \subset \mathrm{Sk}(\mathscr{X}).$$

Proof. Let x be a point of $\mathrm{Sk}(X_K, \omega)$. If $\mathrm{red}_{\mathscr{X}}(x)$ is contained in $\mathscr{X}^{\mathrm{snc}}$, then x lies in $\widehat{\mathscr{X}_\eta}$ and [MN13, 4.4.5] implies that x must lie in $\mathrm{Sk}(\mathscr{X})$, since the restriction of wt_ω to $\widehat{\mathscr{X}_\eta}$ can reach its minimal values only at points of $\mathrm{Sk}(\mathscr{X})$.

Now suppose that $\mathrm{red}_{\mathscr{X}}(x)$ is not contained in $\mathscr{X}^{\mathrm{snc}}$. We will deduce a contradiction with the assumption that x belongs to $\mathrm{Sk}(X_K,\omega)$. Let E be an irreducible component of $\mathscr{X}_s^{\mathrm{snc}}$ whose closure contains $\mathrm{red}_{\mathscr{X}}(x)$, let ξ be the generic point of E and denote by x' the unique point in $\mathrm{red}_{\mathscr{X}}^{-1}(\xi)$. We will prove that $\mathrm{wt}_\omega(x')<\mathrm{wt}_\omega(x)$. Then x cannot belong to the locus $\mathrm{Sk}(X_K,\omega)$ where wt_ω reaches its minimal value. Note that, since $\mathscr{X}$ is $\mathbb{Q}$-factorial, we have

$$|f(x')|\geq |f(x)| \tag{3.3.3}$$

for every element f of the local ring of $\mathscr{X}$ at x.

Replacing ω by its d-fold tensor power $\omega^{\otimes d}$, with d a positive integer, has no influence on the skeleton $\mathrm{Sk}(X_K,\omega)$. Thus we may assume that the divisor

$$mK_{\mathscr{X}}+m(\mathscr{X}_s)_{\mathrm{red}}$$

is Cartier on $\mathscr{X}$ and we denote by $\mathcal{L}$ the associated line bundle. We choose a local generator θ of $\mathcal{L}$ at the point $\mathrm{red}_{\mathscr{X}}(x)$. Note that the pullback of $\mathcal{L}$ to the regular locus $\mathscr{X}_R^{\mathrm{reg}}$ of $\mathscr{X}_R$ is isomorphic to

$$\omega_{\mathscr{X}_R^{\mathrm{reg}}/R}((\mathscr{X}_s^{\mathrm{reg}})_{\mathrm{red}})^{\otimes m}.$$

We fix such an isomorphism. Then we can view ω as a rational section of $\mathcal{L}$ and write $\omega=g\theta$ locally at $\mathrm{red}_{\mathscr{X}}(x)$, with g an element of

$$\mathcal{O}_{\mathscr{X}_R,\mathrm{red}_{\mathscr{X}}(x)}\otimes_R K.$$

Then $\mathrm{wt}_\omega(x')=-\ln|g(x')|$. By (3.3.3), it is enough to show that

$$\mathrm{wt}_\omega(x)>-\ln|g(x)|.$$

Let $h:\mathscr{Y}\to\mathscr{X}$ be a log-resolution of $(\mathscr{X},\mathscr{X}_s)$. Then $\mathrm{Sk}(X_K,\omega)$ is contained in $\mathrm{Sk}(\mathscr{Y})$. We denote by Δ the log pullback of $(\mathscr{X}_s)_{\mathrm{red}}$ to $\mathscr{Y}$. Locally at $\mathrm{red}_{\mathscr{Y}}(x)$, it is explicitly given by

$$\frac{1}{m}(\mathrm{div}(h^*g)-\mathrm{div}_{\mathscr{Y}}(\omega)).$$

Since $\mathscr{X}$ is a *dlt*-model and $\mathrm{red}_{\mathscr{X}}(x)$ does not belong to $\mathscr{X}^{\mathrm{snc}}$, we know that $\Delta<(\mathscr{Y}_s)_{\mathrm{red}}$ locally around $\mathrm{red}_{\mathscr{Y}}(x)$ by Lemma 3.2.3. Therefore, we can write

$$\begin{aligned}\mathrm{wt}_\omega(x) &= v_x(\mathrm{div}_{\mathscr{Y}}(\omega)+m(\mathscr{Y}_s)_{\mathrm{red}})\\ &> v_x(\mathrm{div}_{\mathscr{Y}}(\omega)+m\Delta)\\ &= -\ln|g(x)|.\end{aligned}$$

