THE ESSENTIAL SPECTRUM OF A HANKEL OPERATOR WITH PIECEWISE CONTINUOUS SYMBOL

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A Hankel operator S on a complex Hilbert space with complete orthonormal basis $\{e_n; n = 0, 1, 2, \ldots\}$ is one whose representing matrix has the form

$$S_{ij} = c_{i+j}, \qquad i, j = 0, 1, 2, \ldots .$$

A classical theorem of Nehari [6] shows that a sequence $(c_n)_{n=0}^{\infty}$ defines a bounded Hankel operator if and only if it is the sequence of positive Fourier coefficients of an essentially bounded measurable function ϕ on the unit circle. Hartman subsequently showed that S is compact if and only if ϕ can be chosen to be continuous (see [4] or [1]).

In this note we determine the essential spectrum of S when ϕ is a function possessing left and right limits at every point on the circle.

Notation. Let L^2 be the Hilbert space of square integrable functions on the unit circle T with the usual orthonormal basis $\{z^n; n = 0, \pm 1, \pm 2, \ldots\}$. The unitary operator J on L^2 is defined by $Jz^n = z^{-n}$ and we shall let P denote the orthogonal projection of L^2 onto the Hardy subspace H^2 spanned by $\{z^n; n = 0, 1, 2, \ldots\}$.

For an essentially bounded measurable function ϕ in L^∞, the Toeplitz operator T_ϕ, on H^2, is defined by $T_\phi = PM_\phi \mid H^2$ where M_ϕ is the usual multiplication operator on L^2. We call ϕ the symbol of the Toeplitz operator T_ϕ. The Hankel operator on H^2, with symbol ϕ in L^∞, is defined by $S_\phi = PJM_\phi \mid H^2$.

Let PC denote the collection of functions on T which possess left and right limits at each point. For ϕ in PC and α in T we shall write

$$\phi_\alpha = \frac{1}{2} \lim_{t \to 0+} \{\phi(\alpha e^{it}) - \phi(\alpha e^{-it})\}$$

and call ϕ_α the jump of ϕ at α.

Let T′ denote the non-real points of T and, for $\gamma, \nu \in \mathbb{C}$, let $[\gamma, \nu]$ denote the line segment joining γ and ν. We shall prove the following:

THEOREM 1. Let ϕ be a function in PC. Then

$$\sigma_e(S_\phi) = [0, i\phi_1] \cup [0, i\phi_{-1}] \cup \bigcup_{\alpha \in T'} [-(-\phi_\alpha \phi_{\bar\alpha})^{1/2}, +(-\phi_\alpha \phi_{\bar\alpha})^{1/2}].$$

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In particular notice that a jump at α only contributes to the essential spectrum if it is accompanied by a jump at $\bar{\alpha}$. The key results we shall use are the following two theorems of Gohberg and Krupnik [3] (see also [2] p.20) concerning Toeplitz operators with PC symbols.

THEOREM 2. *Let ϕ and ψ be piecewise continuous functions. Then*

$$T_{\phi\psi} - T_{\phi}T_{\psi}$$

is compact if and only if ϕ and ψ do not have common points of discontinuity.

For ϕ in PC define $\hat{\phi}$ on $T \times [0,1]$ by $\hat{\phi}(e^{it}, s) = s\phi(e^{it-}) + (1-s)\phi(e^{it+})$.

THEOREM 3. *If $\{\phi_{ij}\}_{i=1}^{n}{}_{j=1}^{m}$ are functions in* PC *then* $\sum_i \prod_j T_{\phi_{ij}}$ *is Fredholm if and only if* $\hat{\phi} = \sum_i \prod_j \hat{\phi}_{ij} \neq 0$.

These two theorems can be related to Hankel operators by means of the following formula which has proved useful in other situations (*e.g.*, [9] and [7]). For ϕ in L^∞ let $\tilde{\phi}(z) = \phi(\bar{z})$.

