

**The estimation of the elasticity of substitution
of a CES production function :
Case of Tunisia**

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Abstract

This paper proposes to estimate the elasticity of substitution resulting from the CES (Constant Elasticity of Substitution) production function. The majority of the modellers borrows these elasticities from literature within the framework of the construction of their Computable General Equilibrium models. This choice is carried out because of the non-availability and/or the shortage of the statistics of the economies. This loan is applied to the majority of the developing countries. In the aim to change the tradition of certain modellers, we try with this document to present an estimation of elasticity of substitution resulting from the CES production function applied to the Tunisian economy. During this estimation, we use the statistical data of the period 1983-1994 per 14 sectors which produce tradable goods. The production of these goods uses two factors of production, namely labour and capital. But the non-tradable goods are produced, only, starting from the factor labour. We used the method of Non-linear Least Squares (NLS) to be able to estimate elasticity of substitution of each sector using the GAMS software.

Key words: CGEM, elasticity of substitution, Tunisia, Non-linear Least Square
JEL Code Classification: C68, F1, C1.

1. Introduction

In economic theories, the economists often use the Cobb-Douglas function in their studies. Since the parameters of this function can easily be considered and determined. On the other hand, other economists, like Arrow, Chenery, Minhas and Solow (1961) suggested that the assumption of the unit elasticity of substitution of the Cobb-Douglas production function is not checked from the empirical studies. For this reason, they proposed the CES (Constant Elasticity of substitution) production function. The latter consists with the value-added of a td activity combining two factors of production (Labour and Capital). It is presented as follows:

$$VA_{td} = \left[\left(\varepsilon_{1,td} L_{td} \right)^{\rho_{td}} + \left(\varepsilon_{2,td} K_{td} \right)^{\rho_{td}} \right]^{\frac{-1}{\sigma_{td}}} \quad \text{With} \quad \rho_{td} = \frac{(1 - \sigma_{td})}{\sigma_{td}}$$

$$0 < \sigma_{td} < \infty, \text{ being given } 1 < \rho_{td} < \infty.$$

With:

- VA_{td} : value-added of the td activity (volume)
- $\varepsilon_{1,td}$: index of efficiency for the Labour factor
- $\varepsilon_{2,td}$: index of efficiency for the capital factor
- K_{td} : capital demand of td activity (volume).
- L_{td} : labour demand of td activity (volume).
- ρ_{td} : parameter of substitution in the value-added (CES)
- σ_{td} : elasticity of substitution in the value-added (CES)

This CES function has a specific function: the Cobb-Douglas functions (if $\rho_{td} = 0$) and the Leontief function (if ρ_{td} tends towards the infinite one) and when ρ_{td} is equal to -1 , the elasticity of substitution tends towards the infinite one and the production factors become perfectly substitutable.

Brown and De Cani (1963) gave the general version of the CES function to non-constant returns of scale which leaves the homogeneous latter of degree η_{td} ⁽¹⁾. In our study, $\eta_{td} = 1$ which corresponds to the case of a constant return to scale.

Within the framework of the modelling of Computable General Equilibrium, economists need to consider elasticity corresponding to each function used. The majority of the modellers often prefer to borrow the values of elasticities of the developing economies from literature. Since the estimation of these last idea is carried out from the availability of the statistical data specific to each sector and each country. In the case of developing countries, the non-

(1) The CES production function, by Brown and De Cani (1963) :

$$V^{td} = A^{td} * \left[\alpha^{td} * K^{\rho^{td}} + (1 - \alpha^{td}) * L^{\rho^{td}} \right]^{\frac{-1}{\sigma^{td}}}$$

- if $\eta_{td} = 1$: it is the constant returns to scale.
- if $\eta_{td} > 1$: it is the increasing returns to scale.
- if $\eta_{td} < 1$: it is the decreasing returns to scale.

availability of certain statistics is permanent. But that does not prevent us from applying methods according to our statistical data.