□

Theorem 3.3.4. *If K_X is semi-ample over C and $\mathscr{X}$ is a good minimal dlt-model of X over $\mathscr{C}$, then*

$$\mathrm{Sk}(X_K)=\mathrm{Sk}(\mathscr{X}).$$

Moreover, if m is a positive integer such that $mK_{\mathscr{X}}+m(\mathscr{X}_s)_{\mathrm{red}}$ is Cartier and generated by global sections $\omega_1,\ldots,\omega_r$ over some neighbourhood of s in $\mathscr{C}$, then

$$\mathrm{Sk}(X_K)=\bigcup_{i=1}^{r}\mathrm{Sk}(X_K,\omega_i). \tag{3.3.5}$$

Proof. By Proposition 3.3.2, it is enough to show that $\mathrm{Sk}(\mathscr{X})$ is contained in the right hand side of (3.3.5). Shrinking $\mathscr{C}$ around s if necessary, we can assume that $mK_{\mathscr{X}}+m(\mathscr{X}_s)_{\mathrm{red}}$ is generated by global sections $\omega_1,\ldots,\omega_r$. Then for each point x on $\mathscr{X}_s^{\mathrm{snc}}$, we can choose an index i in $\{1,\ldots,r\}$ such that $\mathrm{div}_{\mathscr{X}^{\mathrm{snc}}}(\omega_i)+m(\mathscr{X}_s)_{\mathrm{red}}$ is an effective divisor on $\mathscr{X}$ and x is not contained in its support. This implies that the weight wt_{ω_i} of ω_i is zero at all points of $\mathrm{Sk}(\mathscr{X})\cap\mathrm{red}_{\mathscr{X}}^{-1}(x)$ and non-negative at all other points of X_K^{an}. Thus $\mathrm{Sk}(\mathscr{X})\cap\mathrm{red}_{\mathscr{X}}^{-1}(x)$ is contained in $\mathrm{Sk}(X_K,\omega_i)$. Varying the point x, we find that $\mathrm{Sk}(\mathscr{X})$ is contained in

$$\bigcup_{i=1}^{r}\mathrm{Sk}(X_K,\omega_i).$$

□

Corollary 3.3.6. *If K_X is semi-ample over C, then the essential skeleton* $\mathrm{Sk}(X)$ *is a strong deformation retract of X_K^{an}.*

Proof. This follows from Theorem 2.2.6(1), Corollary 3.2.9 and Theorem 3.3.4. □

4. The essential skeleton of a Calabi-Yau variety

4.1. **The skeleton is a pseudo-manifold.**

(4.1.1) The case where X is a family of Calabi-Yau varieties over C is of particular interest; the connections with homological mirror symmetry were the main motivation for Kontsevich and Soibelman to define the skeleton in [KS06]. If ω is a volume form on X_K (i.e., a nowhere vanishing differential form of maximal degree) then $\mathrm{Sk}(X_K)=\mathrm{Sk}(X_K,\omega)$ by [MN13, 4.6.4]. We will now prove that the underlying topological space of the essential skeleton $\mathrm{Sk}(X_K)$ is a pseudo-manifold with boundary; this result is implicitly contained in [KK10, Ko11].

(4.1.2) A topological space T is called an n-dimensional *pseudo-manifold with boundary* if it admits a triangulation $\mathscr{T}$ satisfying the following conditions:

1. (dimensional homogeneity) $T=|\mathscr{T}|$ is the union of all n-simplices.
2. (non-branching) Every $(n-1)$-simplex is a face of precisely one or two n-simplices.
3. (strong connectedness) For every pair of n-simplices σ and σ' in $\mathscr{T}$, there is a sequence of n-simplices

$$\sigma=\sigma_0,\sigma_1,\ldots,\sigma_\ell=\sigma'$$

such that the intersection $\sigma_i\cap\sigma_{i+1}$ is an $(n-1)$-simplex for all i.

We say that T is a *closed pseudo-manifold* if we can replace condition (2) by the property that every $(n-1)$-simplex is a face of precisely two n-simplices. A typical example of a 2-dimensional closed pseudo-manifold which is not a manifold is the pinched torus.