LEMMA 4. *For ϕ and ψ in L^∞ we have* $S_\phi S_\psi = T_{\tilde{\phi}\psi} - T_{\tilde{\phi}z}T_{\psi\bar{z}}$.

Proof.

$$\begin{aligned} S_\phi S_\psi &= PJM_\phi PJM_\psi \,|H^2 = PM_{\tilde{\phi}z}(M_{\bar{z}}JPJM_z)M_{\bar{z}\psi}\,|H^2 \\ &= PM_{\tilde{\phi}z}(I-P)M_{\bar{z}\psi}\,|H^2 = T_{\tilde{\phi}\psi} - T_{\tilde{\phi}z}T_{\psi\bar{z}}. \end{aligned}$$

We shall prove Theorem 1 through a series of lemmas which deal with simple symbol functions in PC.

Let $\psi(e^{it}) = i(t-\pi)e^{it}$, $0 \le t < 2\pi$. Then $\psi \in PC$ and has a single jump at 1 where $\psi_1 = -i\pi$. A simple computation shows that

$$(S_\psi z^n, z^m) = (n+m+1)^{-1}, \qquad n, m = 0, 1, 2, \dots,$$

and so S_ψ is the Hankel operator defined by Hilbert's matrix. Magnus [5] has shown that the essential spectrum of this operator is the interval $[0,\pi]$ (*i.e.*, $[0, i\psi_1]$). Alternatively by Lemma 4, $S_\psi^2 = T_{(t-\pi)^2} - T_{(t-\pi)}{}^2$. Theorem 3 can be applied to show that $\sigma_e(S_\psi^2) = [0, \pi^2]$ and, since S_ψ is positive and $\|S_\psi\| \le \|\psi\| \le \pi$, we have proved Magnus's result.

LEMMA 5. *Let ϕ be continuous apart from a jump at α. Then*

(i) *If $\alpha = 1$ then* $\sigma_e(S_\phi) = [0, i\phi_1]$

(ii) *If $\alpha = -1$ then* $\sigma_e(S_\phi) = [0, i\phi_{-1}]$.

Proof. (i) We may write $\phi = \lambda\psi + \theta$ where $\lambda \in \mathbb{C}$ and θ is a continuous function. Since, by Hartman's theorem, S_θ is compact, we have

$$\sigma_e(S_\phi) = \sigma_e(\lambda S_\psi) = [0, i\lambda\psi_1] = [0, i\phi_1].$$

(ii) Let V be the unitary operator on L^2 defined by $(Vf)(e^{it}) = f(-e^{it})$. Then V commutes with J and P and $V^*S_\phi V = S_{\phi'}$ where $\phi'(e^{it}) = \phi(-e^{it})$. By (i)

$$\sigma_e(S_\phi) = \sigma_e(S_{\phi'}) = [0, i\phi'_1] = [0, i\phi_{-1}].$$

LEMMA 6. *Let ϕ be continuous apart from a jump at α and a jump at $\bar\alpha$ where α is a non-real point of* T. *Then $\sigma_e(S_\phi)$ is the line segment*

$$[-(-\phi_\alpha\phi_{\bar\alpha})^{1/2}, +(-\phi_\alpha\phi_{\bar\alpha})^{1/2}].$$

Proof. Let $\phi_\alpha = \lambda_1$ and $\phi_{\bar\alpha} = \lambda_2$. Since we may add a continuous function to the symbol without altering the essential spectrum we may suppose that $\phi(\alpha-) = 0$, $\phi(\alpha+) = 2\lambda_1$, $\phi(\bar\alpha-) = 0$, and $\phi(\bar\alpha+) = 2\lambda_2$. By Lemma 4,