In this paper, we will estimate elasticity of substitution of our CES production function for the case of Tunisia. In general, we have two types of goods: tradable goods and non tradable goods. We will be interested only in the case of tradable goods since the value-added of the non tradable goods (the non-tradable services (according to the Input-Output of Tunisia)) is written only according to the Labour demand of the corresponding activity. Whereas the specification of the value-added used for the production of tradable goods combines both inputs (capital and labour). Thus, we can determine elasticities of substitutions of the two factors only for this type of goods. To reach this estimation, we use the statistical data provided on the one hand by Input-Output tables (I-OT, INS) for the period of 1985-1994, and on the other hand by data made by the I.E.Q. (Institute of Quantitative Economy, Tunis). From this last source, we have the data corresponding to stocks of capital (volume expressed in dinars Millions (DM)) and the occupied working population (O.W.P. expressed in thousands). Whereas for the first source, we could draw that times series of the two factors of production (capital: Great surplus of Exploitation, and Labour : wages of employees) expressed at the current prices, as well as the times series of the value-added expressed to the current prices and the constant prices.

In this spirit, we will use the approach most used in literature for the direct estimation of substitution elasticity. There are several approaches for the estimation of the parameters of the CES production function, but each approach has advantages and disadvantages according to the country in question or the sector concerned. But in our study we used the method of Least Square Non-linear. From this approach, we will try to determine the form of the production function (either CES, or Cobb-Douglas, or Leontief) corresponding to each sector. This procedure is carried out using the GAMS software. With this step, we can establish the specific form of the production function of Tunisia for each product.

2. The empirical model

We present the CES value-added function of a td activity, which uses two factors of production (labour and Capital).

$$VA_{td} = A_{td} * \left[\alpha_{td} * K_{td}^{-\rho_{td}} + (1 - \alpha_{td}) * L_{td}^{-\rho_{td}} \right]^{\frac{-1}{\rho_{td}}}$$

$$\text{avec } \rho_{td} = \frac{(1 - \sigma_{td})}{\sigma_{td}}$$

To facilitate the estimation of this non-linear function, we apply the linearization through the logarithmic form which gives us the following equation:

$$\ln(VA_{td}) = \gamma - \frac{1}{\rho} * \ln(\alpha_{td} L_{td}^{-\rho} + (1 - \alpha_{td}) K_{td}^{-\rho})$$

The method of Least Square Non-linear consists of the minimization of the errors squared under constraint of the preceding equation:

$$\text{Min } \sum_{td} r_{td}^2$$

$$\text{Subject to } \ln(VA_{td}) = \gamma - \frac{1}{\rho} * \ln\left(\alpha_{td} L_{td}^{-\rho} + (1 - \alpha_{td}) K_{td}^{-\rho}\right) + r_{td}$$

With:

td : sector

va_{td} : value-added of the td sector

l_{td} : Labour of the sector td

k_{td} : capital of the sector td

VA_{td} : value-added per capita of the td sector ($\frac{va_{td}}{L}$)

L_{td} : Labour per capita of the td sector ($\frac{l_{td}}{L}$)

K_{td} : capital per capita of the td sector ($\frac{k_{td}}{L}$)

r_{td} : error of the td sector

γ : log of the parameter of efficiency ($\gamma > 0$)

α : parameter of distribution ($0 < \alpha < 1$)

ρ : parameter of substitution of the sector td ($-1 \leq \rho \leq \infty$) and σ : elasticity of substitution of the sector td

3. Statistical data

The estimation of the elasticity of substitution is carried out for 14 products (see: Appendix 1), during the period 1985-1994. From the statistical data provided by the INS of Tunisia as well as I.E.Q., we could collect the data corresponding to the Great Surplus of Exploitation, with the returns of employees, with stocks of capital (volume expressed in DM) and the total of the working population occupied and the occupied working population by the sector (O.W.P. expressed in thousands). And starting from these statistics we can calculate the values of the variables of our program of minimization. With GAMS software, we succeeded to estimate the various parameters of our CES production function ; the elasticity of substitution is determined from the following equation:

$$\sigma_{td} = \frac{1}{(1 + \rho_{td})}$$

4. Empirical results

We recall that we will carry out the method of Least Square Non-linear for the statistics of each sector in our study. To be able to estimate the elasticity of substitution of the CES production function, we solved the minimization programme of the square errors subject to the logarithmic production function.