(4.1.3) For the reader's convenience, we include some basic facts about adjunction for *dlt*-pairs. We refer to Chapter 4 of [Ko13] for more background. Let (Y,Δ) be a *dlt*-pair over k, and let D be a log canonical center of (Y,Δ). Then D is normal, by [Ko13, 4.16]. There is a well defined $\mathbb{Q}$-divisor Δ_D on D, called the *different* of Δ on D [Ko13, 4.18], which is induced by the Poincaré map and satisfies the equation

$$(K_Y+\Delta)|_D = K_D+\Delta_D.$$

In the sequel, whenever we write such an equation it will be understood that Δ_D is the different of Δ on D. The pair (D,Δ_D) is again a *dlt*-pair, by [Ko13, 4.19]. Write $\lfloor\Delta\rfloor=\sum_{i\in I}D_i$. If J is a subset of I and D is a component of $\bigcap_{j\in J}\Delta_j$, it is not hard to see that for every non-empty subset J' of $I\setminus J$, every irreducible component of the intersection

$$D\cap\bigcap_{j\in J'}\Delta_j$$

is a log canonical center of (D,Δ_D) (see [Ko13, 4.19]). Conversely, by repeatedly using inversion of adjunction [Ko13, 4.9], one sees that any log canonical center of (D,Δ_D) is a log canonical center of (Y,Δ), and thus an irreducible component of an intersection $D\cap\bigcap_{j\in J'}\Delta_j$ for some non-empty subset J' of $I\setminus J$.

Theorem 4.1.4. *Assume that K_X is $\mathbb{Q}$-linearly equivalent to 0 over C. Then the underlying topological space of $\mathrm{Sk}(X_K)$ is a a pseudo-manifold with boundary.*

Proof. As we mentioned above, this result is essentially contained in [KK10, Ko11]. Using the terminology there, properties (1)-(3) of a pseudo-manifold all follow from the fact that two minimal log canonical centers of a log crepant structure are $\mathbb{P}^1$-linked in the sense of Definition 9 in [Ko11]. We will now explain this in more detail. We denote by n the relative dimension of X over C.

By Theorem 2.2.6(1), there exists a a good minimal *dlt*-model $\mathscr{X}$ of X over $\mathscr{C}$. By Theorem 3.3.4, we have $\mathrm{Sk}(X_K)=\mathrm{Sk}(\mathscr{X})$. As a triangulation on $\mathrm{Sk}(\mathscr{X})$, we take the first barycentric subdivision of the simplicial structure on $\mathrm{Sk}(\mathscr{X})$. This barycentric subdivision is necessary to guarantee that the intersection of two faces is a codimension one face of both, rather than a union of faces (think of a type I_2 degeneration of elliptic curves, whose skeleton consists of two vertices joined by two edges).

We choose an integer $m>0$ such that $mK_X\sim 0$. Since the divisor $mK_{\mathscr{X}}+m(\mathscr{X}_s)_{\mathrm{red}}$ is semi-ample over $\mathscr{C}$ and trivial over C, we see that $mK_{\mathscr{X}}+m(\mathscr{X}_s)_{\mathrm{red}}$ must be a multiple of $\mathscr{X}_s$ and thus trivial over $\mathscr{C}$. Thus we can apply Theorem 10 in [Ko11] to the *dlt*-pair $(\mathscr{X},(\mathscr{X}_s)_{\mathrm{red}})$ over $\mathscr{C}$. It states that every two minimal log canonical centers D and D^* of $(\mathscr{X},(\mathscr{X}_s)_{\mathrm{red}})$ are $\mathbb{P}^1$-linked. This means, in particular, that they have the same dimension, say $n-d$, and that there exist a sequence of $(n-d+1)$-dimensional log canonical centers $E_1,E_2,\ldots,E_\ell$ and a sequence of $(n-d)$-dimensional log canonical centers $D=D_0,D_1,\ldots,D_\ell=D^*$ such that $D_{i-1},D_i\subset E_i$ for $1\leq i\leq \ell$. In this way, we obtain properties (1) and (3) of a pseudo-manifold with boundary.

If we have two minimal log canonical centers D_1, D_2 of $(\mathscr{X},(\mathscr{X}_s)_{\mathrm{red}})$, contained in an $(n-d+1)$-dimensional log canonical center E, and if we write

$$(K_{\mathscr{X}}+(\mathscr{X}_s)_{\mathrm{red}})|_E=K_E+D_1+D_2+\Delta$$

for some $\Delta\geq 0$, then $(E,D_1+D_2+\Delta)$ is again a *dlt*-pair [Ko13, 4.19]. Moreover, D_1 cannot intersect D_2 or $\lfloor\Delta\rfloor$ because the intersection would be a union of log canonical centers of $(\mathscr{X}, (\mathscr{X}_s)_{\mathrm{red}})$, which contradicts the minimality of D_1. Thus we are in the situation of the second part of the proof of Theorem 10 in [Ko11]. That proof shows that D_1 and D_2 are the only log canonical centers of $(E, D_1+D_2+\Delta)$. Property (2) follows. □

(4.1.5) We can say more in the case where $\mathrm{Sk}(\mathscr{X})$ has maximal dimension, that is, dimension equal to $n = \dim(X_K)$. First, we need a lemma.