$$S_\phi^2 = T_{\bar\phi\phi} - T_{\bar\phi z}T_{\phi\bar z}$$

so that by Theorem 1 $\sigma_e(S_\phi^2)$ is the range of $\widehat{\bar\phi\phi} - \widehat{\bar\phi z}\,\widehat{\phi\bar z}$ on $T \times [0,1]$. Since ϕ is continuous except possibly at α and $\bar\alpha$ it suffices to consider the range of this function on $\alpha \times [0,1]$ and $\bar\alpha \times [0,1]$. Now $\bar\phi\phi$ is continuous near α and $\bar\alpha$ and vanishes at α and $\bar\alpha$, thus $\sigma_e(S_\phi^2)$ is given by the range of $-\hat{\bar\phi}\,\hat\phi$ on $\alpha \times [0,1]$ and $\bar\alpha \times [0,1]$. Since $\hat{\bar\phi}\,\hat\phi\,(\alpha, s) = \hat{\bar\phi}\,\hat\phi\,(\bar\alpha, s) = 4\lambda_1\lambda_2 s(1-s)$ we see that $\sigma_e(S_\phi^2) = [0, -\lambda_1\lambda_2]$.

We now show that $\sigma_e(S_\phi) = -\sigma_e(S_\phi)$ which will complete the proof of the lemma.

Let θ be a function on T which is continuous apart from a proper jump at α. It follows that there exist complex numbers ν_1, ν_2 and a continuous function ϕ' on T such that $\phi = \nu_1\theta + \nu_2\bar{\bar\theta} + \phi'$. Thus, since $S_\theta^* = S_{\bar\theta}$ it will be sufficient to show that $\sigma_e(S) = -\sigma_e(S)$ where $S = \nu_1 S_\theta + \nu_2 S_\theta^*$. Since $S_\theta^2 = T_{\bar\theta\theta} - T_{\bar\theta z}T_{\theta\bar z}$, it follows from Theorem 2 that S_θ^2 is compact. Let π be the homomorphism of $B(H^2)$ into the Calkin algebra A, and let Φ be a faithful representation of A as a C*-algebra of operators on a Hilbert space. Then $(\Phi\circ\pi(S_\theta))^2 = 0$. By a result of Radjavi and Rosenthal ([8] Theorem 1) $(\Phi\circ\pi)(S_\theta)$ has the form $\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$ Thus $(\Phi\circ\pi)(S)$ has the form $\begin{pmatrix} 0 & \nu_1 C \\ \nu_2 C^* & 0 \end{pmatrix}$ and so $\sigma(\Phi\circ\pi(S)) = -\sigma(\Phi\circ\pi(S))$ which implies

$$\sigma(\pi(S)) = -\sigma(\pi(S)),$$

completing the proof.

LEMMA 7. *Let $(a_n)_{n=1}^\infty$ be elements of a complex unital commutative Banach algebra* A *such that* $a = \sum_{n=1}^{\infty} a_n$ *converges in norm and*

$$a_n a_m = a_m a_n = 0 \qquad \textit{for } n \neq m.$$

Then $\sigma(a) \cup \{0\} = \bigcup_{n=1}^{\infty} \sigma(a_n)$.

Proof. Let M be the set of multiplicative linear functionals ϕ on A, so that $\sigma(b) = \{\phi(b); \phi \in M\}$ for b in A. Since $\phi(a) = \sum_{n=1}^{\infty} \phi(a_n)$ and $\phi(a_n) \neq 0$ implies $\phi(a_m) = 0$ for $m \neq n$, the result follows.

We now put together the pieces.

Proof of Theorem 1. We first show that the theorem is true if ϕ is a piecewise continuous function. In this case we can write $\phi = \phi' + \phi'' + \sum_{i=1}^{n} \phi^{(\alpha_i)}$ where ϕ' (resp. ϕ'') is continuous apart from possibly a jump at 1 (resp. -1) and $\phi^{(\alpha_i)}$ is continuous apart from possible jumps at α_i and $\bar{\alpha}_i$. Since, by Theorem 2 and Lemma 4 any pair of operators A, B from $\{\pi(S_{\phi'}), \pi(S_{\phi''}), \pi(S_{\phi(\alpha_i)}); i = 1, \ldots, n\}$ satisfies $AB = 0$ the theorem follows from lemmas 5, 6 and 7.