After having applied this program to each sector using the software GAMS (see appendix 2: case of the sector agriculture and fishing), we gather the numerical values of elasticities of substitution for each sector in the following table:

Table :

Estimate of the elasticity of substitution between K and L

<i>Sectors</i>	<i>Elasticity of substitution</i>
The agricultural produce and fishes	0.966
Products of the food industry	0.737
Materials of construction ceramic and glass	0.213
Machines and materials mechanics and electric	0.291
Chemicals products	0.261
Textiles, clothing and leather	1
products of industries manufactur. Various	0.952
Ores and minerals	0.906
Oil and gas	2.575
Transport and telecommunication	1.556
Electricity and Water	0.956
construction industries	0.972
Trade and various services	0.092
Hotel and restoration	0.277

Source: calculated by the author using the software GAMS

The preceding values of elasticities of substitution for each sector explain well the existence of a CES relation between the two factors of production: Capital and Labour. This relation is confirmed for all the sectors, but it has a specificity which is necessary to mention for the case of the textile, clothing and leather sector. The production function for this last sector appears as Cobb-Douglas, since the elasticity of substitution is equal to 1. On the other hand the production of the other sectors takes the CES form.

We notice that the majority of elasticities are lower than 1, except for the oil and gas sector (2,575) and the Transport and telecommunication sector (1,556).

5. Conclusion

The majority of the current CGE models, the modellers call upon a variety of mathematical functions with specific properties corresponding to each type of problems. These users are also interested in the functions which can be solved in a numerical way, using software (example: EVIEWS, MATLAB, FORTRAN...). But, we point out that the CES function is used in the modelling of Computable General Equilibrium since it is regarded as the general case of the other types of neo-classic function, namely Cobb-Douglas and Leontief. In this document, we were interested in presenting the estimation of elasticities of substitution of the CES production function in the case of Tunisia by applying the method of Least Squares Non-linear to the software GAMS. This estimation led to a specification of the production functions of the 14 sectors in the case of Tunisia. It helps to carry out the case of the other developing countries and also the case of a more detailed desegregation of the production sectors.

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Appendix 1

14 sectors used :

	Aggregate sectors	Disaggregated sectors
AGP	Agriculture and fishing	Agriculture and fishing
MAN	Manufacturing industries	Food industry Construction, ceramic and glass industry Mechanical engineering and electric industry Chemical industry Industry of textile, clothing & leather Industry of various manufacturings
NMAN	non manufacturing industries	Ores and minerals Oil and gas Electricity and water Construction industries
SM	Commercial services	transport and telecommunication Trade and other services Hotel and restoration

Appendix 2

Input of the method of Non-linear Least Squares with the software GAMS: Case of the agriculture and fishing sector

\$ontext

Methode des MOindres Carrée Non-linaire.
Exemple : Estimation de l elasticite de substitution a partir de
la fonction de production de type CES
fait par : Haykel h.s. Gains (juin 2004)
Secteur : Agriculture et pêche
Années : 1985 - 1994

\$offtext

set i 'observations' /i1*i10/;
set j 'parameteres' /L,K,VA/;

table data(i,j)

	L	K	VA
i1	0.299	2.954	0.919
i2	0.295	3.056	0.727
i3	0.290	3.119	0.928
i4	0.284	3.163	0.629
i5	0.279	3.215	0.656
i6	0.273	3.274	0.854
i7	0.268	3.351	0.955
i8	0.262	3.410	0.981
i9	0.256	3.446	0.908
i10	0.250	3.416	0.794

;

parameters

L(i) 'Travail'
K(i) 'Capital'
VA(i) 'valeur ajoutee'

;

L(i)= data(i,'L');
K(i)= data(i,'K');
VA(i)= data(i,'VA');

Variables

gamma 'log du parametre d efficience'
delta 'parametre de distribution'
zeta 'parametre de distribution'
rho 'parametre de substitution'
sigma 'elasticite de substitution'
* eta 'homogeneity parameter'
residual(i) 'terme error'
sse 'somme des erreurs au carre'

;

Equations

```
Fit(i) 'modele non-linaire'
obj   'objective'
;
obj.. sse =e= sum(i,sqr(residual(i)));
fit(i).. log(VA(i)) =e=
      gamma - (1/rho)*log[delta*L(i)**(-rho)+(zeta)*K(i)**(-rho)]
      + residual(i);

*initial values
rho.l = 1;
delta.l=0.01;
*delta.up=+inf;
zeta.lo=0.98;
zeta.up=+inf;
gamma.l=1;
*eta.l= 1;

model nls /obj,fit/;
option nlp=conopt;
*option nlp =minos5;
*option nlp = pathnlp;
solve nls minimizing sse using nlp;
display gamma.l, delta.l,zeta.l,rho.l,sse.l;
sigma.L = 1/(1+rho.L);
display sigma.L;
```