Lemma 4.1.6. *Let $(Z, \Delta = \sum_{i=1}^{j} \Delta_i)$ be a reduced dlt-pair over k such that $K_Z+\Delta$ is Cartier. Let D be a log canonical center and let $\Delta_1, \dots, \Delta_\ell$ be the irreducible components of Δ that contain D. Let U be the maximal open subset of Z where Z is smooth and Δ is a divisor with strict normal crossings. If we write $(K_Z + \Delta)|_D = K_D + \Delta_D$, then Δ_D is equal to the closure of the restriction of $\sum_{i=\ell+1}^{j} \Delta_i|_U$ to $U \cap D$.*

Proof. We first notice that in the above statement, U can be replaced by any smaller open set that meets all the log canonical centers: the closure of the restriction of $\sum_{i=\ell+1}^{j} \Delta_i|_U$ to $U \cap D$ will yield the same divisor on D.

Then by induction, we only need to treat the case where D is a component of Δ, say Δ_1. If we take a log resolution $f : Y \to (Z, \Delta)$ and let $D' = \Delta_1'$ be the birational transform of D, then Δ_D can be computed as follows: if we write $f^*(K_Z + \Delta)|_{D'} = K_{D'} + \Delta_{D'}$ then $\Delta_D = (f|_{D'})_*(\Delta_{D'})$. In particular, as $K_Z + \Delta$ is Cartier, we know that Δ_D is a integral divisor. Since it is effective, and all the components of Δ_D are log canonical centers of (X, Δ), we see that Δ_D must be equal to the closure of the restriction of $\sum_{i=2}^{j} \Delta_i|_U$. □

Theorem 4.1.7. *Assume that k is algebraically closed, K_X is trivial over C and X has an snc-model $\mathscr{Y}$ with reduced special fiber $\mathscr{Y}_s$. Assume moreover that $\mathrm{Sk}(X_K)$ is of dimension $n = \dim(X_K)$. Then $\mathrm{Sk}(X_K)$ is an n-dimensional closed pseudo-manifold.*

Proof. By running MMP for $\mathscr{Y}$ over $\mathscr{C}$, we know that X has a good minimal dlt model $\mathscr{X}$ with reduced special fiber (see [Fu11] or [HX13]). Then one sees as in the proof of Theorem 4.1.4 that $K_{\mathscr{X}}$ is trivial over $\mathscr{C}$. Our assumption on the dimension of $\mathrm{Sk}(X_K)$ implies that the minimal log canonical centers of $(\mathscr{X}, \mathscr{X}_s)$ are points. Let D be a one-dimensional log canonical center, and let D_i $(1 \le i \le \ell)$ be the 0-dimensional log canonical centers contained in D. From Lemma 4.1.6, we know that

$$(K_{\mathscr{X}} + \mathscr{X}_s)|_D = K_D + \sum_{i=1}^{\ell} D_i \sim 0.$$

Thus D is a rational curve and $\ell = 2$, which means that $\mathrm{Sk}(X_K)$ is closed. □

(4.1.8) In Theorem 4.1.7, the condition that $\mathrm{Sk}(X)$ has maximal dimension can not be omitted; for instance, there are examples of semi-stable degenerations of K3-surfaces with trivial relative canonical sheaf where the special fiber is a chain of surfaces, so that the skeleton is homeomorphic to a closed interval. We will now give an interpretation of this condition in terms of the monodromy around $s \in \mathscr{C}$.

Lemma 4.1.9. *Let Y be a connected smooth and proper K-variety and let ω be a non-zero m-pluricanonical form on Y, for some $m > 0$. Let K' be a finite extension of K, set $Y' = Y \times_K K'$ and denote by ω' the pullback of ω to Y'. Then the skeleton* $\mathrm{Sk}(Y,\omega)$ *is the image of* $\mathrm{Sk}(Y',\omega')$ *under the projection morphism* $\pi : (Y')^{\mathrm{an}} \to Y^{\mathrm{an}}$.

Proof. We may assume that K' is Galois over K. Let d be the ramification index of K' over K. We will prove that

$$\mathrm{wt}_{\omega'}(y) = d \cdot \mathrm{wt}_{\omega}(\pi(y)) - d + 1$$

for every divisorial point y on $(Y')^{\mathrm{an}}$ (see [MN13, 2.4.10] for the notion of divisorial point). This immediately implies the statement in the lemma, since $\mathrm{Sk}(X,\omega)$ is the closure of the set of divisorial points where the weight function reaches its minimal value [MN13, 4.5.1].