Suppose now that $\phi \in PC$. We first show that there exists a sequence of piecewise continuous functions $\phi^{(n)}$, $n=1,2,\ldots$, such that

(i) $\phi^{(n)}$ and $\tilde{\phi}^{(m)}$ have no common discontinuities for $n \neq m$.

(ii) $|\psi_\alpha^{(n)}| \leq 2^{-n}$ for $\alpha \in T$, where $\psi^{(n)} = \phi - \sum_{i=1}^{n} \phi^{(i)}$.

Let $\Lambda_n' = \{\alpha: |\phi_\alpha| \geq 2^{-n}\}$. Since $\phi \in PC$ it follows that Λ_n' is finite. Let

$$\Lambda_n'' = \{\bar{\alpha}; \alpha \in \Lambda_n'\} \cup \Lambda_n'$$

and let $\Lambda_1 = \Lambda_1''$, $\Lambda_{n+1} = \Lambda_{n+1}'' \setminus \Lambda_n''$, $n = 1, 2, \ldots$. Now choose $\phi^{(n)}$ to be any piecewise continuous function such that $\phi^{(n)}$ is continuous on $T \setminus \Lambda_n$ and

$$\phi_\alpha^{(n)} = \phi_\alpha \qquad \text{for } \alpha \in \Lambda_n.$$

Then the $\phi^{(n)}$ satisfy (i) and (ii).

By theorem 2 and Lemma 4, (i) shows that $\pi(S_{\phi(n)})\pi(S_{\phi(m)}) = 0$ for $n \neq m$. Also the second condition (ii) shows that $\|\pi(S_{\psi(n)})\| \leq 2.2^{-n}$. This can be seen by first approximating $\psi^{(n)}$ by a piecewise continuous function θ so that $\|\psi^{(n)} - \theta\| \leq \varepsilon$. Since $|\theta_\alpha| \leq 2^{-n} + \frac{1}{2}\varepsilon$ for $\alpha \in T$, there exists a continuous function θ' such that $\|\theta - \theta'\| \leq 2.2^{-n} + \varepsilon$. Thus

$$\|\pi(S_{\psi(n)})\| = \|\pi(S_{\psi(n)-\theta'})\| \leq \|\psi^{(n)} - \theta'\| \leq 2(2^{-n} + \varepsilon).$$

So $\pi(S_\phi) = \sum_{n=1}^{\infty} \pi(S_{\phi(n)})$ and the theorem follows from Lemma 7 and the first part of this proof.

Remarks. Although zero is always a point in the essential spectrum of a Hankel operator it is not always true that the essential spectrum is 'connected to the origin' as in Theorem 1. To see the first half of this statement suppose that A is a left inverse for the Hankel operator S modulo the compacts, so that $AS = I + K$ for some compact operator K. Since $SU^n = U^{*n}S$, where U is the shift on H^2, we have $(I + K)U^n = AU^{*n}S$, and so, for $x \in H^2$, $U^n x = AU^{*n}Sx - KU^n x$. However, $U^n x \to 0$ weakly and so $KU^n x \to 0$ in norm. Since $U^{*n}Sx \to 0$ in norm also, we have a contradiction when $x \neq 0$. The second half of our initial remark follows by considering the Hankel operator $S_{z\bar{\phi}}$ where ϕ is the inner function

$$\phi(z) = \exp\left(\frac{1 + z}{z - 1}\right).$$

It can be shown that $S_{z\bar{\phi}}$ is a self-adjoint partial isometry (a partial symmetry?) and $\sigma(S_{z\bar{\phi}}) = \sigma_e(S_{z\bar{\phi}}) = \{-1\} \cup \{0\} \cup \{1\}$.

Just how arbitrary can the spectrum or essential spectrum of a Hankel operator be? In particular, is any compact subset of the complex plane which contains the origin the spectrum of a Hankel operator?

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