We denote by R' the integral closure of R in K'. Let $\mathscr{Y}'$ be a regular separated R'-scheme of finite type with irreducible special fiber $\mathscr{Y}'_k$, endowed with an isomorphism of K'-schemes $\mathscr{Y}'_{K'} \to Y'$. Let y be the unique point in $\mathrm{red}^{-1}_{\mathscr{Y}'}(\xi)$, where ξ denotes the generic point of $\mathscr{Y}'_k$. Removing a closed subset of $\mathscr{Y}'_k$ if necessary, we can find a regular separated R-scheme of finite type $\mathscr{Y}$ and an isomorphism $\mathscr{Y}_K \to Y$ such that $\mathscr{Y}'$ is an open subscheme of the normalization of $\mathscr{Y} \times_R R'$. Then $\mathrm{red}_{\mathscr{Y}}(\pi(y))$ is a generic point of $\mathscr{Y}_k$.

If we use the notations from (3.2.2) and denote by $(S')^+$ the log scheme associated to $R' \setminus \{0\} \to R'$, then the $(S')^+$-log scheme $(\mathscr{Y}')^+$ is isomorphic to an open log subscheme of the fs base change of $\mathscr{Y}^+$ from S^+ to $(S')^+$. Since log differentials are compatible with fs base change, we can deduce from the description of the weight function in (3.2.2) that

$$\mathrm{wt}_{\omega'}(y) = d \cdot \mathrm{wt}_{\omega}(\pi(y)) - d + 1$$

(the scaling factor d is caused by the renormalization of the discrete valuation on K'). □

Theorem 4.1.10. *Assume that $k = \mathbb{C}$ and denote by n the relative dimension of X over C. Suppose that X is projective over C and that K_X is trivial over C. Let F be a general fiber of the morphism $X \to C$. Then* $\mathrm{Sk}(X_K)$ *has dimension n if and only if the monodromy transformation around $s \in \mathscr{C}$ on $H^n(F(\mathbb{C}),\mathbb{Q})$ has a Jordan block of size $n+1$. If this holds, and $h^{i,0}(F) = 0$ for $0 < i < n$, then* $\mathrm{Sk}(X_K)$ *is a $\mathbb{Q}$-homology sphere.*

Proof. By Lemma 4.1.9 and the Semi-Stable Reduction Theorem we can assume that X has a projective *snc*-model $\mathscr{Y}$ over $\mathscr{C}$ such that $\mathscr{Y}_s$ is reduced. For every integer $i \geq 0$, we denote by

$$\mathbf{H}^i = \mathbb{H}^i(\mathscr{Y}_s, R\psi_{\mathscr{Y}}(\mathbb{Z})) \cong H^i(F(\mathbb{C}),\mathbb{Z})$$

the degree i nearby cohomology of $\mathscr{Y}$ at s; here $R\psi_{\mathscr{Y}}(\mathbb{Z})$ denotes the complex of nearby cycles with $\mathbb{Z}$-coefficients associated to $\mathscr{Y}$. By [St76], the spaces $\mathbf{H}^i$ carry a canonical mixed Hodge structure, whose weight filtration coincides with the monodromy filtration. In particular, there exists a Jordan block of monodromy of size $n+1$ on $\mathbf{H}^n_{\mathbb{Q}}$ if and only if $W_0\mathbf{H}^n_{\mathbb{Q}} \neq 0$.

By [Be09, 5.1] and its proof, the $\mathbb{Q}$-vector space $W_0\mathbf{H}^i_{\mathbb{Q}}$ is canonically isomorphic to the degree i singular cohomology of X_K^{an}, for every $i \geq 0$. Since X_K^{an} is homotopy equivalent to $\mathrm{Sk}(X_K)$ by Corollary 3.3.6, we see that $W_0\mathbf{H}^n_{\mathbb{Q}}$ can only be different from zero if the dimension of $\mathrm{Sk}(X_K)$ is equal to n. We will now prove the converse implication. Suppose that $\mathrm{Sk}(X_K)$ has dimension n and let ω be a relative volume form on X over C such that ω extends to a global section of $\omega_{\mathscr{Y}/\mathscr{C}}(\log \mathscr{Y}_s)$ that generates $\omega_{\mathscr{Y}/\mathscr{C}}(\log \mathscr{Y}_s)$ at at least one generic point of $\mathscr{Y}_s$ (modulo shrinking $\mathscr{C}$, such ω always exists). Then it follows from [MN13, 4.5.5] that $\mathrm{Sk}(X_K)$ is the simplicial subspace of $\mathrm{Sk}(\mathscr{Y})$ spanned by the vertices corresponding to the irreducible components E of $\mathscr{Y}_s$ such that ω generates $\omega_{\mathscr{Y}/\mathscr{C}}(\log \mathscr{Y}_s)$ at the generic point of E. Since $\mathrm{Sk}(X_K)$ has dimension n, we can find such components $E_1, \ldots, E_n$ that intersect in a point. Denote by D the union of n-fold intersection points of components of $\mathscr{Y}_s$. Then by reduction modulo t, ω induces an element of

$$H^0(\mathscr{Y}_s, \omega_{\mathscr{Y}/\mathscr{C}}(\log \mathscr{Y}_s) \otimes \mathcal{O}_{\mathscr{Y}_s})$$

whose image under the Poincaré residue map

$$\mathcal{R} : H^0(\mathscr{Y}_s, \omega_{\mathscr{Y}/\mathscr{C}}(\log \mathscr{Y}_s) \otimes \mathcal{O}_{\mathscr{Y}_s}) \to H^0(\mathscr{Y}_s, \mathrm{Gr}^W_{-n}(\omega_{\mathscr{Y}/\mathscr{C}}(\log \mathscr{Y}_s) \otimes \mathcal{O}_{\mathscr{Y}_s})) \cong H^0(D, \mathcal{O}_D)$$

is different from zero. However, by the degeneration of the Hodge and weight spectral sequences, the image of $\mathcal{R}$ injects into $W_0\mathbf{H}^n_{\mathbb{C}}$. Thus $W_0\mathbf{H}^n_{\mathbb{Q}}$ is non-trivial.

Finally, assume that $\mathrm{Sk}(X_K)$ has dimension n and that $h^{i,0}(X_{\mathrm{gen}}) = 0$ for $0 < i < n$. Then

$$\mathrm{Gr}^0_F \mathbf{H}^i_{\mathbb{C}} \cong H^i(\mathscr{Y}_s, \mathcal{O}_{\mathscr{Y}_s}) = 0$$

for $0 < i < n$ and

$$\mathrm{Gr}^0_F \mathbf{H}^i_{\mathbb{C}} \cong H^i(\mathscr{Y}_s, \mathcal{O}_{\mathscr{Y}_s}) \cong \mathbb{C}$$

for $i = 0, n$ by the degeneration of the Hodge spectral sequence for the limit mixed Hodge structure. Thus $W_0\mathbf{H}^i_{\mathbb{Q}} = 0$ for $0 < i < n$, $W_0\mathbf{H}^0_{\mathbb{Q}} \cong \mathbb{Q}$ and $W_0\mathbf{H}^n_{\mathbb{Q}}$ has dimension at most one; it must have dimension one since we have already proven that it is non-zero. It follows that $\mathrm{Sk}(X_K)$ is a $\mathbb{Q}$-homology sphere. □

4.2. **Removing the algebraicity condition.**

(4.2.1) In this section, we will extend Theorems 3.2.8, 4.1.4, 4.1.7 and 4.1.10 to the case where X is a Calabi-Yau variety over $K = k((t))$ instead of over the curve C. The crucial point is that the skeleton of a Calabi-Yau variety can be computed from the logarithmic structure on the special fiber of any *snc*-model.

Proposition 4.2.2. *Let $\mathscr{Y}$ be a connected regular flat proper R-scheme such that $\mathscr{Y}_k$ is a strict normal crossings divisor. Then for every connected flat proper R-scheme $\mathscr{X}$ and every isomorphism of $R/(t^2)$-schemes*

$$f : \mathscr{Y} \times_R R/(t^2) \to \mathscr{X} \times_R R/(t^2),$$

the following properties hold.

1. *The scheme $\mathscr{X}$ is regular and $\mathscr{X}_k$ is a divisor with strict normal crossings.*
2. *Denote by S^+ the log scheme associated to $R \setminus \{0\} \to R$ and by $\mathscr{Y}^+$ and $\mathscr{X}^+$ the schemes $\mathscr{Y}$ and $\mathscr{X}$ endowed with the divisorial log structures associated to their special fibers. For every integer $d > 0$ we denote by s^+_d the standard log point* $(\mathrm{Spec}\, k, k^* \oplus \mathbb{N})$ *viewed as a log scheme over S^+* via *the morphism of charts $\mathbb{N} \to \mathbb{N} : n \mapsto dn$. If we denote by e the least common multiple of the multiplicities of the components of $\mathscr{Y}_k$, then there exists an isomorphism of log schemes*

$$g : \mathscr{Y}^+ \times_{S^+} s^+_e \to \mathscr{X}^+ \times_{S^+} s^+_e$$

over s_e^+, such that g is compatible with the reduction of f modulo t (meaning that the obvious square in the category of k-schemes commutes).

Proof. It is easy to see that (1) holds, since we can detect regularity by looking at the dimensions of the Zariski tangent spaces at the points of

$$\mathscr{Y}\times_R R/(t^2)\cong \mathscr{Z}\times_R R/(t^2).$$

Moreover, the special fibers of $\mathscr{Y}$ and $\mathscr{Z}$ are isomorphic so that $\mathscr{Z}_k$ is a divisor with strict normal crossings. Point (2) is more subtle and follows from [Ki03, 2.6(2)]. □

Proposition 4.2.3. *Let $\mathscr{Y}$ be a connected regular flat proper R-scheme such that $\mathscr{Y}_K$ has trivial canonical sheaf and $\mathscr{Y}_k$ is a strict normal crossings divisor. Then the skeleta $\mathrm{Sk}(\mathscr{Y})$ and $\mathrm{Sk}(\mathscr{Y}_K)$ only depend on $\mathscr{Y}\times_R R/(t^2)$, in the following sense. Assume that $\mathscr{Z}$ is a regular flat proper R-scheme such that $\mathscr{Z}_K$ has trivial canonical sheaf and there exists an isomorphism of $R/(t^2)$-schemes*

$$f:\mathscr{Y}\times_R R/(t^2)\to \mathscr{Z}\times_R R/(t^2).$$

Then there exists an isomorphism of simplicial spaces $\mathrm{Sk}(\mathscr{Y})\to\mathrm{Sk}(\mathscr{Z})$ that maps $\mathrm{Sk}(\mathscr{Y}_K)$ onto $\mathrm{Sk}(\mathscr{Z}_K)$.

Proof. Reducing f modulo t, we obtain an isomorphism of k-schemes $\mathscr{Y}_k\to\mathscr{Z}_k$ and, by taking the dual intersection complexes, an isomorphism of simplicial spaces with piecewise $\mathbb{Z}$-affine structure $\mathrm{Sk}(\mathscr{Y})\to\mathrm{Sk}(\mathscr{Z})$. We will prove that this isomorphism maps $\mathrm{Sk}(\mathscr{Y}_K)$ onto $\mathrm{Sk}(\mathscr{Z}_K)$.

We use the notations from Proposition 4.2.2(2) and we set $s^+=s_1^+$. We denote by $\mathscr{Y}_k^+$ the log scheme $\mathscr{Y}^+\times_{S^+}s^+$ obtained by restricting the log structure on $\mathscr{Y}^+$ to the special fiber $\mathscr{Y}_k$ of $\mathscr{Y}$. It follows from [IKN05, 7.1] that

$$\Omega := H^0(\mathscr{Y},\omega_{\mathscr{Y}^+/S^+})$$

is a free R-module of rank one and that the reduction map

$$\Omega\otimes_R k\to\Omega_k := H^0(\mathscr{Y}_k,\omega_{\mathscr{Y}_k^+/s^+})$$

is an isomorphism. Let ω be a generator of the R-module Ω and denote by ω_k its image in Ω_k. By (3.2.2), the generic point ξ of an irreducible component E of $\mathscr{Y}_k$ is ω-essential in the sense of [MN13, 4.5.4] if and only if ω_k generates $\omega_{\mathscr{Y}_k^+/s^+}$ at the point ξ. Moreover, the skeleton $\mathrm{Sk}(\mathscr{Y}_K)=\mathrm{Sk}(\mathscr{Y}_K,\omega)$ is the simplicial subspace of $\mathrm{Sk}(\mathscr{Y})$ spanned by the vertices corresponding to such points ξ [MN13, 4.5.5]. However, for every integer $d>0$, the stalk of $\omega_{\mathscr{Y}_k^+/s^+}$ at ξ is generated by global sections if and only if $\omega_{\mathscr{Y}^+\times_{S^+}s_d^+/s_d^+}$ is generated by global sections at any point lying above ξ, by the base change property in [IKN05, 7.1]. The analogous statements hold for $\mathscr{Z}$. Thus it follows from Proposition 4.2.2(2) that the isomorphism $\mathrm{Sk}(\mathscr{Y})\to\mathrm{Sk}(\mathscr{Z})$ maps $\mathrm{Sk}(\mathscr{Y}_K)$ onto $\mathrm{Sk}(\mathscr{Z}_K)$. □

Theorem 4.2.4. *Let X be a geometrically connected, smooth and proper K-variety with trivial canonical sheaf. Then the following properties hold.*

1. *The essential skeleton $\mathrm{Sk}(X)$ is a strong deformation retract of X^{an}.*
2. *If $\mathscr{X}$ is a proper snc-model of X over R, then $\mathrm{Sk}(X)$ is contained in $\mathrm{Sk}(\mathscr{X})$ and can be obtained from $\mathrm{Sk}(\mathscr{X})$ (as a topological subspace of $\mathrm{Sk}(\mathscr{X})$ with piecewise affine structure) by a finite number of elementary collapses.*

(3) *The essential skeleton* $\mathrm{Sk}(X)$ *is a pseudo-manifold with boundary. If* k *is algebraically closed and* $\mathrm{Sk}(X)$ *has dimension* $\dim(X)$*, then it is a closed pseudo-manifold.*

(4) *Assume that* k *is algebraically closed and* X *is projective. Let* σ *be a topological generator of the absolute Galois group* $G(K^a/K)$ *and let* ℓ *be a prime. Then* $\mathrm{Sk}(X_K)$ *has dimension* $n = \dim(X)$ *if and only if the action of* σ *on*

$$H^n_{\text{ét}}(X \times_K K^a, \mathbb{Q}_\ell)$$

has a Jordan block of size $n+1$*. If this holds, and* $h^{i,0}(X) = 0$ *for* $0 < i < n$*, then* $\mathrm{Sk}(X_K)$ *is a* $\mathbb{Q}$*-homology sphere.*

Proof. Let $\mathscr{X}$ be a proper *snc*-model of X over R. By a standard argument based on spreading out and Greenberg Approximation (as explained in [MN13, 5.1.2], for instance) we can find a connected smooth k-curve $\mathscr{C}$, a k-rational point s on $\mathscr{C}$, a uniformizer t in $\mathcal{O}_{\mathscr{C},s}$ and a smooth and proper $\mathscr{C}$-scheme $\mathscr{X}'$ with geometrically connected fibers such that there exists an isomorphism

$$\mathscr{X} \times_R R/(t^2) \to \mathscr{X}' \times_{\mathscr{C}} \operatorname{Spec} \mathcal{O}_{\mathscr{C},s}/(t^2)$$

over $R/(t^2) \cong O_{\mathscr{C},s}/(t^2)$. Inspecting the proof of [MN13, 5.1.2], we see that we can also assume that the relative canonical sheaf of $\mathscr{X}'$ is trivial over $C = \mathscr{C} \setminus \{s\}$ (if the generic fiber of a smooth and proper family over an integral scheme has trivial canonical sheaf, then this holds for all fibers over some dense open subscheme of the base).

By (3.1.4) we know that $\mathrm{Sk}(\mathscr{X})$ is a strong deformation retract of X_K^{an}. Thus by Proposition 4.2.3, it suffices to prove assertions (1)–(3) for $\mathscr{X}' \times_{\mathscr{C}} \operatorname{Spec} K$ instead of X. In this case, they follow from Corollary 3.3.6 and Theorems 3.2.8, 3.3.4, 4.1.4 and 4.1.7.

It remains to prove (4). Invoking the Lefschetz Principle, we may assume that $k = \mathbb{C}$. Taking for $\mathscr{X}$ a projective *snc*-model over R, we can arrange that $\mathscr{X}'$ is projective over $\mathscr{C}$. By Proposition 4.2.2 and the theory of logarithmic nearby cycles [Na98, 3.3] the action of σ on

$$H^n_{\text{ét}}(X \times_K K^a, \mathbb{Q}_\ell)$$

has a Jordan block of size $n+1$ if and only if the corresponding statement holds for $\mathscr{X}'_K$. By Deligne's comparison theorem for étale and complex analytic nearby cycles in [SGA7b, Exp.XIV], it is also equivalent to the property that the monodromy action on the degree n singular cohomology of a general fiber of $\mathscr{X}'$ has a Jordan block of size $n+1$. If $h^{i,0}(X) = 0$ for $0 < i < n$, then we can assume that this also holds for a general fiber of $\mathscr{X}'$, by the proof of [MN13, 5.1.2] (if this property is satisfied by the generic fiber of a smooth and proper family over an integral scheme, then it holds for all fibers over a dense open subscheme of the base, by semi-continuity). Thus the assertion (4) follows from Theorem 4.1.10. □

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